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Thème

Existence and stability for abstract linear and nonlinear evolution problems with single or multiple delay terms

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Dedication

To my father, ***Aissa****, and my mother,* ***Oumelkheir****, who were with me from the first moment of my life to this moment. They have been my greatest supporters in every road I have taken, and their prayers have been the secret of my success. This great achievement is dedicated to you.*

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To all my family and relatives, especially the gentle one in my heart, ***my aunt Fatima****.*

To all my friends whom I cherish their friendship, and my classmates, each by name.

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Abstract:

In this work, we were interested in studying the existence and stability of a general model of evolution equations with a delay term. By the theory of semi-groups and under appropriate conditions on the delayed damping term, we established the global existence and the exponential stability of linear and non-linear evolution systems with a constant delay first and then with a variable and multiple delay term considering a local Lipshitz source term. The exponential stability of the solution was obtained directly from the Duhamel formula of the solution in the three cases cited.

The research aims to understand how delays affect the behaviour and stability of these systems. We terminated this work by illustrating the results obtained with some applications.

Key-words: Abstract evolution equation, Delay term, Exponential stability, Semi-group

Résumé:

Dans ce travail, nous nous sommes intéressés à l'étude de l'existence et de la stabilité d'un modèle très général d'équations d'évolution avec un terme de retard. Par la théorie des semi-groupes et sous des conditions appropriées sur le terme d'amortissement retardé, on a établi l'existence globale et la stabilité exponentielle des systèmes d'évolution linéaires et non linéaires avec un retard constant en premier lieu puis avec un terme de retard variable et multiple en considérant un terme source localement lipshitzien. La stabilité exponentielle de la solution est obtenue directement à partir de la formule de Duhamel de la solution dans les trois cas cités.

La recherche vise à comprendre comment les retards affectent le comportement et la stabilité de ces systèmes. Nous avons terminé ce travail en illustrant les résultats obtenus avec quelques applications.

Mots clés: Équation d'évolution abstraite, Stabilité exponentielle, Semi-groupe, Terme de retard.

ملخص:

في هذا العمل، اهتممنا بدراسة وجود وإستقرار الحل لنموذج عام من معادلات التطور ذات حد التأخير.من خلال نظرية نصف الزمرة وباعتبار شروط مناسبةعلى مصطلح التخميدالمتأخر، أثبتنا الوجود الكلي والإستقرار الأسي لأنظمة خطية وغير خطية مع تأخير ثابت أولاً ثم مع تأخير متغير ومتعدد مع الأخذ في الاعتبار حد منبع ليبشيزي. تم الحصول على الاستقرار الأسي للحل مباشرة من صيغة دوهاميل للحل في الحالات الثلاث المذكورة.

ويهدف البحث إلى فهم كيفية تأثير التأخير على سلوك واستقرار هذه الأنظمة.أنهيناهذاالعمل بتوضيح النتائج التي تم الحصول عليها من خلال بعض التطبيقات.

الكلمات المفتاحية:مسألة التطور الخطي ، نظرية نصف الزمرة،الإستقرار الأسي، حد التأخير.

Contents

Notations

Ω	An open of $\mathbb{R}^n$, of regular boundary
A	A linear operator.
A^*	The adjoint operator of A.
$\mathcal{D}(A)$	The domain of the operator A.
$L(X,Y)$	The space of linear operators de X dans Y.
$\mathcal{L}(X,Y)$	The space of continuous linear operators of X dans Y.
$C([0,+\infty);H)$	The space of continuous functions of $[0,+\infty)$ into a Hilbert space H.
$C^1(0,T;X)$	The space of continuously differentiable functions of $[0,T]$ dans X.
$\mathcal{D}(\Omega),\ C_0^\infty(\Omega)$	The space of infinitely derivable real functions with compact support contained in $\mathcal{D}(\Omega)$.
$\mathcal{D}'(\Omega)$	The space of distributions.
$C_K^\infty(\Omega)$	The set of functions from $C_0^\infty(\Omega)$ to support in K.
$L^p(\Omega)$	The Lebesgue space, $1 \le p \le \infty$.
$\\|.\\|_p$	The norm associated with the Lebesgue space $L^p(\Omega)$.
$W^{m,p}(\Omega),\ H^m(\Omega)$	Sobolev's space.
$W^{1,\infty}(0,+\infty)$	the space of functions on the interval $(0,+\infty)$ that have essentially bounded first derivatives
$H_0^1(\Omega)$	Are the functions $u \in H^1$ such that $u\|_{\partial\Omega} = 0$.
$\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$	The gradient of u.
$u',\ u_t$	The first derivative of u with respect to the times $\frac{\partial u}{\partial t}$.
$u'',\ u_{tt}$	The second derivative of u with respect to the times $\frac{\partial^2 u}{\partial t^2}$.
$L^2(\Omega)$	The space of square-integrable functions in Ω.
$L^1_{\text{loc}}([0,\infty);R)$	The space of locally integrable functions on the interval $[0,+\infty)$ with values in R.

Introduction

In the past, our understanding of dynamic phenomena and changes in natural and engineered systems has relied majorally on ordinary and partial differential equations, which assume immediate responses to system perturbations. However, as scientific knowledge increased and the need for more accurate and complex models grew, it became clear that these simplifying assumptions were insufficient. The advances in scientific understanding have led to the integration of new concepts, particularly delayed differential equations. These can be defined as a type of differential equations that include delays or memory effects in their formulations. They are generally represented as follows

$$x'(t) = f\left(x(t), x\left(t-\tau_1\right), \cdots, x\left(t-\tau_m\right)\right),$$

with τ_j being positive. The presence of the terms $x\left(t-\tau_j\right)$ indicates that the state of the system at time t depends on its state at some previous times $t-\tau_j$. Studies examining delay feedback have shown that delays induce some instabilities [6, 13, 8, 3].

In this document, we discrete a part from the work of [7], namely, we are interested in giving existence and stability results for the below problem

$$\begin{cases} U'(t) = AU(t) + \sum_{i=1}^{n} k_i(t) B_i U\left(t-\tau_i\right) + F(U(t)), & t \in (0, \infty), \\ B_i U\left(t-\tau_i\right) = f_i(t), & t \in \left[-\tau^*, 0\right], \\ U(0) = U_0. \end{cases} \tag{1}$$

where A is the infinitesimal generator of an exponentially stable semigroup $\{S(t)\}_{t\geq 0}$ in the Hilbert space H, The functions $k_i \in L^1_{\text{loc}}\left(\mathbb{R}_+, \mathbb{R}\right)$, for $i = 1, \ldots, n$ and $U_0 \in H$, $f_i \in C\left(\left[-\tau^*, 0\right]; H\right), i = 1, \ldots, n$ are the initial data, for different settings

- ▷ Firstly, in the case linear problem with single constant delay, the function $\tau_i(t)$ be constant and $n = 1$ where the nonlinear source term F satisfies $F = 0$.
- ▷ After that, we return to the system (1) with $F = 0$.
- ▷ As last setting, we study the whole system (1) where the function F satisfies same conditions to be specified later.

For a constant delay feedback coefficient k, the decay results for the abstract model (1) have been recently obtained in [9, 10] where the type of delay considered is constant. They demonstrated when an appropriate smallness constraint on the time-delay feedback is satisfied that, if the C_0 semigroup describing the linear part of the model is exponentially stable then the entire system retains this property. We will show that this result of stability can be extend in the case of feedback varying coefficient, the function k, moreover, in the case of multiple time-varying delay functions τ_i and under suitable conditions on the source term F. In order to establish the well-posedness result and the exponential stability of solution, the theory of semigroups is used. The model (1) is very general where a quite general class of delayed differential systems satisfying this abstract setting, for instance, wave equations and Petrovesky equations with constant or time-varying delay with or without source term, can be rewritten in this framework through a good definition of the operator A, the functions k_i, the operators B_i and the nonlinear term F.

This thesis is organized as follows.

In the **first chapter**, we collected some preliminaries, definitions, theorems, and other auxiliary results used in this thesis. In addition, we presented a reminder of semigroups and evolution equations with some of their solution methods.

The **second chapter** is devoted to studying the global existence and stability results of linear and nonlinear general evolution systems with single or multiple delays. In this chapter, we based on theory of semigroup and Gronwalls lemma.

We cited an abstract nonlinear evolution equation with multiple time varying delays as an application where we verified that this example satisfying the assumptions of the framework considered. This results treated in the **third chapter**.

Chapter 1

Preliminary

The first chapter is a reminder of some mathematical tools. We begin with some definitions and fundamental properties of the theory of operators, which are useful for the future. In addition, we have given some definitions of the spaces L^p , Sobolev's spaces and functional spaces. They are followed by definitions and properties based on the strongly continuous semigroups that are essential to know in order to study our problem. At the end of this chapter, principal classical results in the nonlinear evolution equations have been stated. The main works used are [1, 5, 11, 12].

1.1 Basic theory of functional Analysis

Definition 1.1 (Banach space)**.** A normed vector space E is called a Banach space, if every Cauchy sequence in E converges.

Definition 1.2 (Hilbert space)**.** A Hilbert space is a complete inner product space X. In particular, every Hilbert space is a Banach space with respect to the norm

$$\|x\| = \sqrt{(x,x)}, \qquad \forall x \in X. \tag{1.1}$$

Theorem 1.1 (Cauchy-Schwarz)**.** *If $x, y \in X$, where X is an inner product space, then*

$$|(x,y)| \leq \|x\|\|y\|,$$

where the norm $\|\cdot\|$ is defined in (1.1).

Definition 1.3. Let X, Y be normed spaces with the same scalar field. A mapping T from a subspace of X, called the domain of T and denoted by $\mathcal{D}(T)$, into Y is a linear operator from X to Y if

$$\forall \alpha, \beta \in \mathbb{K}, \forall x, y \in X, \qquad T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

Definition 1.4. Let X and Y be two normed spaces on $\mathbb{K}$. A linear operator A defined from X to Y is a **continuous** operator in $x_0 \in X$, if

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \quad \|x - x_0\|_X \leq \delta \quad \Rightarrow \quad \|A(x) - A(x_0)\|_Y \leq \varepsilon.$$

The space of all continuous linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$.

Definition 1.5. A linear operator T is bounded if there exists a positive constant M, such that

$$\|Tx\|_Y \leq M\|x\|_X, \quad \forall x \in D(T).$$

Definition 1.6. A linear operator T from X to Y is called closed, if for every sequence $\{x_n\}$ in $\mathcal{D}(T)$, we have that, if

$$x = \lim_{n\to\infty} x_n \quad \text{and} \quad y = \lim_{n\to\infty} Tx_n$$

exist, then, $x \in \mathcal{D}(T)$ and $Tx = y$.

Theorem 1.2. *Let X be a normed space and Y a Banach space, then, $\mathcal{L}(X,Y)$ is a Banach space.*

Definition 1.7. Let $T \in \mathcal{L}(H)$ be a bounded linear operator on a Hilbert space H. There exists a unique operator $T^* \in \mathcal{L}(H)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \forall x, y \in X.$$

The operator T^* is called the adjoint of T. A linear operator T in H is said to **self-adjoint** if T is densely defined and $T = T^*$.

Proposition 1.1. *Let H be a Hilbert space, $S : H \to H$ and $T : H \to H$ be bounded linear operators and $\alpha, \beta \in \mathbb{R}$ any two scalars. We then have:*

1. $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$,
2. $(ST)^* = T^* S^*$,
3. $(T^*)^* = T$,
4. $\|T^*\| = \|T\|$,
5. $\|TT^*\| = \|T^*T\| = \|T\|^*$.

A an operator defined on a dense subspace $\mathcal{D}(A) \subset H$ and with values in H.

Definition 1.8 (Positive operator)**.** Let $A : \mathcal{D}(A) \to H$ be self-adjoint. Then, A is positive if

$$\langle Az, z\rangle \geqslant 0, \qquad \forall z \in \mathcal{D}(A).$$

A is strictly positive if for some $m > 0$

$$\langle Az, z\rangle \geqslant m\|z\|^2, \quad \forall z \in \mathcal{D}(A).$$

Definition 1.9 (Square root of an operator)**.** If $A \in \mathcal{L}(H)$ is positive, then there exists a unique positive operator $A^{\frac{1}{2}}$, called the square root of A, such as $\left(A^{\frac{1}{2}}\right)^2 = A$ Moreover, $A^{\frac{1}{2}}$ commutes with every operator that commutes with A.

1.1.1 Lebesgue spaces and Sobolev spaces

Definition 1.10. Let Ω be a non empty and open subset in $\mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{R}$. As in the one-dimensional case, the set $\operatorname{supp}\varphi$, defined by

$$\operatorname{supp}\varphi = \overline{\{x \in \Omega; \varphi(x) \neq 0\}},$$

is called the support of the function φ. Let $\mathcal{D}(\Omega)$ be the set of C^∞ functions from Ω to $\mathbb{R}$ with compact supports included in Ω. Let $\alpha \in \mathbb{N}^n$ be a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\varphi \in \mathcal{D}(\Omega)$. We define

$$D^\alpha \varphi = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

It is evident that $D(\Omega)$ is a vector space over $\mathbb{R}$.

Definition 1.11. A distribution on $\mathcal{D}(\Omega)$, we mean a real-valued, linear continuous functional defined on $\mathcal{D}(\Omega)$. We denote by $\mathcal{D}'(\Omega)$ the set of all distributions on $\mathcal{D}(\Omega)$. If $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$ we denote $(u, \varphi) = u(\varphi)$.

