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Faculty of Science and Technology
قسم الرياضيات والإعلام الآلي
Department of Mathematics and Informatics

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Thesis of end study, with a view to obtaining the diploma

Master

Domaine: Mathematics and Informatics, **Field:** Mathematics
Specialty: Functional Analysis and Applications

Theme

The Rational Limit Cycle of a Special Case of Abel Differential Equations

Presented by:

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University Year: 2023/2024

Dedications

I dedicate this modest work to:

- *My dear father and mother and the whole family.*
- *My husband, to my daughters: Anfal, Rofaida and to my son "Imran".*
- *My Professors.*

Boukhebibi Sarra

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Thank Allah the Glorious for all his graces and for giving me the strength, focus and patience to accomplish this work. I would like to express my deep gratitude to my thesis director, Professor Kina Abdelkrim, for proposing this thesis topic to me, directing me and providing me invaluable and judicious advice. I would like to thank all the members of the honorable jury for evaluating my work.

Finally, I would like to thank everyone who, directly or indirectly, contributed to the success of this work.

الملخص:

تهدف هذه المذكرة إلى دراسة الأنظمة التفاضلية كثيرة الحدود، مع التركيز على استقرارها، ودوراتها الحدية، وتطبيقات محددة في معادلات آبل

$$\frac{dy}{dx} = A(t)y^2 + B(t)y^3$$

يتم تقديم دراسة شاملة للأنظمة التفاضلية كثيرة الحدود المستوية، بما في ذلك تعريفاتها، وتوصيف النقاط المفردة، واستقرار هذه النقاط. ونتطرق في المذكرة أيضا لمعايير مثل Dulac- Bendixon يتم دراسة الدورات الحدية غير التافهة في معادلات آبل، بما في ذلك وجودها وعددها وتعددها.

وتشمل النتائج الرئيسية لعمل السيد Changjian وآخرين تحت عنوان:

"On the Rational Limit Cycles of Abel Equations"

توضيح وجود دورات حدية غير تافهة لمعادلات آبل في ظل ظروف معينة على درجات كثيرات الحدود A(t) وB(t)

ثم يظهر أن تعدد الدورات الحدية غير التافهة يمكن أن يكون غير محدود، باستخدام دراسة معادلات آبل المضطربة.

الكلمات المفتاحية: الأنظمة التفاضلية كثيرة الحدود، الدورات الحدية، معادلات آبل.

Abstract :

This work aims to study polynomial differential systems, focusing on their stability, limit cycles, and specific applications in Abel equations

$$\frac{dy}{dx} = A(t)y^2 + B(t)y^3$$

A comprehensive study of planar polynomial differential systems is presented, including their definitions, characterizations of singular points, and the stability of these points. Nontrivial limit cycles in Abel equations are analyzed, including their existence, number, and multiplicity. We also discuss criteria such as Bendixson-Dulac to determine the presence of limit cycles.
Key findings from the work of Mr. Changjian and others include:
"**On the Rational Limit Cycles of Abel Equations**" The existence of nontrivial limit cycles for Abel equations under certain conditions on the degrees of the polynomials A(t) and B(t) is demonstrated. Then is shown that the multiplicity of nontrivial limit cycles can be unbounded, using the study of perturbed Abel equations.

Keywords : Polynomial differential systems, Limit cycles, Abel equations.

Résumé:

Ce mémiore vise à étudier les systèmes différentiels polynomiaux, en se concentrant sur leur stabilité,leurs cycles limites et leurs applications spécifiques dans les équations d'Abel

$$\frac{dy}{dx} = A(t)y^2 + B(t)y^3 \quad .$$

Une étude complète des systèmes différentiels polynomiaux planaires est présentée, incluant leurs définitions, caractérisations des points singuliers et la stabilité de ces points. Les cycles limites non triviaux dans les équations d'Abel sont analysés, y compris leur existence, leur nombre et leur multiplicité. Dans ce mémiore, nous discutons de critères tels que Dulac-Bendixon pour déterminer l'existence de cycles limites. Les principales conclusions du travail de M. Changjian et d'autres comprennent :
« **On the Rational Limit Cycles of Abel Equations** »: L'existence de cycles limites non triviaux pour les équations d'Abel sous certaines conditions sur les degrés des polynômes A(t) et B(t) est démontrée. On montre ensuite que la multiplicité des cycles limites non triviaux peut être illimitée, en utilisant l'étude des équations d'Abel perturbées.

Mots clés : Systèmes différentiels polynomiaux, Cycles limites, Équations d'Abel.

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GENERAL INTRODUCTION

Differential equations are first encountered in mathematics in the works of Leibnitz (1646–1716) and Newton (1642–1727), towards the end of the 17th century. First of all, differential equations are closely related to solving geometric problems, Newtonian physics (point dynamique and planet motion), and formalized differential and integral calculus. They quickly establish themselves as a useful tool for analyzing natural phenomena and a source of inquiries about mathematical concepts like function. Following Newton and Leibnitz, among others, were the brothers Bernoulli, Jacob (1657–1705), and Johan (1667–1748), who followed earlier research and introduced the first differential equations of their own. Additional first-order equations with special derivatives that are frequently related to classification.

In this work, we are interested in an important aspect of the qualitative theory of planar polynomial differential systems, namely limit cycles. Such systems are fundamental in understanding various natural and engineered systems, including electrical circuits, biological systems, and mechanical systems. The core of this research is the analysis of stability and limit cycles within these systems, providing insights into their long-term behavior and stability.

A key concept explored in this work is the limit cycle, a closed trajectory in the phase space that neighboring trajectories either spiral into or away from over time. This concept is crucial in many applications, as it can represent stable periodic behaviors in real-world systems. Understanding the conditions under which limit cycles exist and their stability properties is essential for predicting the behavior of nonlinear systems.