Definition 1.12. the derivative of order α of the function u in the sense of distributions over $\mathcal{D}(\Omega)$ is the distribution $\mathcal{D}^\alpha u$ defined by

$$(\mathcal{D}^\alpha u, \varphi) = (-1)^{|\alpha|}\int_\Omega u D^\alpha \varphi d\omega.$$

for each $\varphi \in \mathcal{D}(\Omega)$, where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ is the length of the multi-index α.

Definition 1.13. Let Ω be an open set of $\mathbb{R}^n$, equipped with the Lebesgue measure dx. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values in $\mathbb{R}$, it is provided with the norm

$$\|u\|_{L^1} = \int_\Omega |u(x)|dx.$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{f : \Omega \to \mathbb{R}, f \text{ measurable and } \int_\Omega |f(x)|^p dx < +\infty\right\},$$

equipped with norm

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}.$$

We also define the space $L^\infty(\Omega)$

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ measurable}, \exists c > 0, \text{ so that } |f(x)| \leq c \text{ a.e. on } \Omega\},$$

it will be equipped with the essential-sup norm

$$\|u\|_{L^\infty} = ess \sup_{x\in\Omega}|u(x)| = \inf\{c; |u(x)| \leq c \text{ a.e. on } \Omega\}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{\text{loc}}(\Omega)$ if $\mathbf{1}_K f \in L^p(\Omega)$ for any compact $K \subset \Omega$.

Theorem 1.3. ***(Hölder's inequality)****.Assume that $f \in L^p$ and $g \in L^{p'}$ with $1 \leq p \leq \infty$. Then $fg \in L^1$ and*

$$\int |fg| \leq \|f\|_p \|g\|_{p'}.$$

Definition 1.14. Let Ω be an open set of $\mathbb{R}$, and $1 \leq i \leq n$. A function $u \in L^1_{\text{loc}}(\Omega)$ has an i^{th} weak derivative in $L^1_{\text{loc}}(\Omega)$ if there exists $f_i \in L^1_{\text{loc}}(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega u(x)\partial_i\varphi(x)dx = -\int_\Omega f_i(x)\varphi(x)dx.$$

This leads to say that the i^{th} derivative within the meaning of distributions of u belongs to $L^1_{\text{loc}}(\Omega)$, we write

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i.$$

Definition 1.15. Let Ω be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}, 1 \leq p \leq +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega);\ \text{such that } \partial_i u \in L^p(\Omega), 1 \leq i \leq n\right\},$$

where $\partial_i u$ is the i^{th} weak derivative of $u \in L^1_{\text{loc}}(\Omega)$. For $1 \leq p < +\infty$ we define the space $W_0^{1,p}(\Omega)$ as being the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$, and we write

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}.$$

Definition 1.16. Let Ω be an open set of $\mathbb{R}^n, m \geq 2$ integer number and p real number such that $1 \leq p$, we define the space $W^{m,p}(\Omega)$ as following

$$W^{m,p}(\Omega) = \left\{u \in L^p(\Omega),\ \text{such that } \partial^\alpha u \in L^p(\Omega), \forall\alpha, |\alpha| \leq m\right\},$$

where $\alpha \in \mathbb{N}^n, |\alpha| = \alpha_1 + \ldots + \alpha_n$ the length of α and $\partial^\alpha u = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$.
The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial^\alpha u\|_{L^p}.$$

For $p = 2$, the space $W^{m,2}(\Omega)$ is noted $H^m(\Omega)$.

Proposition 1.2 (Poincaré's inequality). *Suppose I is a bounded interval. Then there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{W^{1,p}(I)} \leq C\|u'\|_{L^p(I)} \quad \forall u \in W_0^{1,p}(I).$$

In other words, on the space $W_0^{1,p}$, the quantity $\|u'\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}$ norm.

1.1.2 The functional spaces

Definition 1.17. Let $(X, \|.\|_X)$ be a real Banach space. We define the space $C(0,T;X)$ by

$$C(0,T;X) = \left\{f\ : (0,T) \to X\ ; \quad \text{with } f \text{ continuous } \right\}.$$

equipped with the norm

$$\|u\|_{C(0,T;X)} = \max_{t\in[0,T]} \|u(t)\|_X.$$

Definition 1.18. A function $f : (0,T) \to X$ is called strongly differentiable at $t_0 \in (0,T)$ if there exists an element

$$\frac{df}{dt}(t_0) \in X \quad \text{such that} \quad \lim_{h\to 0} \left\| \frac{1}{h} \left(f(t_0+h) - f(t_0) - \frac{df}{dt}(t_0) \right) \right\|_X = 0.$$

$\frac{df}{dt}(t_0)$ is called the strong derivative of f en t_0.

Definition 1.19. Let $0 < T < \infty$ and let $(X, \|.\|_X)$ be a real Banach space. We denote by $D(0,T;X)$,the set of continuous functions with compact support in $(0,T)$ with values in X.

Definition 1.20. A function $f : (0,T) \to X$ be an integrable function if there exists a sequence of functions $(f_n)_n$, $n \in \mathbb{N}$ belonging to $D(0,T;X)$ such that

$$\lim_{n\to\infty} \int_0^T \|f_n(s) - f(s)\|_X \, ds = 0.$$

Theorem 1.4 (Bochner)**.** *A measurable function $f : (0,T) \to X$ is integrable if and only if the application $t \to \|f(t)\|_X$, which is defined from $(0,T)$ into $\mathbb{R}^+$ is integrable, in this case ,we have*

$$\left\| \int_0^T f(s)\, ds \right\|_X \leq \int_0^T \|f(s)\|_X \, ds.$$

Definition 1.21. Let $1 \leq p < \infty$, The Lebesgue space $L^p(0,T;X)$ is the set of classes of measurable functions $f : (0,T)X$ such that the application $t \to \|f(t)\|_X$ belongs to $L^p(X)$. The space $L^p(0,T;X)$ is a normed space equipped with the norm

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p \, dt \right)^{\frac{1}{p}}.$$

For $p = \infty$,

$$L^\infty(0,T;X) = \{ f : (0,T) \to X; \text{ measurable and } \quad \exists\, C > 0 \,:\, \|f(t)\|_X \leq C \text{ a.e} \},$$

equipped with the norm

$$\|f\|_{L^\infty(0,T;X)} = \inf \{ C > 0 \,; \quad \|f(t)\|_X \leq C \quad \text{a.e} \quad t \in (0,T) \}.$$

Proposition 1.3.

1. *$L^p(0,T;X)$ is a Banach space, for $(1 \leq p \leq \infty)$.*
2. *If X is a Hilbert space with inner product $\langle\, .\,,\, .\, \rangle_X$ then, $L^2(0,T;X)$ is also a Hilbert space with inner product*

$$\langle u, v \rangle_{L^2(0,T;X)} = \int_0^T \langle u(t), v(t) \rangle_X dt.$$

3. *For $1 \leq q \leq r \leq \infty$, we have $L^r(0,T;X) \hookrightarrow L^q(0,T;X)$ with continuous injection.*

Definition 1.22. Let $u, w \in L^1(0,T;X)$. The function w is called the generalized derivative of order n of u on $(0,T)$ if

$$\int_0^T \varphi^{(n)}(t)\, u(t)\, dt = (-1)^n \int_0^T \varphi(t)\, w(t)\, dt \quad \forall \varphi \in \mathcal{D}(0,T;X).$$

Definition 1.23. The Sobolev space $H^1(0,T;X)$ is the space of functions $u : (0,T) \to X$ such that

$$u \in L^2(0,T;X) \quad \text{and} \quad u' \in L^2(0,T;X).$$

The space $H^1(0,T;X)$ is a Banach space equipped with the norm

$$\|u\|_{H^1(0,T;X)} = \left(\|u\|_{\ L^2(0,T;X)} + \|u'\|_{\ L^2(0,T;X)} \right)^{1/2}.$$

Given an integer $m \geq 2$,we define by recurrence the space

$$H^m(0,T;X) = \left\{ u \in H^{m-1}(0,T;X)\,;\ u' \in H^{m-1}(0,T;X) \right\},$$

equipped with the norm

$$\|u\|_{H^m(0,T;X)} = \|u\|_{L^2(0,T;X)} + \sum_{\alpha=1}^{m} \left\| u^{(\alpha)} \right\|_{L^2(0,T;X)}.$$

Proposition 1.4. *If $f \in L^p(0,T;X)$ and $\dfrac{\partial f}{\partial t} \in L^p(0,T;X)$with $(1 \leq p \leq \infty)$, So f is continuous from $[0,T]$ to X after a possible modification on a negligible set of $(0,T)$.*

Definition 1.24 (locally integrable function)**.** A function u defined almost everywhere on Ω is said to be locally integrable on Ω provided $u \in L^1(\Omega)$ for every open $U \subset \Omega$. In this case we write $u \in L^1_{\text{loc}}(\Omega)$.

Lemma 1.1 (Gronwall's Lemma)**.** *Let $\alpha \geq 0$ and $\beta \in C(0,T;\mathbb{R})$ such that $\beta(t) \geq 0$, $\forall t \in [0,T]$. if $u \in C(0,T;\mathbb{R})$ is a function such that:*

$$u(t) \leq \alpha + \int_0^T \beta(s)u(s)ds, \quad \forall t \in [0,T].$$

then,

$$u(t) \leq \alpha e^{\int_0^T \beta(s)ds}, \quad \forall t \in [0,T].$$

Definition 1.25 (Gâteaux Differentiability)**.** Let $f : U \longrightarrow Y$ be a map, Let $x_0 \in U$.The function f is a Gâteaux Differentiable function at x_0 if

1. f is differentiable at x_0 in every direction $v \in U \backslash \{0\}$.
2. There exists a bounded linear map $A \in \mathcal{L}(X,Y)$ such that $f'(x_0, v) = A(v)$ for all v element of $X \backslash \{0\}$.

In this case, the map $f'(x_0)$ is called the Gâteaux differential of f at x_0 and is denoted by $D_G f(x_0)$ or $f'_G(x_0)$.

In orther words, f is Gâteaux differentiable at x_0 if there exists a bounded linear map $A \in \mathcal{L}(X,Y)$ such that:

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = A(v), \quad \forall v \in X \backslash \{0\}.$$

1.2 Basic theory of semigroups

1.2.1 Uniformly continuous semigroups

Definition 1.26. Let X be a Banach space. A one parameter family $\{T(t)\}$, $0 \leq t < \infty$, of bounded linear operators from X into X, is a semigroup of bounded linear operator on X if

1. $T(0) = I$ (I is the identity operator on X).
2. $T(t+s) = T(t)T(s)$, for every $t, s \geq 0$ (the semigroup property).

Definition 1.27. A semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$, is uniformly continuous if

$$\lim_{t\longrightarrow 0} \|T(t) - I\| = 0.$$

if $\{T(t)\}_{t\geq 0}$ is a uniformly continuous semigroup of bounded linear operators then

$$\lim_{x\longrightarrow t} \|T(s) - T(t)\| = 0.$$

Definition 1.28. The linear operator A defined by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{t\longrightarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \left.\frac{d^{+}T(t)x}{dt}\right|_{t=0}, \quad \forall x \in \mathcal{D}(A),$$

is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$, $\mathcal{D}(\mathcal{A})$ is the domain of A.

Theorem 1.5. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Remark 1.1. a semigroup $\{T(t)\}_{t\geq 0}$ has a unique infinitesimal generator. If $T(t)$ is uniformly continuous its infinitesimal generator is a bounded linear operator. On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\}_{t\geq 0}$. Is this semigroup unique? This result is given by the following theorem:

Theorem 1.6. *Let $\{T(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\longrightarrow 0} \frac{T(t) - I}{t} = A = \lim_{t\longrightarrow 0} \frac{S(t) - I}{t},$$

then $T(t) = S(t)$, for $t \geq 0$.

Corollaire 1.1. *Let $\{T(t)\}_{t\geq 0}$ be a uniformly continuous semigroup of bounded linear operators, then*

a. *There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.*

b. *There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.*

c. *The operator A in the part (b) is an infinitesimal generator of $\{T(t)\}_{t\geq 0}$.*

d. *$t \to T(t)$ is differentiable in norm, and*

$$\frac{dT(t)}{dt} = AT(t) = T(t)A.$$

1.2.2 Strongly continuous semi-groups

In the following, we assume that A is a bounded linear operator of X in X.

Definition 1.29. A semigroup $\{T(t)\}_{t\geq 0}$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t\longrightarrow 0} T(t)x = x \quad \text{for every} \quad x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

Theorem 1.7. *Let $\{T(t)\}_{t\geq 0}$ be a C_0 semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \quad \textit{for} \quad 0 \leq t < \infty. \tag{1.2}$$

Definition 1.30. The growth bound of the strongly continuous semigroup T is the number $\omega_0(T)$ defined by

$$\omega_0(T) = \inf_{t\in(0,\infty)} \frac{1}{t} \log \|T_t\|.$$

Clearly ,$\omega_0(T) \in [-\infty, \infty)$.

Proposition 1.5. *Let T be a strongly continuous semigroup on X, with growth bound $\omega_0(T)$. Then*

1. $\omega_0(T) = \lim_{t\to\infty} \frac{1}{t} \log \|T_t\|$,
2. *For any $\omega > \omega_0(T)$ there exists an $M_\omega \in [1, \infty)$ such that*

$$\|T_t\| \leqslant M_\omega e^{\omega t}, \quad \forall t \in [0, \infty).$$

3. *The function $\varphi : [0, \infty) \times X \to X$ defined by $\varphi(t, z) = T_t z$ is continuous .*

Definition 1.31. Let$\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on X, with growth bound $\omega_0(T)$. This semigroup is called exponentially stable if $\omega_0(T) < 0$.

We give some properties on the $\mathcal{C}_0-$semigroups.

Theorem 1.8. *Let $\{T(t)\}_{t\geq 0}$ be a $\mathcal{C}_0$ semigroup and let A be its infinitesimal generator. Then*

1. *For $x \in X$,*

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x.$$

2. *For $x \in X$,*

$$\int_0^1 T(s)xds \in \mathcal{D}(A) \quad \textit{and} \quad A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

3. *For* $x \in \mathcal{D}(A)$,

$$T(t)x \in \mathcal{D}(A) \quad \textit{and} \quad \frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

4. *For* $x \in \mathcal{D}(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Theorem 1.9. *Let* $\{T(t)\}_{t\geq 0}$ *and* $\{S(t)\}_{t\geq 0}$ *be* C_0 *semigroups of bounded linear operators with infinitesimul generators* A *and* B *respectively. If* $A = B$ *then ,*

$$T(t) = S(t), \quad \textit{for all } t \geq 0.$$

Theorem 1.10. *If* A *is the infinitesimal generator of a* C_0 *semigroup* $\{T(t)\}_{t\geq 0}$ *then,the domain of* A*,* $\mathcal{D}(A)$ *is dense in* X *and* A *is a closed linear operator.*

1.2.3 Hille-Yosida and Lumer-Phillips theorem

The focus of this paragraph is to provide an analysis of the $\mathcal{C}_0-$semigroups of contractions, which is commonly referred to as the Lumer Phillips theorem. To begin, it is necessary to establish some preliminary information.

Definition 1.32. A one-parameter family $\{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is a contraction semigroup on X provided

$$T(t) \leq 1, \quad \forall t \geq 0.$$

Definition 1.33. Let X be a Banach space and let X^* be its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by

$$F(x) = \left\{x^* : x^* \in X^* \quad \text{and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\right\}.$$

Definition 1.34. A linear operator A is dissipative if for every $x \in D(A)$ there is a $x^* \in F(x)$ such that

$$\operatorname{Re}\langle Ax, x^*\rangle \leq 0.$$

Theorem 1.11. *A lineair operator* A *in dissipatios if and only if*

$$\|(\lambda I - A)x\| \geq \lambda\|x\| \quad \textit{for all} \quad x \in \mathcal{D}(A) \textit{ and } \lambda > 0.$$

Theorem 1.12 (Hille-Yosida)**.** *A linear (unbounded) operator. A is the infinitesimal generator of a* C_0 *semigroup of contractions* $\{T(t)\}_{t\geq 0}$*,*$t \geq 0$*if and only if*

1. A *is closed and* $\overline{\mathcal{D}(A)} = X$.

2. *The resolvent set* $\rho(A)$ *of* A *contains* $\mathbb{R}^+$*and for every* $\lambda > 0$,

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}.$$

Theorem 1.13 (Lumer Phillips)**.** *Let A be a linear operator with dense domain $\mathcal{D}(A)$in X.*

1. *If A is dissipative and there is a$\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$, of $\lambda_0 I - A$ is X, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

2. *If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x), \operatorname{Re}\langle Ax, x^*\rangle \leq 0$.*

1.3 Nonlinear Evolution Equations

1.3.1 The homogeneous Cauchy problem

Let X be a Banach space and let A be a linear operator from $\mathcal{D}(A) \subset X$ into X. Given $u_0 \in X$ the abstract Cauchy problem, with initial data u_0, consists of find a solution u to the following initial value problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t), \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

We mean by a solution an X, valued function u such that u is continuous for $t \geq 0$, continuously differentiable and $u \in \mathcal{D}(A)$ for $t > 0$ and (1.3) is satisfied.

Theorem 1.14. *If A is the infinitesimal generator of a differentiable semigroup, then, for every $u_0 \in X$, the initial value problem* (1.3) *has a unique solution.*

Proof. see ([11], Ch.4 ,P.104) □

1.3.2 The Inhomogeneous Cauchy problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t) \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where $f : [0, T[\to X$.

We will assume throughout this section that A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$. Consequently the corresponding homogeneous equation, i.e., the equation with $f \equiv 0$, has a unique solution for every initial value $u_0 \in \mathcal{D}(A)$.

Definition 1.35. A function $u : [0, T[\to X$ is a (classical) solution of(1.4) if

1. u is contimuous for $[0, T]$.

2. u is continuously differentiable function $]0, T[$.

3. $u(t) \in \mathcal{D}$, for $t \in]0, T[$ and (1.4) is satisfied on $[0, T[$.

Theorem 1.15. *If $f \in L^1(0, T; X)$, then for all $x \in X$,the initial value problem* (1.4) *has at most one solution. If it has a solution , this solution is given by*

$$u(t) = T(t)u_0 + \int_0^t T(t - s)f(s)ds.$$

Proof. see ([11], Ch.4 ,P.106) □

Definition 1.36. Let A be the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$. Let $u_0 \in X$ and $f \in L^1(0,T;X)$. The function $u \in C([0,T];X)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T,$$

is the mild solution of the initial value problem (1.4) on $[0,T]$ and the last estimate is called Duhamel's formula.

1.3.3 Nonlinear Evolution Equations

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t,u(t)), & t > t_0, \\ u(t_0) = u_0, \end{cases} \tag{1.5}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0$, on a Banach space X and $f : [t_0,T] \times X \to X$ is continuous in t and satisfies a Lipschitz condition in u.

Remark 1.2. The initial value problem(1.5) does not necessarily have a solution of any kind. However, if it has a classical or strong solution , shows that this solution u satisfies the integral equation

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^t T(t-s)f(s,u(s))ds. \tag{1.6}$$

Definition 1.37. A continuous solution u of the integral equation (1.6) will be called a mild solution of the initial value problem (1.5).

We start with the following classical result which assures the existence and uniqueness of mild solutions of (1.5) for Lipschitz continuous functiuns f.

Theorem 1.16. *Let $f : [t_0,T] \times X \to X$ be continuous in t on $[t_0,T]$ and unifirrmly Lipschitz continuous (with constant L) on X. If $-A$ is the infinitesimul generator of a C_0 semigroup $T(t), t \geq 0$, on X then for every $u_0 \in X$ the Initidal value problem (1.5)has a unique mild solution $u \in C([t_0,T] : X)$.*

Theorem 1.17. *Let $f : [0,\infty \mid \times X \to X$ be continuous in t for $t \geq 0$ and locally Lipschitz contimuuns in u, uniformly in t on boumded intervals. If $-A$ is the infinitesimal generutor of a $\mathcal{C}_0$ semigroup $T(t), t \geq 0$ on X then for every $u_0 \in X$ there is a $t_{max} \leq \infty$ such that, the initial value problem*

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t,u(t)), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

has a unique mild solution u on $[0, t_{\max}[$. Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \longrightarrow t_{max}} \|u(t)\| = \infty.$$

Theorem 1.18. *(Regularity). Let $-A$ be the infinitesimal generator of a C_0 semigroup $T(t), t \geq 0$ on X. If $f : [t_0,T] \times X \to X$ is continuously differentiable from $[t_0,T] \times X$ into X then the mild solution of (1.5) with $u_0 \in \mathcal{D}(A)$ is a classical solution of the initial value problem.*

Chapter 2

Existence and stability of abstract linear and nonlinear evolution systems with a single or multiple delay term

This chapter investigates the global existence and the energy decay for abstract linear evolution problems with delay terms. We started by considering a constant delay and ended with a nonlinear version of the previous setting where multiple time-independent delays replaced the constant delay. Based on semigroup theory, the existence and uniqueness of the solution are established for both systems under specific conditions. Furthermore, exponential stability results are derived directly by estimating the solution formula. The analysis encompasses both the case of a single constant delay and time-varying delays. A nonlinear model with a local Lipschitz condition on the nonlinearity is also considered.

2.1 Abstract linear evolution equation with a constant delay

2.1.1 Setting of problem

Let H be a Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, A an operator from H into itself which generates a $\mathcal{C}_0$-semi group exponentially stable.

The objective of this section is to study the well-posedness and the exponential stability results for the following abstract delayed evolution equation:

$$(P_1) : \begin{cases} U'(t) = AU(t) + k(t)BU(t-\tau), & \forall t \in (0, \infty), \\ U(0) = U_0, \\ BU(t-\tau) = f(t), & \forall t \in (0, \tau), \end{cases}$$

where $\tau > 0$ is a fixed delay parameter, $k : [0, \infty) \to \mathbb{R}$ is a function belonging to $L^1_{\text{loc}}([0, \infty); \mathbb{R})$ and $U_0 \in H, f \in C(0, \tau; H)$ are the initial datums.

For getting the intended results, we assume the following assumptions:

(H_1) A is the infinitesimal generator of an exponentially stable semigroup $\{S(t)\}_{t\geq 0}$ in the Hilbert space H which means

$$\exists M, \omega > 0, \quad \|S(t)\|_{\mathcal{L}(H)} \leq Me^{-wt}, \quad \forall t \geqslant 0. \tag{2.1}$$

(H_2) $B : H \to H$ is a continuous linear operator.

2.1.2 Well-posedness

The existence of the solution of the system (P_1) is given by the following proposition

Proposition 2.1. *Given $U_0 \in H$ and a continuous function $f : [0, \tau] \to H$, the problem (P_1) has a unique (weak) solution given by Duhamel's formula, for all $t \geq 0$*

$$U(t) = S(t)U_0 + \int_0^t k(s)S(t-s)BU(s-\tau)ds. \tag{2.2}$$

Proof. The desired result is to establish the existence and uniqueness of a continuous solution U for the system (P_1) on the entire time domain $[0, +\infty)$. We achieve this by dividing the time domain into intervals of length τ and analyzing each interval iteratively.

For the first interval $[0, \tau]$, we define

$$G_1(t) = k(t)BU(t-\tau), \quad t \in [0, \tau],$$

to integrate the time delay and rewrite the system (P_1) as a standard non-homogeneous problem

$$\begin{cases} U'(t) = AU(t) + G_1(t), & \forall t \in (0, \tau), \\ U(0) = U_0. \end{cases} \tag{2.3}$$

We have that $G_1 \in L^1((0, \tau); H)$ since $k \in L^1_{loc}([0, \infty); \mathbb{R})$ and $f \in C([0, \tau]; H)$, by Theorem 1.15, the system (2.3) admit a unique solution $U \in C([0, \tau]; H)$ on this interval satisfying the Duhamel's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)G_1(s)ds, \quad \forall t \in [0, \tau],$$

then, it gives

$$U(t) = S(t)U_0 + \int_0^t k(s)S(t-s)BU(s-\tau)ds, \quad \forall t \in [0, \tau]. \tag{2.4}$$

Similarly, for the next interval $[\tau, 2\tau]$, setting

$$G_2(t) = k(t)BU(t-\tau),$$

where $U(t-\tau)$ for $t \in [\tau, 2\tau]$ is obtained from the first step. Therefore, the model (P_1) may also rewritten as inhomogeneous Cauchy problem in the interval $[\tau, 2\tau]$. We may recast the model (P_1) as follows

$$\begin{cases} U'(t) = AU(t) + G_2(t) & \text{in} \quad (\tau, 2\tau), \\ U(\tau) = U(\tau^-), \end{cases} \tag{2.5}$$

with $U(\tau^-)$ is a notation of the value of U in $t = \tau$, it can be estimated from (2.4).