The work is organized as follows:

- **Preliminary Concepts**: This chapter covers the basic definitions and theorems related to planar polynomial differential systems. Concepts such as vector fields, phase portraits, equilibrium points, and their stability are discussed. Key theorems like the Hartman-Grobman theorem .

- **On the theorie of limit cycles**: This chapter delves into the stability and existence of limit cycles. It includes criteria for the presence of limit cycles, such as Bendixson's and Dulac's criteria, and discusses their application to various types of nonlinear systems.

- **Rational Limit Cycles of Abel Equations**: The final chapter focuses on Abel equations, a specific type of polynomial differential system. The existence and properties of rational limit cycles within these equations are examined, along with their implications for Hilbert's 16th problem.

Overall, this work aims to advance the understanding of polynomial differential systems and their limit cycles, providing valuable tools and insights for both theoretical studies and practical applications.

CHAPTER 1

PRELIMINARY CONCEPTS

1.1 Planar polynomial differential systems

Some basic concepts for the qualitative theory of polynomial differential systems are covered in this chapter. The concepts of vector field, flux, phase portrait, equilibrium point, and linearization of non-linear differential systems at equilibrium points will all be covered. We will begin with polynomial differential systems. Next,we study the equilibrium points' characteristics and stability,then integrability and invariant curves.The basic theorems are reviewed , which include the Hartman-Grobman theorem, the existence and uniqueness theorem, and the theorems of the Lyapunov and Poincaré stability method of classification.

Definition : 1 *[28] A real planar polynomial differential system of degree m is a system*

$$\begin{cases} \dot{x} = P(x, y), \\ \\ \dot{y} = Q(x, y), \end{cases} \tag{1.1}$$

where P and Q are real polynomials in the variables x and y. The degree of the system (1.1) is the maximum of the degrees of the polynomials P and Q. As usual the dot denotes derivative with respect to the independent variable t.

1.1.1 Vector field

Definition : 2 *[28] The vector field associated to the polynomial differential system (1.1) is an application*

$$\mathcal{X} : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$M(x, y) \longmapsto \frac{d\overrightarrow{M}}{dt} = \begin{pmatrix} p(x, y) \\ \\ Q(x, y) \end{pmatrix}$$

Vector fields are essential for understanding the behavior of solutions to differential equations. They are used to define the local flow generated by a vector field .

In summary, vector fields provide a geometric interpretation of differential equations, allowing us to study the behavior of the solutions in a visual and intuitive way.

The graphical representation of a vector field on the plane consists in drawing a number of well chosen vectors $(x, \mathcal{X}(x))$ as in (1.1).

Remark: 1 *[28] We note $\mathcal{X} = (P, Q)$ the vector field associated with the system (1.1) . It is also written in the form:*

$$\mathcal{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

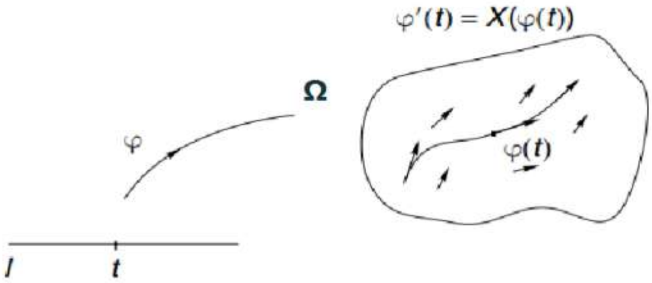


Figure 1.1: An integral curve [7].

Example: 1 *We give the following example;*

where all the following figures were drawn using the phase plane plotter [30]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases} \tag{1.2}$$

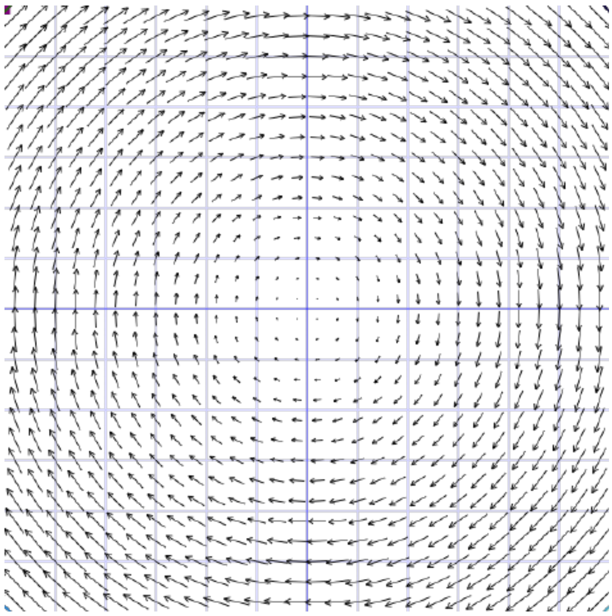


Figure 1.2: Vector field associated with the system (1.2).

Example: 2 *We give the following example*

$$\begin{cases} \dot{x} = y - 2x^2 + 1, \\ \dot{y} = y^2 - 3x. \end{cases} \tag{1.3}$$

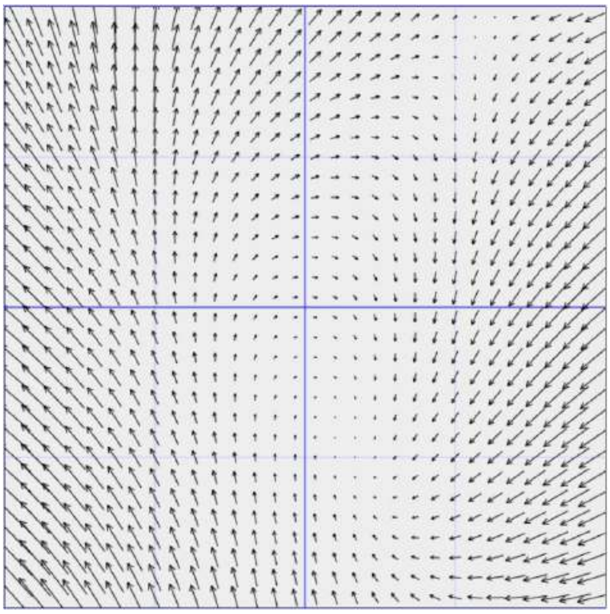


Figure 1.3: Vector field associated with the system (1.3).