By using a classical result of abstract Cauchy problems, there exist a unique continuous solution $U : [\tau, 2\tau] \to H$ satisfying

$$U(t) = S(t-\tau)U\left(\tau^-\right) + \int_\tau^t k(s)S(t-s)BU(s-\tau)ds, \quad t \in [\tau, 2\tau]. \tag{2.6}$$

Combining the partial solutions (2.4) and (2.6), we construct a unique continuous solution $U : [0, 2\tau] \to \mathbb{R}$ satisfying the following formula of Duhamel

$$U(t) = S(t)U_0 + \int_0^t k(s)S(t-s)BU(s-\tau)ds, \quad t \in [0, 2\tau].$$

By repeating this process on subsequent intervals and extending the solutions, we ultimately prove the existence and uniqueness of the desired solution $U \in C([0,\infty); H)$ on the whole time domain given by (2.2). □

2.1.3 Exponential stability

In this subsection, we establish the exponential stability directly from the Duhamel's formula (2.2). This result is stated in the following theorem

Theorem 2.1. *Assume that there exist two constants $\omega' \in (0, \omega)$ and $\gamma \in \mathbb{R}$ such that, for every $t \geq 0$*

$$M\|B\|e^{\omega\tau} \int_0^t |k(s+\tau)|ds \leq \gamma + \omega' t. \tag{2.7}$$

Then, there exists a positive constant M' such that the solution decays exponentially, for all $t \geq 0$, we have

$$\|U(t)\| \leq M' e^{-(\omega-\omega')t}. \tag{2.8}$$

Proof. According to Proposition 2.1, the system (P_1) admit a unique solution given by

$$U(t) = S(t)U_0 + \int_0^t k(s)S(t-s)BU(s-\tau)ds, \quad t \in \mathbb{R}_+.$$

Passing by the norm, we get

$$\begin{aligned}\|U(t)\| &= \left\|S(t)U_0 + \int_0^t k(s)S(t-s)BU(s-\tau)ds\right\|, \\ &\leq \|S(t)\| \, \|U_0\| + \int_0^t |k(s)| \|S(t-s)\| \|BU(s-\tau)\| ds,\end{aligned}$$

so, We obtain

$$\begin{aligned}\|U(t)\| \leq& \|S(t)\| \, \|U_0\| + \int_0^\tau |k(s)| \|S(t-s)\| \|BU(s-\tau)\| ds \\ &+ \int_\tau^t |k(s)| \|S(t-s)\| \|BU(s-\tau)\| ds,\end{aligned}$$

from the third equation in the system (P_1) and by (H_2), we have

$$\begin{aligned}\|U(t)\| \leq& \|S(t)\| \, \|U_0\| + \int_0^\tau |k(s)| \|S(t-s)\| |f(s)| ds \\ &+ \int_\tau^t |k(s)| \|S(t-s)\| \|B\| \|U(s-\tau)\| ds.\end{aligned}$$

Thus, by using the fact that the semigroup $\{S(t)\}_{t\geq 0}$ is exponentially stable, the inequality (2.1), we arrive at

$$\|U(t)\| \leq Me^{-\omega t}\|U_0\| + \int_0^{\tau} |k(s)|Me^{-\omega(t-s)}|f(s)|\,ds + \int_{\tau}^{t} |k(s)|Me^{-\omega(t-s)}\|B\|\|U(s-\tau)\|\,ds.$$

Then,

$$\|U(t)\| \leq Me^{-\omega t}\|U_0\| + M\int_0^{\tau} |k(s)|e^{\omega(s-t)}|f(s)|\,ds + M\|B\|\int_{\tau}^{t} |k(s)|e^{\omega(s-t)}\|U(s-\tau)\|\,ds,$$

which is

$$e^{\omega t}\|U_t\| \leq M\|U_0\| + M\int_0^{\tau} |k(s)|e^{\omega s}|f(s)|\,ds + M\|B\|\int_{\tau}^{t} |k(s)|e^{\omega s}\|U(s-\tau)\|\,ds.$$

Therefore,

$$e^{\omega t}\|U_t\| \leq M\|U_0\| + Me^{\omega\tau}\int_0^{\tau} |k(s)|e^{\omega(s-\tau)}\|f(s)\|ds + M\|B\|e^{\omega\tau}\int_{\tau}^{t} |k(s)|e^{\omega(s-\tau)}\|U(s-\tau)\|ds, \tag{2.9}$$

Now, for all t ≥ 0, we take

$$u(t) := e^{\omega t}\|U(t)\|, \alpha := M\|U_0\| + Me^{\omega\tau}\int_0^{\tau} |k(s)|e^{\omega(s-\tau)}\|f(s)\|ds \tag{2.10}$$

and

$$\beta(t) := M\|B\|e^{\omega\tau}|k(t+\tau)|. \tag{2.11}$$

On the other hand, the last term in (2.9) be can estimate by making the following change of variable $\delta = s-\tau$, it result

$$\int_0^{t-\tau} |k(\delta+\tau)|e^{\omega\delta}\|U(\delta)\|ds,$$

then,

$$M\|B\|e^{\omega\tau}\int_0^{t-\tau} |k(s+\tau)|e^{\omega(s)}\|U(s)\|ds. \tag{2.12}$$

By combining (2.9), (2.12), (2.10)and (2.11), we obtain

$$u(t) \leq \alpha + \int_0^{t-\tau} \beta(s)u(s)ds, \quad \forall t \geq 0,$$

Therefore, such as $\beta \geq 0$ and $u \geq 0$,

$$u(t) \leq \alpha + \int_0^{t} \beta(s)u(s)ds, \quad \forall t \geq 0.$$

We apply Lemma 1.1, we get, for all $t \geq 0$

$$u(t) \leq \alpha e^{\int_0^t B(s)ds}.$$

From the definition of u, we attain

$$e^{\omega t}\|U(t)\| \leq \alpha e^{\int_0^t B(s)ds}, \quad \forall t \geq 0.$$

Lastly, we found

$$\|U(t)\| \leq \alpha e^{\int_0^t \beta(s)ds - \omega t}, \quad \forall t \geq 0. \tag{2.13}$$

On the other hand, by (2.11) and(2.7), we acquire

$$\begin{aligned}\int_0^t \beta(s)ds - \omega t &= \int_0^t M\|B\|e^{\omega\tau}|k(s+\tau)|ds - \omega t, \\ &\leq \gamma + \omega' t - \omega t = \gamma - (\omega - \omega')\, t.\end{aligned}$$

Consequently, the inequality (2.13) becomes

$$\|U(t)\| \leq \alpha e^{\gamma} e^{-(\omega-\omega')t} \leq M' e^{-(\omega-\omega')t},$$

for all $t \geq 0$ where $M' = \alpha e^{\gamma}$.

Hence, We obtain the exponential stability estimate of the solution of the system (P_1). Otherwise, the proof of the Theorem 2.1 is done. □

2.2 Abstract linear evolution equation with multiple time varying delays

2.2.1 Setting of problem

Let consider the following abstract evolution equation with multiple time dependent delays

$$(P_2) : \begin{cases} U'(t) = AU(t) + \sum_{i=1}^{n} k_i(t) B_i U\left(t - \tau_i(t)\right), & \forall t \in \ (0, \infty), \\ B_i U(t) = f_i(t), & \forall t \in \left[-\tau^*, 0\right], \\ U(0) = U_0, \end{cases}$$

where

1. As previously, the operator A generates an exponentially stable semigroup $\{S(t)\}_{t\geq 0}$ in a Hilbert space H.
2. $\forall i = 1, \dots, n, \tau_i$ be the time delayed functions with τ^* will be defined later.
3. The functions $k_i : \mathbb{R}_+ \to \mathbb{R}$ are belonging to $L^1_{\text{loc}}\left(\mathbb{R}_+, \mathbb{R}\right)$, for $i = 1, \dots, n$.
4. $U_0 \in H, f_i \in C\left(\left[-\tau^*, 0\right]; H\right), i = 1, \dots, n$ are the initial data.

For studying the system (P_2), we'll need to make the next assumptions :

(H_3) $\forall i = 1, \dots, n, B_i$ are continuous linear operator from H to itself.

(H_4) The time-varying delay functions $\tau_i : \mathbb{R}_+ \to (0, +\infty), i = 1, \dots, n$ belonging to $W^{1,\infty}$, moreover, we suppose, for each $i = 1, \dots, n$,

$$0 \leq \tau_i(t) \leq \bar{\tau}_i, \tag{2.14}$$

and

$$\tau_i'(t) \leq c_i < 1, \tag{2.15}$$

for almost everywhere $t > 0$, with appropriate constants $\bar{\tau}_i$ and c_i. Let give the parameter τ^* by

$$\tau^* := \max_{i=1,\dots,n} \left\{\tau_i(0)\right\}, \tag{2.16}$$

Remark 2.1. We get from (2.15) that

$$(t - \tau_i(t))' = 1 - \tau_i'(t) > 0, \qquad \text{a.e.} \qquad \forall t > 0, \tag{2.17}$$

and therfore,

$$t - \tau_i(t) \geq -\tau_i(0), \quad \forall t \geq 0. \tag{2.18}$$

2.2.2 Well-posedness

For the existence and uniqueness result of solution in this case, we refer to the following theorem

Theorem 2.2. *Given $U_0 \in H$ and continuous functions $f_i : [-\tau^*, 0] \to H, i = 1, \ldots, n$, the problem (P_2) has a unique (weak) solution given by*

$$U(t) = S(t)U_0 + \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i U\left(s - \tau_i(s)\right) ds, \quad \forall t \geq 0. \tag{2.19}$$

Proof. Let us firstly consider that for all $i = 1, \ldots, n$, the time delays functions $\tau_i(t)$ are bounded from below by positive constants $\underline{\tau}_i$, which means

$$\tau_i(t) \geq \underline{\tau}_i, \qquad \text{for every} \qquad t \geq 0. \tag{2.20}$$

Similarly to the the proof of Proposition (2.2), let us use the method of steps by considering an interval of length τ_m in which given as follow

$$\tau_{\min} := \min\left\{\underline{\tau}_i, i = 1, \ldots, n\right\} > 0. \tag{2.21}$$

By (2.20) and(2.21) , we have

$$\tau_i(t) \geq \underline{\tau}_i \geq \tau_{\min} > 0, \quad \forall t \geq 0,$$

so,

$$t - \tau_i(t) \leq t - \underline{\tau}_i \leq t - \tau_{\min} \quad \forall t \geq 0 \text{ and } i = 1, \ldots, n.$$

Therefore, if $t \in [k\tau_{\min}\,, (k+1)\tau_{\min}\,]$, we've got

$$k\tau_{\min} \leq t \leq (k+1)\tau_{\min},$$

then,

$$t - \tau_i(t) \leq k\tau_{\min}\,, \quad \forall i = 1, \ldots, n.$$

Let pass to the general case. For each fixed positive number $0 \leq \epsilon \leq 1$, let define the modified time delay functions $\tau_i^\epsilon(t)$ by

$$\tau_i^\epsilon(t) := \tau_i(t) + \epsilon, \quad t \geq 0, \quad i = 1, \ldots, n. \tag{2.22}$$

Moreover, we extend the initial data f_i to continuous functions $\tilde{f}_i : [-\tau^* - 1, 0] \to H$ with the constant τ^* given by (2.16). Now, we introduce the next model

$$\begin{cases} U_\epsilon'(t) = AU_\epsilon(t) + \sum_{i=1}^n k_i(t) B_i U\left(t - \tau_i^\epsilon(t)\right), & \forall t \in (0, \infty), \\ B_i U_\epsilon(t) = \tilde{f}_i(t), & \forall t \in \left[-\tau^* - 1, 0\right], \\ U_\epsilon(0) = U_0. \end{cases} \tag{2.23}$$

This system has a unique and continuous solution $U_\epsilon(\cdot) \in C([0, +\infty); H)$, since $\tau_i^\epsilon(t) \geq \epsilon > 0$, for each t and i, and as $\tilde{f}_i$ are continuous functions. The solution satisfying the following formula

$$U_\epsilon(t) = S(t)U_0 + \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i U_\epsilon\left(s - \tau_i^\epsilon(s)\right) ds, \tag{2.24}$$

for all $t \geq 0$ and $i = 1, \dots, n$.

Now, for $\epsilon_1, \epsilon_2 \leq 1$ and by using (2.24), we have

$$
\begin{aligned}
U_{\epsilon_1}(t) - U_{\epsilon_2}(t) =& \left[S(t)U_0 + \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) ds \right] \\
& - \left[S(t)U_0 + \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) ds \right], \\
=& \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i \left[U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) \right] ds, \\
=& \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i \left[U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}(s)\right) \right] ds \\
& + \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i \left[U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) \right] ds, \\
=& \sum_{i=1}^n \int_0^t S(t-s) k_i(s) B_i \left[U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}(s)\right) \right] ds \\
& + \sum_{i=1}^n \int_0^t S(t-s) k_i(s) B_i \left[U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) \right] ds.
\end{aligned}
$$

Passing by the norm, we obtain

$$
\begin{aligned}
\|U_{\epsilon_1}(t) - U_{\epsilon_2}(t)\| \leq & \left\| \sum_{i=1}^n \int_0^t S(t-s) k_i(s) B_i \left[U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}(s)\right) \right] ds \right\| \\
& + \left\| \sum_{i=1}^n \int_0^t S(t-s) k_i(s) B_i \left[U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) \right] ds \right\|.
\end{aligned}
$$

Therefore, by (2.1), we arrive at

$$
\begin{aligned}
\|U_{\epsilon_1}(t) - U_{\epsilon_2}(t)\| \leq & \int_0^t M e^{-\omega(t-s)} \sum_{i=1}^n |k_i(s)| \, \|B_i\| \, \left\| U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}(s)\right) \right\| ds \\
& + \int_0^t \sum_{i=1}^n M e^{-\omega(t-s)} |k_i(s)| \, \|B_i\| \, \left\| U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) \right\| ds.
\end{aligned}
$$

Under the assumption (H_3), it result

$$
e^{\omega t} \|U_{\epsilon_1}(t) - U_{\epsilon_2}(t)\| \leq M \left(I_1 + I_2 \right), \tag{2.25}
$$

with

$$
I_1 := \int_0^t e^{\omega s} \sum_{i=1}^n |k_i(s)| \, \|B_i\| \, \left\| U_{\epsilon_1}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}(s)\right) \right\| ds
$$

and

$$
I_2 := \sum_{i=1}^n \int_0^t e^{\omega s} |k_i(s)| \, \|B_i\| \, \left\| U_{\epsilon_2}\left(s - \tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s - \tau_i^{\epsilon_2}(s)\right) \right\| ds.
$$

Let firstly estimate I_1, for

$$
\sigma = s - \tau_i^{\epsilon 1}(s) \quad \text{and} \quad \varphi_i(s) := s - \tau_i(s) = \epsilon_1 + \sigma, \tag{2.26}
$$

we have,

$$I_1 = \sum_{i=1}^{n} \int_{-\tau_i^{\epsilon 1}(0)}^{t-\tau_i^{\epsilon 1(t)}} e^{\omega\left(\sigma+\tau_i^{\epsilon 1}(s)\right)} \left|k_i\left(\varphi_i^{-1}\left(\epsilon_1+\sigma\right)\right)\right| \|B_i\| \, \|U_{\epsilon_1}(\sigma) - U_{\epsilon_2}(\sigma)\| \, \frac{d\sigma}{1-\tau_i'(s)},$$