1.1.2 Solution and periodic solution

Definition : 3 *[8] The differential system (1.1) solution is referred to as any differentiable function*

$$\varphi : \quad I \subset \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longrightarrow \varphi(t) = (\varphi_1(t), \varphi_2(t)),$$

where I is $\mathbb{R}$ a interval which satisfies the following criteria:

1. *for $t \in I, (t, \varphi(t)) \in \Omega$ open of $\mathbb{R}$*

2. *for all*

$$t \in I : \begin{cases} \frac{d\varphi_1(t)}{dt} = P(\varphi_1(t), \varphi_2(t)), \\ \frac{d\varphi_2(t)}{dt} = Q(\varphi_1(t), \varphi_2(t), \end{cases}$$

Definition : 4 *[8] We call periodic solution of system(1.1)any solution $\varphi(t)$=($\varphi_1(t)$,$\varphi_2(t)$), for which there exists a real T such that: $\forall t \in I$, $\varphi_1(t+T)$=$\varphi_1(t)$ et $\varphi_2(t+T)$=$\varphi_2(t)$. The period of this solution is the smallest $T > 0$ which is suitable.*

Definition : 5 *[7] The graph of a solution $\varphi : I \longrightarrow \mathbb{R}^2$ is an integral curve or trajectory of the differential system (1.1).*

1. *The image φ of $\mathbb{R}$ is an orbit of the differential system (1.1).*

2. *The space $\mathbb{R}^2$ or the solutions take their values (where each point represents a distinct state of the system) is called the phase space.*

Remark: 2 *[7] If the period of $X(t)$ is T, then the solution likewise has a ped $KT, K \in \mathbb{Z}$.*

Definition : 6 *[7] The representation of a solution, denoted by $X(t), t \in I$, on the plane $\mathbb{R}^2$ is the orbit of the system .*

Definition : 7 *[8] We call the phase plane associated with system (1.1), the open $\Omega \subseteqq \mathbb{R}^2$ where their solutions have values.*

Example: 3 *The harmonic oscillator is governed by the differential equation*

$$\ddot{x} + w^2 x = 0,$$

Witch can be written in the form of the following systems:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -w^2 x, \end{cases} \tag{1.4}$$

is system integrates easily since $\frac{dy}{dx} = -w^2\frac{x}{y}$ *,this gives a set of solutions* $y^2 + w^2x^2 = c$ *which* $c \in \mathbb{R}$ *.In other words this system has a continuous one-parameter family of periodic solutions represented in this phase plane by ellipses.*

1.1.3 Phase portrait

Trajectories that show the system's evolution across time are usually included in the phase portrait, along with crucial locations like equilibrium points, limit cycles, and other significant characteristics. The stability, periodicity, and general dynamics of the system described by the differential equations can all be understood by examining the phase portrait.

Definition : 8 *[22] A phase portrait is a phase space geometric representation of a dynamic system's trajectories, where each set of initial conditions is represented by a curve or a point.*

Example: 4 *The associated phase portrait with the system (1.2)*

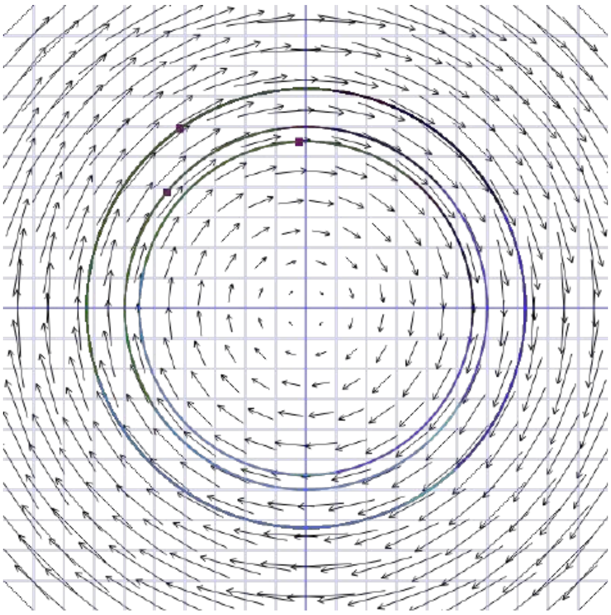


Figure 1.4: Phase portrait associated with the system (1.2).

Example: 5 *The associated phase portrait with the system (1.3)*

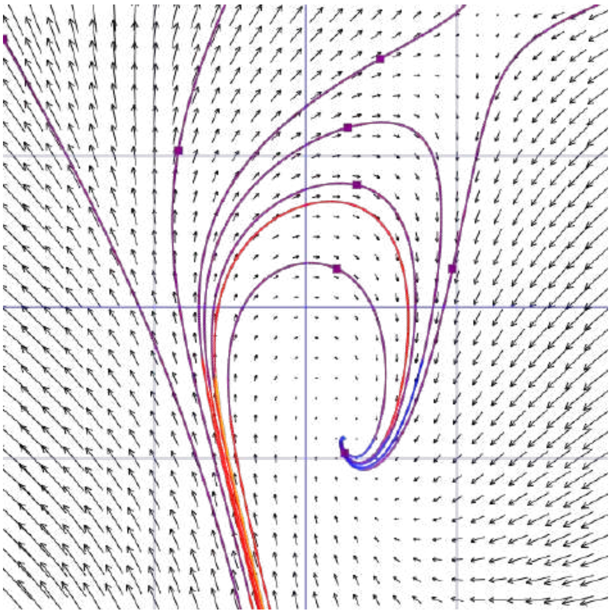


Figure 1.5: Phase portrait associated with the system (1.3) .

1.2 Singular point

Definition : 9 *[28] A point* (x^*, y^*) *called a singular point (respectively regular point) of system (1.1) if it satisfies*

$$P(x^*, y^*) = Q(x^*, y^*) = 0.$$

Remark: 3 *[28] For the vector field, the concept of an equilibrium point is equivalent to that of a single point. When examining the vector field in isolation, we prefer to refer to it as a solitary point, and when considering the trajectories, we speak of a point of equilibrium.*

Example: 6 *Let the system:*

$$\begin{cases} \dot{x} &= (x-2)(1+x+y), \\ \dot{y} &= (y-1)(x-1), \end{cases}$$

then the singular points are : (2,1),(-2,1),(1,0).

1.2.1 Linearization and Jacobian matrix

Most existing systems in nature are nonlinear. The most natural approach to studying the behavior of trajectories of a nonlinear autonomous differential system near a singular point is to reduce it to the study of the associated linear system.
We denote by $J_x(x^*, y^*)$ the Jacobian matrix associated with the vector field$mathcalX$ in the neighborhood of a singular point (x^*, y^*) defined by

$$J_x(x^*, y^*) = \begin{pmatrix} \frac{\partial P}{\partial x}(x^*, y^*) & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) \end{pmatrix} \tag{1.5}$$

The linearization of the nonlinear system (1.1) is given by:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x}(x^*, y^*) & \frac{\partial P}{\partial y}(x^*, y^*) \\ \frac{\partial Q}{\partial x}(x^*, y^*) & \frac{\partial Q}{\partial y}(x^*, y^*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

1.2.2 Classification of singular points

Definition : 10 *[7] Singular point stability is determined by the general stability theorems. Therefore, if the real eigenvalues (or real components of complex eigenvalues) are negative, the equilibrium point is asymptotically stable. Two examples of such equilibrium places are stable focus and stable node. If the real part of at least one eigenvalue is posi-*

tive, the corresponding equilibrium point is unstable. It might be a saddle for example. Lastly, we are dealing with the classical stability in the Lyapunov sense where the singular point is a center, or simply imaginary roots. Let's assume that the eigenvalues of the Jacobian matrix (1.5) are denoted by λ_1 and λ_2

1. *If $0 < \lambda_1 < \lambda_2$ then the singular point is unstable. is an "unstable node".*

2. *If $\lambda_1 < 0 < \lambda_2$ then it is said that the singular point is a "saddle point".*

3. *If $\lambda_1 < \lambda_2 < 0$ then the singular point is asymptotically stable. It is said that the singular point is a "stable node".*

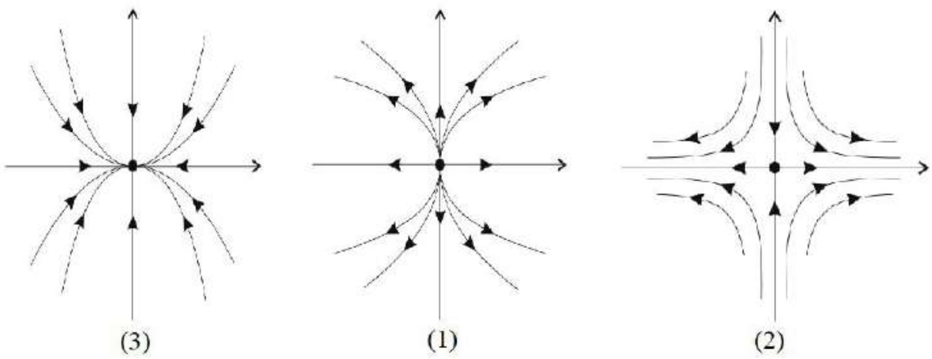


Figure 1.6: Real non-zero and distinct eigenvalues [22]

If λ_1 and λ_2 are distinct complex numbers (where their imaginary parts are not zero) (see figure 1.5):

1. If $Re(\lambda_1), Re(\lambda_2) < 0$ then the singular point is asymptotically stable.

   It is said that the point is a stable spiral sink "stable focus".

2. If $Re(\lambda_1), Re(\lambda_2) = 0$ then the singular point is a center.

3. If $Re(\lambda_1), Re(\lambda_2) > 0$ then the singular point is unstable.

   It is said that the point is an unstable spiral source "unstable focus".

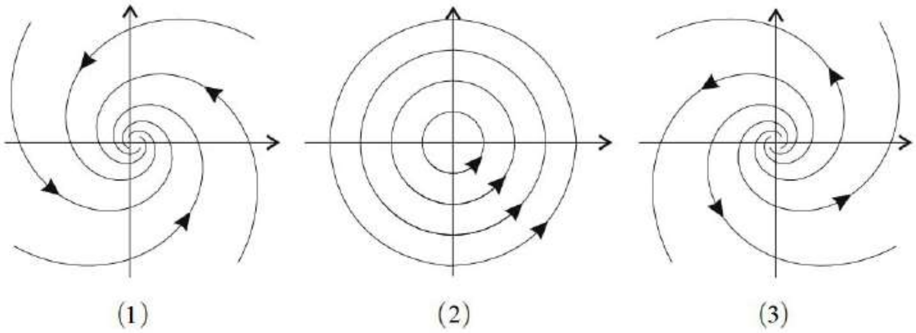


Figure 1.7: Complex eigenvalues[22]

If λ_1 and λ_2 are equal:

J_x is diagonalizable .