Using (2.15) and (2.22), we get

$$I_1 \leq \sum_{i=1}^{n} \int_{-\tau_i^{\epsilon 1}(0)}^{t-\tau_i^{\epsilon 1}(t)} \frac{e^{\omega(\tau_i(s)+\epsilon_1)}}{1-c_i} e^{\omega\sigma} |k_i\left(\varphi_i^{-1}\left(\sigma+\epsilon_1\right)\right)| \, \|B_i\| \, \|U_{\epsilon_1}(\sigma) - U_{\epsilon_2}(\sigma)\| \, ds,$$

from (2.14) and as $\epsilon_1 \leq 1$, we achieve

$$I_1 \leq \sum_{i=1}^{n} \int_{-\tau_i^{\epsilon 1}(0)}^{t-\tau_i^{\epsilon 1(t)}} \frac{e^{\omega(\bar{\tau}_i+1)}}{1-c_i} e^{\omega\sigma} \left|k_i\left(\varphi_i^{-1}\left(\sigma+\epsilon_1\right)\right)\right| \|B_i\| \, \|U_{\epsilon_1}(\sigma) - U_{\epsilon_2}(\sigma)\| \, ds,$$

Consequently,

$$I_1 \leq \sum_{i=1}^{n} \int_{-\tau_i^{\epsilon 1}(0)}^{t} \frac{e^{\omega(\bar{\tau}_i+1)}}{1-c_i} e^{\omega\sigma} \left|k_i\left(\varphi_i^{-1}\left(\sigma+\epsilon_1\right)\right)\right| \|B_i\| \, \|U_{\epsilon_1}(\sigma) - U_{\epsilon_2}(\sigma)\| \, ds,$$

which is

$$I_1 \leq \sum_{i=1}^{n} \frac{e^{\omega(\bar{\tau}_i+1)}}{1-c_i} \|B_i\| \int_0^t e^{\omega s} \left|k_i\left(\varphi_i^{-1}\left(s+\epsilon_1\right)\right)\right| \|U_{\epsilon_1}(s) - U_{\epsilon_2}(s)\| \, ds. \tag{2.27}$$

Secondly, let estimate I_2, we have

$$I_2 = \sum_{i=1}^{n} \int_0^t e^{\omega s} |k_i| \, \|B_i\| \, \|U_{\epsilon_2}\left(s-\tau_i^{\epsilon_1}(s)\right) - U_{\epsilon_2}\left(s-\tau_i^{\epsilon_2}(s)\right)\| \, ds.$$

Since $U_{\epsilon_2}(\cdot) \in C\left(\mathbb{R}_+, H\right)$, then it is locally uniformly continuous, for every $x \in \mathbb{R}_+$there exists a neighbourhood $\vartheta \subset \mathbb{R}_+$such that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x_1 - x_2| \leq \delta \quad \Rightarrow \quad \|U_{\epsilon_1}\left(x_1\right) - U_{\epsilon_2}\left(x_2\right)\| \leq \varepsilon,$$

for all $x_1, x_2 \in \vartheta$. Let take $x_1 = s - \tau_i^{\epsilon_1}(s)$ and $x_2 = s - \tau_i^{\epsilon_2}(s)$, we obtain

$$x_1 - x_2 = \tau_i(s) + \epsilon_1 - \left(\tau_i(s) + \epsilon_2\right) = \epsilon_1 - \epsilon_2.$$

Then, for a fixed $T > 0$, we deduce

$$I_2 \leq \sum_{i=1}^{n} \int_0^t \varepsilon \, |k_i| \, \|B_i\| \, e^{\omega s} ds, \leq C\left(T, \epsilon_1 - \epsilon_2\right).$$

So, we get the following estimate

$$I_2 \leq C\left(T, \epsilon_1 - \epsilon_2\right). \tag{2.28}$$

Now, using (2.27) and (2.28) in (2.25), we conclude

$$\begin{aligned} e^{\omega t} \|U_{\epsilon_1}(t) - U_{\epsilon_2}(t)\| \; &\leq M \sum_{i=1}^{n} \frac{e^{\omega(\bar{\tau}_i+1)}}{1-c_i} \|B_i\| \int_0^t e^{\omega s} \left|k_i\left(\varphi_i^{-1}\left(s+\epsilon_1\right)\right)\right| \|U_{\epsilon_1}(s) - U_{\epsilon_2}(s)\| \, ds \\ &\quad + MC\left(T, \epsilon_1 - \epsilon_2\right), \end{aligned} \tag{2.29}$$

Next, applying Lemma 1.1 for

$$u(t) = e^{\omega t} \left\| U_{\epsilon_1}(t) - U_{\epsilon_2}(t) \right\|, \quad \alpha = MC\left(T, \epsilon_1 - \epsilon_2\right)$$

and

$$\beta(s) = M \sum_{i=1}^{n} \frac{e^{\omega(\bar{\tau}_i+1)}}{1-c_i} \left\| B_i \right\| \left| k_i\left(\varphi_i^{-1}\left(s+\epsilon_1\right)\right) \right| ds.$$

We find

$$\left\| U_{\epsilon_1}(t) - U_{\epsilon_2}(t) \right\| \leq MC(T, \epsilon_1 - \epsilon_2) e^{\int_0^T \sum_{i=1}^n \frac{e^{\omega(\bar{\tau}_i+1)}}{1-c_i} \|B_i\| e^{\omega s} |k_i(\varphi_i^{-1}(s+\epsilon_1))| ds}.$$

Therefore, we can confirm that for $\epsilon \longrightarrow 0$ The functions U_ϵ converge locally uniformly to a function $U \in C([0, +\infty); H)$ that satisfies 2.19 , and this concludes the proof.

□

2.2.3 Exponential stability

The next theorm proves that the solution of the problem (P_2) is exponentially stable under a suitable condition on the parameters of the system.

Theorem 2.3. *Assume that there exist two constants $\omega' \in (0, \omega)$ and $\gamma \in \mathbb{R}$ such that, for all $t \geq 0$,*

$$M \sum_{i=1}^{n} \frac{e^{\omega \bar{\tau}_i}}{1-c_i} \left\| B_i \right\| \int_0^t \left| k_i\left(\varphi_i^{-1}(s)\right) \right| ds \leq \gamma + \omega' t, \tag{2.30}$$

Hence, there exists a constant $M' > 0$ such that the solution of (P_2) is exponentially stable, i.e.

$$\|U(t)\| \leq M' e^{-(\omega-\omega')t}, \quad \forall \quad t \geq 0. \tag{2.31}$$

Proof. From the formula of Duhamel, the estimate (2.19) and by(2.1), it follows

$$\begin{aligned} \|U(t)\| &\leq \|S(t)U_0\| + \left\| \int_0^t S(t-s) \sum_{i=1}^n k_i(s) B_i U\left(s - \tau_i(s)\right) ds \right\|, \\ &\leq Me^{-\omega t} \|U_0\| + \int_0^t Me^{-\omega(t-s)} \left\| \sum_{i=1}^n k_i(s) B_i U\left(s-\tau_i(s)\right) \right\| ds, \\ &\leq Me^{-\omega t} \|U_0\| + \int_0^t Me^{-\omega t + \omega s} \sum_{i=1}^n |k_i(s)| \left\| B_i U\left(s - \tau_i(s)\right) \right\| ds. \end{aligned}$$

So, from (2.14), we deduce

$$\begin{aligned} e^{\omega t}\|U(t)\| &\leq M\|U_0\| + M \sum_{i=1}^n \int_0^t e^{\omega s} |k_i(s)| \left\| B_i U\left(s-\tau_i(s)\right) \right\| ds, \\ &\leq M\|U_0\| + M \sum_{i=1}^n e^{\omega \overline{\tau_i}} \int_0^t e^{\omega(s-\tau_i(s))} |k_i(s)| \left\| B_i U\left(s-\tau_i(s)\right) \right\| ds. \end{aligned}$$

Now, let consider the following change of variable

$$\varphi_i(s) := s - \tau_i(s) = \sigma, \tag{2.32}$$

for every $i = 1, \cdots, n$. Based on (2.15), we note that φ_i are invertible functions and we get

$$\begin{aligned}
&\int_0^t e^{\omega(s-\tau_i(s))} |k_i(s)| \, \|B_i U(s-\tau_i(s))\| \, ds \\
&= \int_{-\tau_i(0)}^{t-\tau_i(t)} e^{\omega\sigma} \left| k_i\left(\varphi^{-1}(\sigma)\right)\right| \|B_i U(\sigma)\| \frac{d\sigma}{1-\tau_i'(s)}, \\
&\leq \frac{1}{1-c_i} \int_{-\tau_i(0)}^{t-\tau_i(t)} e^{\omega\sigma} \left| k_i\left(\varphi^{-1}(\sigma)\right)\right| \|B_i U(\sigma)\| \, d\sigma.
\end{aligned}$$

Otherwise,

$$\begin{aligned}
e^{\omega t}\|U(t)\| \;\leq\; & M\|U_0\| + M\sum_{i=1}^n \frac{e^{\omega\overline{\tau_i}}}{1-c_i}\left(\int_{-\tau_i(0)}^0 e^{\omega s}\left|k_i\left(\varphi^{-1}(s)\right)\right| \|B_i U(s)\| \, ds,\right. \\
& \left. + \int_0^{t-\tau_i(t)} e^{\omega s}\left|k_i\left(\varphi^{-1}(s)\right)\right| \|B_i U(s)\| \, ds\right), \\
& \leq M\|U_0\| + M\sum_{i=1}^n \frac{e^{\omega\overline{\tau_i}}}{1-c_i}\int_{-\tau_i(0)}^0 e^{\omega s}\left|k_i\left(\varphi^{-1}(s)\right)\right| \|f_i(s)\| \, ds \\
& + M\sum_{i=1}^n \frac{e^{\omega\overline{\tau_i}}}{1-c_i}\int_0^{t-\tau_i(t)} e^{\omega s}\left|k_i\left(\varphi^{-1}(s)\right)\right| \|B_i U(s)\| \, ds, \\
& \leq M\|U_0\| + M\sum_{i=1}^n \frac{e^{\omega\overline{\tau_i}}}{1-c_i}\int_{-\tau_i(0)}^0 e^{\omega s}\left|k_i\left(\varphi^{-1}(s)\right)\right| \|f_i(s)\| \, ds \\
& + M\sum_{i=1}^n \frac{e^{\omega\overline{\tau_i}}}{1-c_i}\int_0^{t} e^{\omega s}\left|k_i\left(\varphi^{-1}(s)\right)\right| \|B_i\| \, \|U(s)\| ds.
\end{aligned} \tag{2.33}$$

Furthermore, for all $t \geq 0$, setting in the case

$$u(t) := e^{\omega t}\|U(t)\|,$$

$$\alpha := M\left(\|U_0\| + \sum_{i=1}^n \frac{e^{\omega\bar{\tau}_i}}{1-c_i}\int_{-\tau_i(0)}^0 e^{\omega s}\left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|f_i(s)\| \, ds\right)$$

and

$$\beta(t) = M\sum_{i=1}^n \frac{e^{\omega\bar{\tau}_i}}{1-c_i}\left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|B_i\|.$$

As a result, the inequality (2.33) takes the form

$$u(t) \leq \alpha + \int_0^t \beta(s)u(s)ds, \quad \forall t \geq 0.$$

From the inequality of Gronwall, see Lemma 1.1, we arrive at

$$u(t) \leq \alpha e^{\int_0^t \beta(s)ds},$$

for all $t \geq 0$. So,

$$\|U(t)\| \leq \alpha e^{\int_0^t \beta(s)ds - \omega t}, \quad \forall t \geq 0.$$

Similar to the proof of Theorem 2.1 and by considering the condition (2.30), we establish the exponential stability estimate of the solution in the case of multiple time-varying delays functions. This completes the proof. □

2.3 Nonlinear evolution equation with multiple time varying delay

This section is devoted to establish the exponential stability results for a nonlinear model with a multiple time-varying delays function as before.

2.3.1 Setting problem

Lastly, let consider the next system

$$(P_3):\begin{cases} U'(t)=AU(t)+\sum_{i=1}^{n}k_i(t)B_iU\left(t-\tau_i(t)\right)+F(U(t)), & t\in(0,\infty),\\ B_iU\left(t-\tau_i\right)=f_i(t), & t\in\left[-\tau^*,0\right],\\ U(0)=U_0. \end{cases}$$

As before, we keep that

1. $\{S(t)\}_{t\geq 0}$ is an exponentially stable semigroup in a Hilbert space H generated by the operator A.
2. The delayed functions $\tau_i\in W^{1,\infty}\left(\mathbb{R}_+,\mathbb{R}_+^*\right),\forall i=1,\ldots,n$ satisfy (2.14) and (2.15). τ^* is defined in (2.16).
3. The functions $k_i:\mathbb{R}_+\to\mathbb{R}$ are belonging to $L^1_{\text{loc}}\left(\mathbb{R}_+,\mathbb{R}\right)$, for $i=1,\ldots,n$.
4. $U_0\in H, f_i\in C\left(\left[-\tau^*,0\right];H\right), i=1,\ldots,n$ are the initial data.