1. If $\lambda > 0$ then we say that the point is a "source in star".

2. If $\lambda < 0$ then we say that the point is a "sink in star".

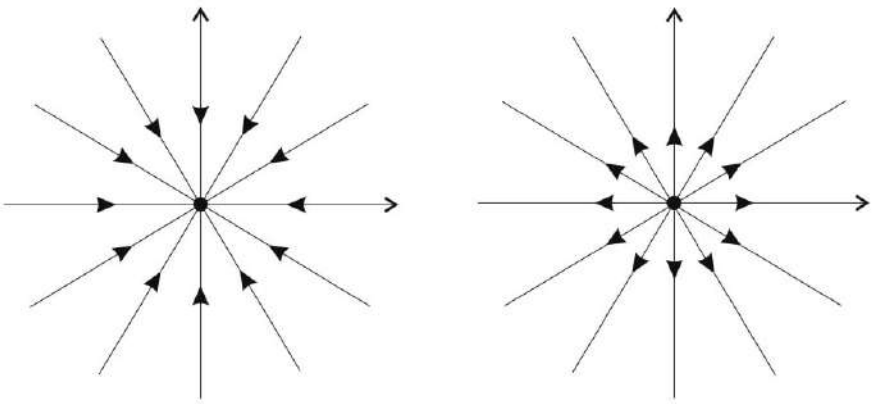


Figure 1.8: Eigenvalues are $\lambda_1 = \lambda_2$ [22]

A is not diagonalizable

1. If $\lambda > 0$ then the point is called a source.

2. If $\lambda < 0$ then the point is called a sink.

3. If $\lambda = 0$ then all points on the line kv are equilibrium points.

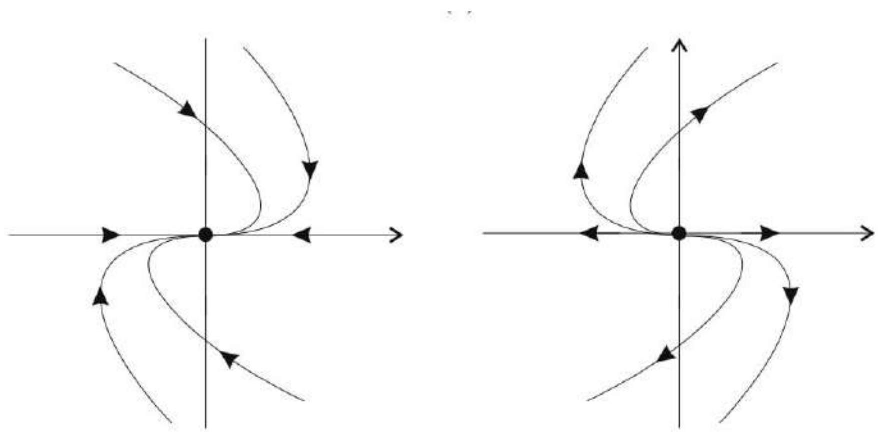


Figure 1.9: Real non-zero and distinct eigenvalues [22]

Remark: 4 *[7] We recall that a singular point p is hyperbolic if the eigenvalues of the linear part of the system at* (x^*, y^*) *have non zero real part.*

Theorem: 1 *[28] Let* (x^*, y^*) *be a hyperbolic singular point of system (1.9). Then in a neighborhood of the trajectories of* (x^*, y^*) *the solutions of system (1.9) have the same shape as the trajectories of the solutions of its linearization.*

Example: 7

$$\begin{cases} \dot{x} = -3x - y^2, \\ \\ \dot{y} = 3x^4 - y, \end{cases} \tag{1.6}$$

The system (1.6) has a single singular point which is the origin (0, 0). The Jacobian matrix associated with this system is given by:

$$J_x(x^*, y^*) = \begin{pmatrix} -3 & -2y \\ \\ 12x^3 & -1 \end{pmatrix} \tag{1.7}$$

the system linearized at this point is

$$\begin{pmatrix} \dot{x} \\ \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ \\ y \end{pmatrix}$$

. This system has two negative real eigenvalues $\lambda_1 = -3 and \lambda_2 = -1$. Then the singular point (0, 0) is a stable node.

Definition : 11 *[28] The singular point in the system (1.1) is called hyperbolic if the eigenvalues of the matrix $J_x(x^*, y^*)$ have real part different from 0. Otherwise, the singular point is said to be non-hyperbolic.*

1.2.3 Topological equivalence

Definition : 12 *[7] A homeomorphism of $\mathbb{R}^2$ is a bijective continuous map $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ whose inverse bijection is continuous.*

Definition : 13 *[7] Two autonomous systems in the plane*

$$\begin{cases} \dot{x} = P_1(x(t), y(t)), \\ \dot{y} = Q_1(x(t), y(t)), \end{cases} \tag{1.8}$$

and

$$\begin{cases} \dot{x} = P_2(x(t), y(t)), \\ \dot{y} = Q_2(x(t), y(t)). \end{cases} \tag{1.9}$$

Defined on two open sets U and V in $\mathbb{R}^2$,respectively are topologically equivalent if there exists a homeomorphism.

$$h : U \longrightarrow V$$

, such that h transforms the orbits of (A) into orbits of (B) and preserves the orientation of the orbits. The following theorem allows us to reduce the study of a differential system in the neighborhood of a hyperbolic singular point to the study of a linear system topologically equivalent to (1.9) in the neighborhood of the origin.

1.2.4 Theorem of Hartman-Grobman

The Hartman-Grobman theorem, sometimes referred to as the linearization theorem, is an important finding in the mathematical study of dynamical systems.Grasp the behavior of systems close to equilibrium points requires a grasp of this theorem. It asserts that a nonlinear system can be spatially topologically identical to its linearization around a hyperbolic equilibrium point under specific circumstances.

Theorem: 2 *(Hartman-Grobman,1967)[28]*

Let's assume that the Jacobian matrix at the singular point has two eigenvalues λ_1 and λ_2 such that the solutions of system (1.1) are approximately given by the solutions of the linearized system (1.1) in the vicinity of the singular point. In other words, the phase portrait of the linearized system constitutes, in the vicinity of this equilibrium point, a good approximation of that of the system (1.1).

Remark: 5 *[7] If $Re(\lambda_{1,2}) = 0$,the singular point (x^*, y^*) is called a center for the linearized system. Determining its nature in the case of system (1.1) requires further investigations: this is the center problem.*

1.2.5 Stability of an singular point

Definition : 14 *[7] Let (x^*, y^*) is singular point of the system (1.1). Note by $X^* = (P(x^*, y^*), Q(x^*, y^*))$ and $X(t) = (P(x(t), y(t), Q(x(t), y(t))))$.*

- *(x^*, y^*) is called stable if*

$$\forall \varepsilon > 0, \exists \eta > 0, \|X(t_0) - X^*\| < \eta \Rightarrow \forall t > 0, \|X(t_0) - X^*\| < \varepsilon.$$

- *(x^*, y^*) is called asymptotically stable if it is stable and : $\forall \varepsilon > 0, \exists \delta > 0, \alpha > 0 and \beta > 0$ such as:*

$$\lim_{t \longrightarrow +\infty} \|X(t) - X^*\| = 0.$$

- (x^*, y^*) *is called exponentially stable if*

$$\|X(t_0) - X^*\| < \varepsilon \forall, t > 0 : \|X(t_0) - X^*\| < \alpha \|X(t_0) - X^*\| e^{-\beta t}.$$

1.2.6 Stability in the sense of Lyapunov

The concept of Lyapunov stability is rooted in the idea of finding Lyapunov functions that certify the stability or asymptotic stability of equilibrium points, with positive definite and negative definite functions playing key roles in stability analysis.

Definition : 15 *[22] Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a $\mathbb{C}^1$ function $\dot{X} F(X)$, and X_0 a differential system,with singular points of system a Lyapunov $\mathbb{C}^1$ function such as*

$$V(X_0) = 0, \forall X \neq X_0 : V(X) > 0,$$
$$\forall X \neq X_0 : \overrightarrow{-gradV(X)} \dot{X} < 0.$$

Theorem: 3 *[22] Let X_0 an equilibrium and is a Lyapunov function. Then*

- $\dot{V}(X) \leq 0$ *then X_0 is lyapunov stable .*
- $\dot{V}(X) \leq 0$ *then X_0 is asymptotically stable.*
- $\dot{V}(X) > 0$ *then X_0 is unstable .*

Remark: 6 *[22] T_0 distinguish between the first and second cases, the Lyapunov function is usually called strict Lyapunov function.*

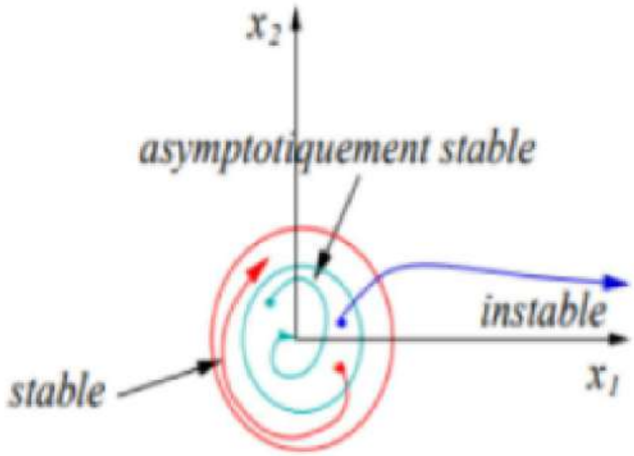


Figure 1.10: Different types of stability in the sense of Lyapunov[22]

Example: 8 *[22] Let the simplest possible example*

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = y. \end{cases}$$

$$V(x,y) = \frac{x^2+y^2}{2},$$

which is a positive definite function on all $\mathbb{R}^2$*.*

$$\dot{V}(x,y) = -(x^2+y^2),$$

which is negative for all $(x,y) \neq 0$*. We conclude that the equilibrium* $(\hat{x},\hat{y}) = (0.0)$ *is asymptotically stable. Moreover, since* $\dot{V} < 0$ *for all* $\mathbb{R}^2$ *and clearly* $V(x,y) \longrightarrow \infty$ *if* $\sqrt{x^2+y^2} \longrightarrow \infty$*,the equilibrium is globally asymptotically stable.*

1.2.7 Stability in the Poincaré sense

Compared to Lyapunov's concept, stability in the Poincaré sense is more precisely defined as the distance from a point M(x, y) to a periodic solution.

Definition : 16 *[33] Let $X(t) = x(x(t), y(t))$ a solution of system (1.1) a periodic solution $\phi(t)$ of this system is stable in the poincaré sense (or orbitally stable) if :*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|X(t_0) - \phi(t_0)\| < \delta \Longrightarrow d(t) = \inf_{t \in [0,t]} \|X(t) - \phi(t)\| < \varepsilon,$$

for $t \in [t_0, +\infty]$, $\phi(t)$ is asymptotically stable if it is stable and if in addition $\lim_{t \longrightarrow +\infty} d(t) = 0$.