In order to state the desired results, let suppose that the functional F is locally Lipschitz

(H_5) $F:H\longrightarrow H$ is a nonlinear locally Lipschitz function, i.e. for each positive constant r, there exists a positive constant $L(r)$ such that

$$\|F(U)-F(V)\|_H\leq L(r)\|U-V\|_H, \tag{2.34}$$

for all $U,V\in H$ with $\|U\|_H\leq r,\|V\|_H\leq r$, moreover, $F(0)=0$.

2.3.2 Well posedness

The global existence and uniqueness of solution of system (P_3) is given in the next result.

Theorem 2.4. *Let assume that*

1. *There exist $\omega'\in(0,\omega)$ and $\gamma\in\mathbb{R}$ such that (2.30) is satisfied.*
2. *There exist $\rho>0$ and $C_\rho>0$ with $L\left(C_\rho\right)<\frac{\omega-\omega'}{M}$.*

Then, if $U_0\in H$ and $f_i\in C\left(\left[-\tau^*,0\right];H\right)$, for $i=1,\ldots,n$ verify

$$\|U_0\|_H^2+\sum_{i=1}^{n}\int_{-\tau^*}^{0}|k_i(s)|\cdot\|f_i(s)\|_H^2\,ds<\rho^2, \tag{2.35}$$

then the problem (P_3) has a unique global solution $U\in C([0,+\infty);H)$ satisfying

$$\|U(t)\|\leq C_\rho,\quad \forall t>0. \tag{2.36}$$

Proof. Note that the system (P_3) is a nonlinear inhomogeneous Cauchy problem, from the previous results in the well-poseness of problem (P_2) and by using the fact that F is locally Lipschitz function, the system (P_3) admit a unique local solution according to Theorem 1.16. Under the condition (2.35), we get the global existence by applying the similar method as in [2]. For more details, we will adopt this method in the applications part, see 3 □

2.3.3 Exponential stability

The decay result associated to the problem (P_3) is ensuring by the following theorem:

Theorem 2.5. *Under the assumptions of Theorem 2.4, there exists a constant $\tilde{M} > 0$ such that the solution of (P_3) satisfy the following estimate of decay*

$$\|U(t)\| \leq \tilde{M} e^{-(\omega-\omega'-ML(C_\rho))t}, \quad \forall t \geq 0.$$

Proof. Starting by Duhamel's formula which given as follows

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\left(\sum_{i=1}^n k_i(s)B_iU(s-\tau_i(s)) + F(u(s))\right) ds,$$

for all $t \geq 0$. Moreover, as previously, we have

$$\begin{aligned} \|U(t)\| &\leq Me^{-\omega t}\|U_0\| + Me^{-\omega t}\int_0^t e^{\omega s}\sum_{i=1}^n |k_i(s)| \|B_iU(s-\tau_i(s))\| ds \\ &+ Me^{-\omega t}\int_0^t e^{\omega s}\|F(U(s))\| ds. \end{aligned}$$

Thus, by (2.14), we get

$$\begin{aligned} e^{\omega t}\|U(t)\| \leq\ & M\|U_0\| + M\int_0^t e^{\omega\bar{\tau}_i} e^{\omega s-\tau_i(s)}\sum_{i=1}^n |k_i(s)| \|B_iU(s-\tau_i(s))\| ds \\ &+ Me^{-\omega t}\int_0^t e^{\omega s}\|F(U(s))\| ds, \end{aligned}$$

Using the same change of variable,

$$\varphi_i(s) := s - \tau_i(s) = \sigma,$$

and repeating the same argument in the previous section, we arrive at

$$\begin{aligned} e^{\omega t}\|U(t)\| &\leq M\|U_0\| + M\sum_{i=1}^n \frac{e^{\omega\bar{\tau}_i}}{1-c_i}\int_{-\tau_i(0)}^0 e^{\omega s}\left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|f_i(s)\| ds \\ &+ M\sum_{i=1}^n \frac{e^{\omega\bar{\tau}_i}}{1-c_i}\int_0^t e^{\omega s}\left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|B_i\| \|U(s)\| ds \\ &+ Me^{-\omega t}\int_0^t e^{\omega s}\|F(U(s))\| ds. \end{aligned}$$

From (2.34) and by using the fact that $F(0) = 0$ and since the solution satisfies (2.36), we achieve

$$e^{\omega t}\|U(t)\| \leq M\|U_0\| + M\sum_{i=1}^n \frac{e^{\omega\bar{\tau}_i}}{1-c_i}\int_{-\tau_i(0)}^0 e^{\omega s}\left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|f_i(s)\| ds \tag{2.37}$$

$$+ M\sum_{i=1}^n \frac{e^{\omega\bar{\tau}_i}}{1-c_i}\int_0^t e^{\omega s}\left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|B_i\| \|U(s)\| ds \tag{2.38}$$

$$+ ML(C_\rho)\int_0^t e^{\omega s}\|U(s)\| ds, \tag{2.37}$$

for all $t \geq 0$. At this position, we set

$$u(t) := e^{\omega t}\|U(t)\|,$$
$$\alpha := M\,\|U_0\| + M\sum_{i=1}^{n}\frac{e^{\omega\bar{\tau}_i}}{1-c_i}e^{\omega s}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|f_i(s)\|\,ds$$

and

$$\beta(t) := \sum_{i=1}^{n}\frac{e^{\omega\bar{\tau}_i}}{1-c_i}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|B_i\|\,.$$

Consequently,

$$u(t) \leq \alpha + \int_0^t \beta(s)u(s)ds + ML\left(C_\rho\right)\int_0^t u(s)ds, \quad \forall t \geq 0.$$

So,

$$u(t) \leq \alpha + \int_0^t \left[\beta(s) + ML\left(C_\rho\right)\right]u(s)ds,$$

Applying Gronwall's inequality, see Lemma 1.1, we conclude that

$$u(t) \leq \alpha e^{\int_0^t[\beta(s)+ML(C_\rho)]ds}.$$

Finally, by (2.30), we have

$$\|U(t)\| \leq \tilde{M}e^{-(\omega-\omega'-ML(C_\rho))t}, \quad \forall t \geq 0.$$

with $\tilde{M} = \alpha e^{\gamma}$. As $L\left(C_\rho\right) < \frac{\omega-\omega'}{M}$, we conclude that the solution decays exponentially. □

Chapter 3

Application

In this chapter, we give an abstract semilinear evolution equation with multiple time varying delays as an application that illustrate the results obtained.

Consider the following second-order evolution equation

$$u_{tt} + \mathcal{A}u + CC^{*}u_t = \nabla\Psi(u) + \sum_{i=1}^{n} k_i(t) D_i D_i^{*} u_t \left(t - \tau_i(t)\right), t \in (0, \infty), \tag{3.1}$$

With

$$\begin{cases} u(0) = u_0, u_t(0) = u_1, \\ D_i D_i^{*} u_t(t) = f_i(t), \end{cases} \quad t \in \left[-\tau^{*}, 0\right], i = 1, \ldots, n \ , \tag{3.2}$$

Where

1. $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ a positive self adjoint operator with a compact inverse in a real Hilbert space $\mathcal{H}$ (we denote by $\|.\|$the norm in $\mathcal{H}$). Moreover, V the domain of the operator $\sqrt{\mathcal{A}} = \mathcal{A}^{1/2}$.
2. $\forall i = 1, \ldots, n, W_i$ be real Hilbert spaces and $C : W_0 \to \mathcal{H}, D_i : \mathcal{H} \to W_i$ are bounded linear operators.
3. The functions k_i, τ_i and the constant τ^{*} are defined as before.
4. The initial data $(u_0, u_1, f_i) \in V \times \mathcal{H} \times C\left(\left[-\tau^{*}, 0\right]; \mathcal{H}\right)$.
5. The estimates (2.21) and (2.20) are satisfied.

To ensure the well-posedness assumption and the exponential decay estimate of problem (3.1), we consider the following assumptions:

(A_1) The bounded linear operators $C : W_0 \to \mathcal{H}$ and $D_i : \mathcal{H} \to W_i$ satisfy

$$a\|u\|_{W_i}^2 \leq \|C^{*}u\|_{W_0}^2 \tag{3.3}$$

and

$$\|D_i^{*}u\|_{W_i}^2 \leq d_i\|u\|^2, \tag{3.4}$$

for all $u \in \mathcal{H}$ and $i = 1, \ldots, l$, with appropriate positive constants d_i and a.

(A_2) Let $\Psi : V \to \mathbb{R}$ be a functional having a Gâteaux derivative $D\Psi(u)$ in every $u \in V$ such that

$$\forall u \in V, \exists c(u) \in \mathbb{R}_+ : \quad |D\Psi(u)(v)| \leq c(u)\|v\|, \quad \forall v \in V. \tag{3.5}$$

where $D\Psi(u)$ is the Gâteaux derivative of the functional Ψ at u. Thus, Ψ can be extended to the whole $\mathcal{H}$ and we denote by $\nabla\Psi(u)$ the unique vector representing $D\Psi(u)$ in the Riesz isomorphism, namely

$$\langle \nabla\Psi(u), v\rangle = D\Psi(u)(v), \quad \forall v \in \mathcal{H}.$$

Moreover, for all $r > 0$, there exist $L(r) > 0$ such that

$$\|\nabla\Psi(u) - \nabla\Psi(v)\| \leq L(r)\|\sqrt{\mathcal{A}}(u-v)\|, \tag{3.6}$$

for all $u, v \in V$ satisfying $\|\sqrt{\mathcal{A}}u\| \leq r$ and $\|\sqrt{\mathcal{A}}v\| \leq r$.

Furthermore, $\Psi(0) = 0$, $\nabla\Psi(0) = 0$, and there exists an increasing continuous function h such that, $\forall u \in V$,

$$\|\nabla\Psi(u)\| \leq h(\|\sqrt{\mathcal{A}}u\|)\|\sqrt{\mathcal{A}}u\|. \tag{3.7}$$

For $U := (u, v)^T$ where $v = u_t$, the problem (3.1) can be rewritten as the system (P_3) with operator A is given by

$$A := \begin{pmatrix} 0 & 1 \\ -\mathcal{A} & -CC^* \end{pmatrix}.$$

The Hilbert space H by

$$H := V \times \mathcal{H},$$

While $B_i, i = 1, \ldots, n$ and F are defined as follows

$$B_iU := (0, D_iD_i^*v)^T \quad \text{and} \quad F(U) := (0, \nabla\Psi(u))^T.$$

Under the above assumptions on Ψ, it implies that the function F satisfies(2.34) and $F(0) = 0$. Moreover, the operator A generates an exponentially stable semigroup $S(t)_{t\geq 0}$, as shown in the next Lemma.

Lemma 3.1. *The operator A is the infinitesimal generator of C_0-semigroup $S(t)_{t\geq 0}$ on H.*

Proof. We will prove that the linear operator A is the infinitesimal generator of C_0-semigroup $S(t)_{t\geq 0}$ on H based on the theorem of Lummer-Phillips.

Firstly, A is a dissipative operator. Indeed, for $U = (u, v) \in H$, we got

$$\begin{aligned}
\left\langle A\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\rangle_H &= \left\langle \begin{pmatrix} v \\ -\mathcal{A}u - CC^*v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\rangle_H, \\
&= \langle v, u\rangle_V + \langle -\mathcal{A}u - CC^*v, v\rangle, \\
&= \langle \sqrt{\mathcal{A}}v, \sqrt{\mathcal{A}}u\rangle - \langle \sqrt{\mathcal{A}}\sqrt{\mathcal{A}}u, v\rangle - \langle C^*v, C^*v\rangle, \\
&= -\langle C^*v, C^*v\rangle, \\
&= -\|C^*v\|^2 \leq 0.
\end{aligned}$$

Thus, we deduce that A is a dissipative operator.

Secondly, The operator $\lambda I - A$ is surjective, for any $\lambda > 0$. Indeed, for all $\varphi = (\varphi_1, \varphi_2)^t \in H$, there exist $U = (u, v)^t \in H$ such that

$$(\lambda I - A)U = \varphi.$$

This last is equivalent to

$$\begin{cases} \lambda u - v = \varphi_1, \\ \lambda + \mathcal{A}u + CC^* v = \varphi_2. \end{cases} \tag{3.8}$$

from the first equation in the previous system, we have

$$v = \lambda u - \varphi_1.$$

So,

$$\lambda(\lambda u - \varphi_1) + \mathcal{A}u + CC^*(\lambda u - \varphi_1) = \varphi_2,$$

Consequently,

$$\lambda^2 u + \mathcal{A}u + \lambda CC^* u = \varphi_2 + \lambda\varphi_1 + CC^*\varphi_1.$$

For $\lambda = 1$, we obtain

$$u + \mathcal{A}u + CC^* u = \varphi_2 + \varphi_1 + CC^*\varphi_1, \tag{3.9}$$

for $\phi \in V$, we got

$$< u + \mathcal{A}u + CC^* u, \phi > = < \varphi_2 + \varphi_1 + CC^*\varphi_1, \phi > . \tag{3.10}$$

Thus, we put

$$a(u, \phi) = < u + \mathcal{A}u + CC^* u, \phi >$$

and

$$b(\phi) = < \varphi_2 + \varphi_1 + CC^*\varphi_1, \phi > .$$

Now, We prove that (3.10) is a continuous and coercive bilinear form.

First, we show that a is bilinear form, $\forall \alpha \in \mathbb{R}, \forall u_1, u_2 \in V$, then

$$a(\alpha u_1 + u_2, \phi) = < \alpha u_1 + u_2 + \mathcal{A}(\alpha u_1 + u_2) + CC^*(\alpha u_1 + u_2), \phi >,$$

by the linearity of the inner product and the linearity of $\mathcal{A}$ and CC^*, we have

$$\alpha < u_1 + \mathcal{A}u_1 + CC^* u_1, \phi > + < u_2 + \mathcal{A}u_2 + CC^* u_2, \phi > = \alpha a(u_1, \phi) + a(u_2, \phi),$$

Moreover, $\forall \phi_1, \phi_2 \in V$, we have

$$\begin{aligned} a(u, \alpha\phi_1 + \phi_2) &= < (u + \mathcal{A}u + CC^* u), (\alpha\phi_1 + \phi_2) >, \\ &= \alpha < u + \mathcal{A} + CC^* u_1, \phi_1 > + < u + \mathcal{A} + CC^*, \phi_2 >, \\ &= \alpha a(u, \phi_1) + a(u, \phi_2), \end{aligned}$$

So, a is bilinear form.

Secondly, a is continuous form, indeed

$$\begin{aligned} |a(u, \phi)| &= | < (u + \mathcal{A}u + CC^* u), \phi > |, \\ &\leq \|u + \mathcal{A}u + CC^* u\| \, \|\phi\|. \end{aligned}$$

Using the Hölder inequality and as $\mathcal{A}$ and CC^* are continuous operators, we get

$$\begin{aligned}|a(u,\phi)| &\leq \|u\|\|\phi\| + \|\mathcal{A}u\|\|\phi\| + \|CC^*u\|\,\|\phi\|,\\ &\leq (\|u\| + \|\mathcal{A}\|u\| + \|CC^*\|\|u\|))\,\|\phi\|,\\ &\leq C\|u\|\phi\|.\end{aligned}$$

Now, we demonstrate that a is coercive,

$$\begin{aligned}a(u,u) &=< u,u > + < \mathcal{A}u,u > + < CC^*u,u >,\\ &\geq \|\sqrt{\mathcal{A}}u\|^2 \geq \|u\|_V^2.\end{aligned}$$

Thus, by applying Lax-Milgram theorem, we conclude that (3.10) has a solution unique $u \in V$ such that, the equation $a(u,\phi) = b(\phi)$ is verified, so, there exist $U = (u,v)^t \in H$ such that

$$(\lambda I - A)U = \varphi.$$

Otherwise, the operator A is m-dissipative operator. Therefore, according to Theorem 1.13, A is an infinitesimal generator of a C_0 semigroup $S(t)_{t\geq 0}$ on H. □

Remark 3.1. A generates a C_0 semigroup $S(t)_{t>0}$ on H exponentially stable, for some conditions on the damping operator CC^*, we refer to [4], see also ([6], Ch. 5).

Now, let define the energy functional associated to problem (3.1) as follows, for all $t \geq 0$

$$\begin{aligned}E(t) =& \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\sqrt{\mathcal{A}}u\|^2 - \Psi(u)\\ &+ \frac{1}{2}\sum_{i=1}^n \frac{1}{1-c_i}\int_{t-\tau_i(t)}^t \left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|D_i^*u_t(s)\|_{W_i}^2\,ds.\end{aligned} \tag{3.11}$$

We will demonstrate that the problem (3.1) satisfies the well-posedness assumption and the exponential decay estimate of Theorem 2.5 for small initial data under a suitable condition between the constant a and the functions k_i in (3.3). We need the next lemma.

Lemma 3.2. *Assume that $k_i(t) = k_i^1(t)+k_i^2(t)$ with $k_i^1 \in L^1([0,+\infty))$ and $k_i^2 \in L^\infty(0,+\infty)$ for $i = 1,\ldots,n$. Furthermore, assume that*

$$\left\|k_i^2\right\|_\infty \leq \frac{2a}{nd_i}\cdot\frac{1-c_i}{2-c_i}, \quad i = 1,\ldots,n. \tag{3.12}$$

Then, for any solution u of problem(3.1), defined on $[0,T)$ for some $T>0$, and satisfying $E(t) \geq \frac{1}{4}\|u_t(t)\|^2$ for all $t \in [0,T)$, we have

$$E(t) \leq CE(0), \quad \forall t \in [0,T), \tag{3.13}$$

with

$$C = e^{\sum_{i=1}^n d_i \int_0^{+\infty}\left(\frac{1}{1-c_i}\left|k_i^1\left(\varphi_i^{-1}(s)\right)\right| + \left|k_i^1(s)\right|\right)ds}. \tag{3.14}$$

Proof. By differentiating the energy functional, we obtain

$$\begin{aligned}E'(t) \;=\;& \langle u_{tt},u_t\rangle + \langle\mathcal{A}u,u_t\rangle - \langle\nabla\psi(u),u_t\rangle + \frac{1}{2}\sum_{i=1}^n\frac{1}{1-c_i}\left|K_i\left(\varphi_i^{-1}(t)\right)\right|\|D_i^*u_t(t)\|_{w_i}^2 -\\ &\frac{1}{2}\sum_{i=1}^n\frac{1-\tau_i'(t)}{1-c_i}\left|K_i\left(\varphi_i^{-1}(t-\tau_i(t))\right)\right|\|D_i^*u_t(t-\tau_i(t))\|_{w_i}^2,\end{aligned}$$

for all $t \geq 0$. By using (3.1), we have

$$\langle u_{tt} + \mathcal{A}u - \nabla\psi(u), u_t\rangle = \left\langle -CC^*u_t + \sum_{i=1}^{n} k_i(t)D_iD_i^*u_t\left(t-\tau_i(t)\right), u_t\right\rangle.$$

Thus,

$$\begin{aligned} E'(t) &= -\left\|C^*u_t(t)\right\|_{W_0}^2 + \sum_{i=1}^{n} k_i(t)\left\langle D_i^*u_t\left(t-\tau_i(t)\right), D_i^*u_t(t)\right\rangle \\ &+ \sum_{i=1}^{n} \frac{1}{2\left(1-c_i\right)}\left|k_i\left(\varphi^{-1}(t)\right)\right| \left\|D_i^*u_t(t)\right\|_{W_i}^2 \\ &- \sum_{i=1}^{n}, \frac{\left(1-\tau_i'(t)\right)}{2\left(1-c_i\right)}\left|k_i\left(\varphi^{-1}\left(t-\tau_i(t)\right)\right)\right| \left\|D_i^*u_t\left(t-\tau_i(t)\right)\right\|_{W_i}^2. \end{aligned}$$

According to (2.32), we deduce

$$\begin{aligned} E'(t) &= -\left\|C^*u_t(t)\right\|_{W_0}^2 + \sum_{i=1}^{n} k_i(t)\left\langle D_i^*u_t\left(t-\tau_i(t)\right), D_i^*u_t(t)\right\rangle \\ &+ \sum_{i=1}^{n} \frac{1}{2\left(1-c_i\right)}\left|k_i\left(\varphi^{-1}(t)\right)\right| \left\|D_i^*u_t(t)\right\|_{W_i}^2, \\ &- \sum_{i=1}^{n} \frac{\left(1-\tau_i'(t)\right)}{2\left(1-c_i\right)}\left|k_i(t)\right| \left\|D_i^*u_t\left(t-\tau_i(t)\right)\right\|_{W_i}^2. \end{aligned}$$

By using the Cauchy-Schwarz's and Young's inequality, it result

$$\sum_{i=1}^{n} k_i(t)\left\langle D_i^*u_t\left(t-\tau_i(t)\right), D_i^*u_t(t)\right\rangle \leq \sum_{i=1}^{n} \frac{|k_i(t)|}{2}\left[\left\|D_i^*u_t\left(t-\tau_i(t)\right)\right\|_{W_i}^2 + \left\|D_i^*u_t(t)\right\|_{W_i}^2\right].$$

Hence,

$$\begin{aligned} E'(t) &\leq -\left\|C^*u_t(t)\right\|_{W_0}^2 + \frac{1}{2}\sum_{i=1}^{n}\left[\frac{1}{\left(1-c_i\right)}\left|k_i\left(\varphi^{-1}(t)\right)\right| + |k_i(t)|\right]\left\|D_i^*u_t(t)\right\|_{W_i}^2 \\ &+ \frac{1}{2}\sum_{i=1}^{n}\left(1-\frac{\left(1-\tau_i'(t)\right)}{\left(1-c_i\right)}\right)|k_i(t)|\left\|D_i^*u_t\left(t-\tau_i(t)\right)\right\|_{W_i}^2. \end{aligned}$$

By using (2.15), we get

$$\frac{1}{2}\sum_{i=1}^{n}\left(1-\frac{\left(1-\tau_i'(t)\right)}{\left(1-c_i\right)}\right)|k_i(t)|\left\|D_i^*u_t\left(t-\tau_i(t)\right)\right\|_{W_i}^2 < 0.$$

Then,for all $t \geq 0$

$$E'(t) \leq -\left\|C^*u_t(t)\right\|_{W_0}^2 + \frac{1}{2}\sum_{i=1}^{n}\left[\frac{1}{\left(1-c_i\right)}\left|k_i\left(\varphi^{-1}(t)\right)\right| + |k_i(t)|\right]\left\|D_i^*u_t(t)\right\|_{W_i}^2.$$

Using the definition of $k_i = k_i^1 + k_i^2$, we arrive at

$$\begin{aligned} E'(t) &\leq -\left\|C^*u_t(t)\right\|_{W_0}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{1-c_i}\left|k_i^2\left(\varphi^{-1}(t)\right)\right| + \left|k_i^2(t)\right|\right)\left\|D_i^*u_t(t)\right\|_{W_i}^2 \\ &+ \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{1-c_i}\left|k_i^1\left(\varphi^{-1}(t)\right)\right| + \left|k_i^1(t)\right|\right)\left\|D_i^*u_t(t)\right\|_{W_i}^2. \end{aligned}$$

From (3.3) and (3.4), we attain

$$\begin{aligned}
&- \|C^* u_t(t)\|_{W_0}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{1-c_i}\left|k_i^2\left(\varphi^{-1}(t)\right)\right| + \left|k_i^2(t)\right|\right)\|D_i^* u_t(t)\|_{W_i}^2 \\
\leq &- a\|u_t(t)\|^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{2-c_i}{1-c_i}\left\|k_i^2\right\|_\infty \|D_i^* u_t(t)\|_{W_i}^2, \\
\leq &\sum_{i=1}^{n}\left(\frac{-a}{n} + \frac{1}{2d_i}\frac{2-c_i}{1-c_i}\left\|k_i^2\right\|_\infty\right)\|u_t(t)\|^2,
\end{aligned}$$

Under the condition (3.12), we have

$$\sum_{i=1}^{n}\left(\frac{-a}{n} + \frac{1}{2d_i}\frac{2-c_i}{1-c_i}\left\|k_i^2\right\|_\infty\right)\|u_t(t)\|^2 < 0.$$

As a result,

$$E'(t) \leq \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{1-c_i}\left|k_i^1\left(\varphi^{-1}(t)\right)\right| + \left|k_i^1(t)\right|\right)\|D_i^* u_t(t)\|_{W_i}^2.$$

By (3.4), we conclude

$$E'(t) \leq \frac{1}{2}\sum_{i=1}^{n}\left(\frac{d_i}{1-c_i}\left|k_i^1\left(\varphi^{-1}(t)\right)\right| + \left|k_i^1(t)\right|\right)\|u_t(t)\|^2.$$

Integrate the last estimate, we get

$$E(t) \leq E(0) + \int_0^t \frac{1}{2}\sum_{i=1}^{n}\left(\frac{d_i}{1-c_i}\left|k_i^1\left(\varphi^{-1}(s)\right)\right| + \left|k_i^1(s)\right|\right)\|u_t(s)\|_{W_i}^2\, ds, \tag{3.15}$$

From (3.15) and (3.11), we obtain

$$E(t) \leq E(0) + \int_0^t K(s)E(s)ds,$$

With

$$K(t) := \sum_{i=1}^{t}\left(\frac{d_i}{1-c_i}\left|k_i^1\left(\varphi_i^{-1}(t)\right)\right| + \left|k_i^1(t)\right|\right).$$

Now, Gronwall's inequality yields

$$E(t) \leq E(0)e^{\int_0^t K(s)ds}.$$

This last implies (3.13) with C defined in (3.14). The proof of Lemma 3.2 is now complete. □

Proposition 3.1. *Assume that* $k_i(t) = k_i^1(t) + k_i^2(t), i = 1, \ldots, n,$ *where* $k_i^1 \in L^1([0, +\infty))$ *and* $k_i^2 \in L^\infty(0, +\infty)$ *with* (3.12)*. Then, the system 3.1 satisfies the assumptions of well-posedness result, Theorem 2.4.*

Proof. Initially, we limit our scope to the period $[0, \tau_{\min}]$,Where $\tau_{\min}$ represents the constant specified in equation (2.21). At this time, the model can be recast in abstract form

$$\begin{cases} U'(t) = AU(t) + \sum_{i=1}^{n} k_i(t)G_i(t) + F(U(t)) & \text{in} \quad (0, \infty), \\ U(0) = U_0, \end{cases} \tag{3.16}$$

with $G_i(t) = (0, f_i(t - \tau_i(t))), i = 1, \dots, n$

Let use classical methods from nonlinear semigroup theory to prove the existence of a unique mild solution defined on a maximal interval $[0, \delta)$ where $\delta \leq \tau_{\min}$. We aim to demonstrate that with sufficiently small initial data, the solution extends globally and adheres to a specific constraint. Although, this approach is inspired by [2], we encounter additional challenges due to the time-varying nature of $k_i(\cdot)$ resulting that the energy function is not necessary decreasing. Firstly, we note that

$$\text{if} \quad h\left(\left\|\sqrt{\mathcal{A}}u_0\right\|\right) < \frac{1}{2}, \quad \text{then} \quad E(0) > 0. \tag{3.17}$$