Remark: 7 *[33] Stability in the Lyapunov sense implies stability in the Poincaré sense but the converse is not true.*

1.3 Invariant curve

The study of differential systems benefits greatly from using invariant curves, which can disclose crucial structural characteristics of the underlying dynamical system and shed light on the qualitative behavior of solutions.

Definition : 17 *[15] let $U : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ a function from class $\mathbb{C}^1$*

$$C_U = \{(x, y) \in \Omega / U(x, y) = 0\},$$

is referred to as an invariant curve if a function K of class $\mathbb{C}^1$ exists in Ω that satisfies the following relation, and that function is known as the cofactor:

$$P(x, y)\frac{\partial U}{\partial x} + Q(x, y)\frac{\partial U}{\partial x} = K(x, y)U(x, y),$$

for all $(x, y) \in \Omega$.

Example: 9 *[10] The curve defined by the equation $x^2 + y^2 = 0$ and $2x^2 + y^2 - 1 = 0$, is* *an invariant curve for the system :*

$$\begin{cases} \dot{x} &= -(x-y)(x^2-xy+y^2)+x(2x^4+2x^2y^2+y^4), \\ \dot{y} &= -(x+y)(2x^2-xy+2y^2)+y(2x^4+2x^2y^2+y^4), \end{cases}$$

the curves $x^2+y^2-1=0$, *and* $2x^2+y^2-1=0$, *have the cofactors* $2(2x^4+2x^2y^2+y^4-x^2-2y^2)$ *and* $2(2x^4+2x^2y^2+y^4-x^2+xy-2y^2)$, *respectively.*

Remark: 8 *[15]*

- *When the cofactor* $K(x,y)$ *is an invariant curve of polynomial cofactor.*
- *On the invariant curve, the gradient* $(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y})$ *of* U *is orthogonal to the vector field* $\mathcal{X}(P,Q)$, *so the vector field is tangent to the invariant curve at every point on this curve, hence this name is inspired by the fact that this curve is formed by solutions (or trajectories) of the vector field* $\mathcal{X}$.

1.3.1 Algebraic invariant curve

Definition : 18 *[15] An invariant curve* $U(x,y)=0$ *is said to be algebraic of degree* m *if* U *is a polynomial of degree* m*, otherwise we say that it is non-algebraic or transcendent. As you may recall the divergence of the system (1.1) is represented by the div notation.*

$$div(x,y) = \frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}.$$

Definition : 19 *[15] An algebraic curve* $U(x,y)=0$ *is irreducible, if* $U(x,y)$ *is a irreducible polynomial in the ring* $\mathbb{R}[x,y]$.

Theorem: 4 *[20] We consider the system (1.1) and* $\Gamma(t)$ *a periodic orbit of period* $T>0$

We suppose that $U : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an invariant curve

$$\Gamma(t) = \{(x, y) \in \Omega : U(x, y) = 0\},$$

and $K(x, y) \in \mathbb{C}^1$ is the cofactor specified in the equation (1,3) of the invariant curve $U(x, y) = 0$ .We suppose that $p \in \Omega$ such as $U(p) = 0$ and $\nabla U(p) \neq 0$ so P is singular point of systeme (1.3) and

$$\int_0^T div(\Gamma(t))dt = \int_0^T K(\Gamma(t))dt. \tag{1.10}$$

1.4 Integrability and first integral

Definition : 20 *[3] The $\mathbb{R}$-polynomial system (1.1) is integrable on an open subset Ω of $\mathbb{R}^2$ if there exists a nonconstant analytic function $H : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ called a first integral of the system on Ω , which is constant on all solution curves $(x(t), y(t))$ of system (1.2) contained in Ω i.e. $H(X(t)) = cste \quad \forall t \in I$ onstant for all values of t for which the solution $(x(t), y(t))$ is defined and contained in Ω . Clearly H is a first integral of system (1.1) on U if and only if $XH = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} \equiv 0$ on Ω .*

1.4.1 Integrating factors

Definition : 21 *[3] Let Ω be an open subset of $\mathbb{R}^2$ and let $R : \Omega \longrightarrow \mathbb{R}$ be an analytic function which is not identically zero on Ω. The function R is an integrating factor of the $\mathbb{R}$-polynomial system (1.1) if one Factors of the following three equivalent conditions holds :*

1. $\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}$.

2. $div(RP, RQ) = 0$.

3. $XR = -Rdiv(P, Q)$.

Definition : 22 *[3] The first integral H associated to the integrating factor R is given by*

$$H(x, y) = \int R(x, y)P(x, y)dy + h(x). \tag{1.11}$$

where h is chosen such that $\frac{\partial H}{\partial x} = -RQ$ then

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \quad \dot{y} = RQ = -\frac{\partial H}{\partial x}. \tag{1.12}$$

In (1.11)we suppose that the domain of integration Ω is well adapted to the specific expression. Conversely, given a first integral H of system (8.1)we always can find an integrating factor R for which (1.1) holds.

Proposition: 1 *[7] If the $\mathbb{R}$-polynomial system (8.1) has two integrating factors R_1 and R_2 on the open subset Ω of $\mathbb{R}^2$, then in the open set $\Omega \backslash \{R_2 = 0\}$ the function R_1/R_2 is a first integral, provided R_1/R_2 is non-constant.*

1.4.2 Exponential factors

Definition : 23 *[1] Let $g, h \in \mathbb{C}$ be relatively prime polynomials. The function $F = e^{g/h}$ is an exponential factor of system (1.1) if $XF/F = L \in \mathbb{C}[x, y]$. In this case, L is called the cofactor of F . It has degree at most $m - 1$. The expression which defines L is often written as*

$$P\frac{\partial e^{g/h}}{\partial x} + Q\frac{\partial e^{g/h}}{\partial y} = Le^{g/h}. \tag{1.13}$$

Remark: 9 *An exponential factor appears when an invariant algebraic curve has a geometric multiplicity greater than one.The exponential factors play the same role as invariant* *algebraic curves to obtain a first integral for the polynomial system. For more details on exponential factors than the ones given in this section.*

1.4.3 Inverse integrating factor:

The inverse integrating factor is an important tool in the study of the existence and non-existence of limit cycles.