This deduction stems from the assumption (3.7) concerning the function Ψ, i.e.

$$|\Psi(u)| \leq \int_0^1 |\langle \nabla\Psi(su), u\rangle| ds,$$

applying Cauchy-Schwarz, we have

$$|\Psi(u)| \leq \int_0^1 \|\nabla\Psi(su)\| \|\sqrt{\mathcal{A}}u\| ds,$$

by (3.7), we get

$$\begin{aligned} |\Psi(u)| &\leq \int_0^1 h(\|\sqrt{\mathcal{A}}su\|)\ \|\sqrt{\mathcal{A}}su\| \|\sqrt{\mathcal{A}}u\| ds, \\ &\leq \|\sqrt{\mathcal{A}}u\|^2 \int_0^1 h(s\|\sqrt{\mathcal{A}}u\|)\ s\ ds. \end{aligned}$$

As $s \leq 1$ and h is increasing function, we have

$$\begin{aligned} |\Psi(u)| &\leq \|\sqrt{\mathcal{A}}u\|^2 \int_0^1 h(\|\sqrt{\mathcal{A}}u\|)\ s\ ds, \\ &\leq \|\sqrt{\mathcal{A}}u\|^2 h(\|\sqrt{\mathcal{A}}u\|) \int_0^1\ s\ ds. \end{aligned}$$

So,

$$|\Psi(u)| \leq \frac{1}{2} h(\|\sqrt{\mathcal{A}}u\|)\ \|\sqrt{\mathcal{A}}u\|^2. \tag{3.18}$$

Now, from (3.11), we have

$$\begin{aligned} E(0) &= \frac{1}{2}\|u_t(0)\|^2 + \frac{1}{2}\|\sqrt{\mathcal{A}}u(0)\|^2 - \Psi(u(0)) \\ &+ \frac{1}{2}\sum_{i=1}^{l} \frac{1}{1 - c_i} \int_{-\tau_i(0)}^{0} \left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|D_i^* u_t(s)\|_{W_i}^2\, ds. \end{aligned}$$

By (3.17) and(3.18), it results

$$\begin{aligned} E(0) &\geq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\left\|\sqrt{\mathcal{A}}u_0\right\|^2 - \frac{1}{4}\left\|\sqrt{\mathcal{A}}u_0\right\|^2 \\ &+ \frac{1}{2}\sum_{i=1}^{l} \frac{1}{1 - c_i} \int_{-\tau_i(0)}^{0} \left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|D_i^* u_t(s)\|_{W_i}^2\, ds, \\ &\geq \frac{1}{2}\|u_1\|^2 + \frac{1}{4}\left\|\sqrt{\mathcal{A}}u_0\right\|^2 \\ &+ \frac{1}{2}\sum_{i=1}^{l} \frac{1}{1 - c_i} \int_{-\tau_i(0)}^{0} \left|k_i\left(\varphi_i^{-1}(s)\right)\right| \|D_i^* u_t(s)\|_{W_i}^2\, ds > 0. \end{aligned}$$

Let prove that if

$$h\left(\left\|\sqrt{\mathcal{A}}u_0\right\|\right) < \frac{1}{2} \quad \text{and} \quad h\left(2C^{1/2}E^{1/2}(0)\right) < \frac{1}{2}. \tag{3.19}$$

where C is the constant defined in (3.14), then, we have

$$\begin{aligned} E(t) >& \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{1}{4}\|\sqrt{\mathcal{A}}u(t)\|^2 \\ &+ \frac{1}{2}\sum_{i=1}^{l}\frac{1}{1-c_i}\int_{t-\tau_i(t)}^{t}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|D_i^*u_t(s)\|_{W_i}^2\,ds, \quad \forall t\in[0,\delta). \end{aligned} \tag{3.20}$$

Let us prove by contradiction, we consider the supremum r of all $s \in [0,\delta)$, such that the equation (3.20) holds true, for all $t \in [0, s]$. We assume that $r < \delta$. As a consequence of continuity, we have

$$\begin{aligned} E(r) \ &\geq \ \frac{1}{2}\left\|u_t(r)\right\|^2 + \frac{1}{4}\|\sqrt{\mathcal{A}}u(r)\|^2 \\ &+ \frac{1}{2}\sum_{i=1}^{l}\frac{1}{1-c_i}\int_{r-\tau_i(r)}^{r}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|D_i^*u_t(s)\|_{W_i}^2\,ds \geq 0, \end{aligned} \tag{3.21}$$

Therefore, we infer from (3.21) and (3.13) that

$$\begin{aligned} \frac{1}{4}\|\sqrt{\mathcal{A}}u(r)\|^2 \ &\leq E(r) \leq CE(0), \\ \|\sqrt{\mathcal{A}}u(r)\|^2 &\leq 2(E(r))^{\frac{1}{2}} \leq 2(CE(0))^{\frac{1}{2}}, \end{aligned}$$

using the fact that h is increasing, we've got

$$h(\|\sqrt{\mathcal{A}}u(r)\|) \leq h\left(2E^{1/2}(r)\right) \leq h\left(2C^{1/2}E^{1/2}(0)\right) < \frac{1}{2}.$$

Using(3.18) in the definition of $E(r)$, this gives

$$\begin{aligned} E(r) \ &= \ \frac{1}{2}\left\|u_t(r)\right\|^2 + \frac{1}{2}\|\sqrt{\mathcal{A}}u(r)\|^2 - \Psi(u(r)) \\ &+ \frac{1}{2}\sum_{i=1}^{l}\frac{1}{1-c_i}\int_{r-\tau_i(r)}^{r}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\cdot\|D_i^*u_t(s)\|_{W_i}^2\,ds, \\ &> \ \frac{1}{2}\left\|u_t(r)\right\|^2 + \frac{1}{4}\|\sqrt{\mathcal{A}}u(r)\|^2 \\ &+ \frac{1}{2}\sum_{i=1}^{l}\frac{1}{1-c_i}\int_{r-\tau_i(r)}^{r}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|D_i^*u_t(s)\|_{W_i}^2\,ds, \end{aligned}$$

contradicting the maximality of r. This implies $r = \delta$.

At this position, let define

$$\rho := \frac{1}{2C^{1/2}}h^{-1}\left(\frac{1}{2}\right). \tag{3.22}$$

So, (3.19) is satisfied, for all $u_0 \in V, u_1 \in \mathcal{H}, f_i \in C\left([-\tau^*, 0], W_i\right), i = 1, \ldots, n$, satisfying

$$\left\|\sqrt{\mathcal{A}}u_0\right\|^2 + \|u_1\|^2 + \frac{1}{2}\sum_{i=1}^{l}\frac{1}{1-c_i}\int_{-\tau_i(0)}^{0}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|f_i(s)\|_{W_i}^2\,ds < \rho^2,$$

Indeed, this condition implies $\left\|\sqrt{\mathcal{A}}u_0\right\| < \rho$ and then, observing that $C > 1$. Hence, we conclude

$$h\left(\left\|\sqrt{\mathcal{A}}u_0\right\|\right) < h(\rho) = h\left(\frac{1}{2C^{1/2}}h^{-1}\left(\frac{1}{2}\right)\right) < \frac{1}{2}.$$

Moreover, from (3.19), we get the estimate

$$\begin{aligned} E(0) \ &\leq \frac{3}{4}\left\|\sqrt{\mathcal{A}}u_0\right\|^2 + \frac{1}{2}\|u_1\|^2 \\ &\quad + \frac{1}{2}\sum_{i=1}^{l}\frac{1}{1-c_i}\int_{-\tau_i(0)}^{0}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|f_i(s)\|_{W_i}^2\,ds < \rho^2, \end{aligned}$$

(3.22) gives

$$h\left(2C^{1/2}E^{1/2}(0)\right) < h\left(2C^{1/2}\rho\right) < h\left(\frac{2C^{1/2}}{2C^{1/2}}\left(h^{-1}(1/2)\right)\right) = \frac{1}{2}.$$

We conclude that (3.19) holds, so(3.20) satisfied. Moreover,

$$\begin{aligned} 0 \ &< \ \frac{1}{4}\|u_e(t)\|^2 + \frac{1}{4}\|\sqrt{A}u(t)\|^2 \\ &\quad + \frac{1}{4}\sum_{i=1}^{t}\frac{1}{1-c_i}\int_{t-\tau_i(t)}^{t}\left|k_i\left(\varphi_i^{-1}(s)\right)\right|\|D_i^* u_t(s)\|_{W_i}^2\,ds, \\ &\leq \ E(t), \\ &\leq \ CE(0) \leq C\rho^2. \end{aligned}$$

for all $t \in [0,\delta]$.

By taking the solution at time $t = \delta$ as initial datum, we can extend the solution of problem (3.16). Follow the same argument as before, we may expand the solution to include the entire interval $[0,\tau_{\min}]$, and the solution satisfies

$$h\left(\left\|\sqrt{\mathcal{A}}u\left(\tau_{\min}\right)\right\|\right) \leq h\left(2E^{1/2}\left(\tau_{\min}\right)\right) \leq h\left(2\bar{C}^{1/2}E^{1/2}(0)\right) < \frac{1}{2},$$

where the estimate (3.13) has been applied on the whole interval $[0,\tau_{\min}]$.

Then on the second interval $[\tau_{\min}, 2\tau_{\min}\}$, after we obtain the solution $U(\cdot)$ on the interval $[0,\tau_{\min}]$, we may rephrase the problem (3.1) using $G_i(t) = (0, D_i D_i^* u_t\left(t-\tau_i(t)\right))$ to put it in the abstract form (3.16). Note that.

$$\text{for} \ \ t \in [\tau_{\min}, 2\tau_{\min}] \quad \text{it results} \quad t-\tau_i(t) \leq \tau_{\min}).$$

To obtain a global solution satisfying (3.13) , we can repeat the same deduction on earh interval of length $\tau_{\min}$. □

Conclusion

This work has presented a systematic analysis of linear and nonlinear abstract, delayed evolution models. The solution's global existence and exponential decay estimates have been established in the case of a single constant delay and multiple time-varying delay functions based on classical results in the theory of semigroups of nonlinear evolution equations and undersome conditions on the initial datum. Duhamel's formula and Gronwall's inequality have played a main role in both results. Moreover, the same results of existence and stability have been established for a nonlinear abstract, delayed system by considering a nonlinear source term satisfying some condition of Lipschitz.

Through careful proofs and theoretical setting, we have highlighted the importance of understanding the relation between system parameters, time delays, and feedback coefficients in ensuring stability and decay properties.

By giving a second-order abstract semilinear evolution equation with multiple time-varying delays, we have illustrated the comprehensive analysis presented in this work, in other words, the decay results of energy together with delayed effects in equations of evolution.

Bibliography

[1] R. A. Adams. Sobolev spaces, Academic Press, New York, 1975

[2] F. Alabau-Boussouira, P. Cannarsa and D. Sforza. Decay estimates for second order evolution equations with memory. Journal of Functional Analysis, 254 (2008) 13421372.

[3] A. Batkai, S. Piazzera. Semigroups for delay equations, Research Notes in Mathematics, 10. AK Peters, Ltd., Wellesley, MA, 2005

[4] C. Bardos, G. Lebeau and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM journal on control and optimization 30.5 (1992): 1024-1065.

[5] H. Brezis, Analyse fonctionnelle, Masson, Halsted Press, 1983.

[6] R. Datko, J. Lagnese, and MP. Polis. An example on the effect of time delays in boundary feedback stabilization of wave equations. SIAM journal on control and optimization 24.1 (1986): 152-156.

[7] V. Komornik, C.Pignotti. Energy decay for evolution equations with delay feedbacks. Mathematische Nachrichten. 295.2 (2022): 377-394.

[8] S. Nicaise, P. Pignotti. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM Journal on Control and Optimization. 45.5 (2006): 1561-1585.

[9] S. Nicaise, C. Pignotti. Exponential stability of abstract evolution equations with time delay. Journal of Evolution Equations. 15 (2015): 107-129.

[10]] S. Nicaise, C. Pignotti. Well-posedness and stability results for nonlinear abstract evolution equations with time delays. Journal of Evolution Equations. 18 (2018): 947-971.

[11] A. Pazy. Semigroups of linear operators and applications to partial differential equations, Vol. 44 of Applied Math. Sciences. Springer Verlag, New York, 1983.

[12] M. Tucsnak, G. Weiss. Observation and control for operator semigroups, Springer Science and Business Media, 2009.

[13] G. Q. Xu, S. P. Yung and L. K.Li. Stabilization of wave systems with input delay in the boundary control. ESAIM: Control, optimisation and calculus of variations. 12.4 (2006): 770-785.

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شهادة الترخيص بالإيداع

أنا الأستاذة : **شلاوة حورية**

بصفتي مشرف والمسؤول عن تصحيح مذكرة تخرج ماستر الموسومة بـــــ

Existence and stability for abstract linear and nonlinear evolution problems with single or multiple delay terms

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