Definition : 24 *[20] A non-zero function $V : \Omega \longrightarrow \mathbb{R}$ is an inverse integrating factor of the system (1.1) on an open $\Omega \subset \mathbb{R}^2$ if $V \in \mathbb{C}^1(\Omega), V \neq 0$ on Ω which satisfies the following equation*

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial x}\right) V.$$

Theorem: 5 *[8] Let V an inverse integrating factor (1.1) on an open $\Omega \in \mathbb{R}^2$*

1. *the function $\frac{1}{V}$ on $\Omega \backslash V = 0$ is an inverse integrating factor of system (1.1)in addition, the function*

$$I(x,y) = -\int \frac{P(x,y)}{V(x,y)}dy + \int \left(\frac{Q(x,y)}{V(x,y)} + \frac{\partial}{\partial x}\int \frac{P(x,y)}{V(x,y)}dy\right) dx,$$

   is a first integral of the system (1.1) .

2. *If the system (1.1) has a first integral I, then the function*

$$V_1(x,y) = \frac{P}{-\frac{\partial I}{\partial y}} = \frac{Q}{-\frac{\partial I}{\partial y}},$$

   is an integrating factor of system (1.1).

1.4.4 Darboux first integral:

The Darboux first integral is a significant class of first integrals exponential factors and the algebraic invariant curve can be used to define these functions.

Definition : 25 *[5] $x = (P, Q)$ of degree n is a polynomial planar complex vector field that admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors. K_i for $i = 1, ..., P$ and q exponential factor $\exp\left(\frac{g_i}{h_i}\right)$ with cofactors L_j for $j = 1, ..., q$ so :*

1. *There exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ if the function*

$$f_1^{\lambda_1} ... f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} ... \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q}. \tag{1.14}$$

   is a first integral of the vector field $\mathcal{X}$.

2. *It exists $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -div(P, Q)$ if the fanction (1.14) is a first integral of the vector field $\mathcal{X}$.*

Definition : 26 *[5] The Darboux function is the name given to the function (1.14). We say that the polynomial system (1.1) has a Darboux first integral if its first integral takes the form (1.14).*

Definition : 27 *[20] A Liouville function is a function that can be expressed by quadratures of elementary functions. The following theorem gives a relationship between the first integral and the inverse integrating factor.*

Theorem: 6 *[8] If a polynomial system has a Liouville first integral, then it has an inverse Darboux integrating factor.*

CHAPTER 2

ON THE THEORIE OF LIMIT CYCLES

The existence of limit cycles was first detected by Poincaré en 1882.These cycles are characterized by their stability and are crucial for simulating the behavior of numerous oscillatory systems found in real life. Neighboring trajectories are drawn to a stable or attractive limit cycle as time approaches infinity. On the other hand, when time gets closer to negative infinity, an unstable limit cycle repels nearby trajectories. Semi-stable limit cycles behave in both repellent and appealing ways. Trajectories' stability and the existence of nearby trajectories that spiral toward or away from the limit cycle are among the properties of trajectories that are analyzed in the study of limit cycles. By forecasting the existence or absence of these periodic orbits, the Poincaré-Bendixson theorem is an important tool in determining the existence of limit cycles in two-dimensional nonlinear dynamical systems. In this chapter, we are particularly interested in the study of limit cycles. Then, we present the main results on existence and non-existence.

Definition : 28 *[8] A limit cycle of system (1.1) is an isolated periodic solution in the set of all the periodic solutions.*

Example: 10 *[28] Either the system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + x(1 - x^2 - y^2). \end{cases} \tag{2.1}$$

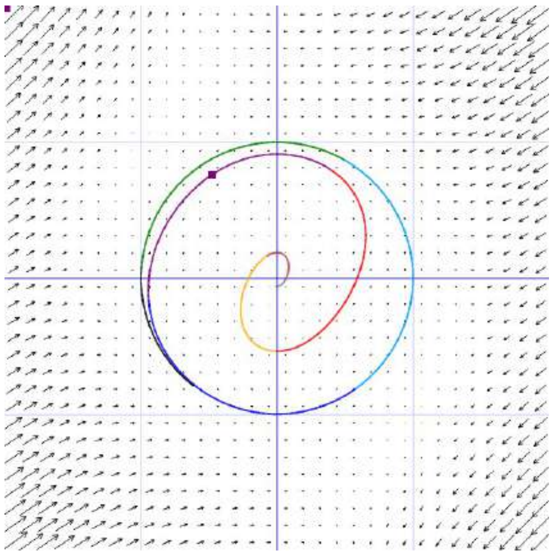


Figure 2.1: The phase portait of system (2.1)

Example: 11 *[31] Either the system*

$$\begin{cases} \dot{x} = 2x - y - 2x(x^2 + y^2), \\ \dot{y} = x + 2y - 2y(x^2 + y^2), \end{cases} \tag{2.2}$$

In polar coordinates $x = r\cos(\theta)$ *and* $y = rsin(\theta)$*, with* $r > 0$*, it becomes :*

$$\begin{cases} \dot{r} = 2r(1 - r^2), \\ \dot{\theta} = 1. \end{cases} \tag{2.3}$$

Hence

$$\dot{r} = 0 \Longrightarrow r = \pm 1 \quad or \quad r = 0.$$

Since $r > 0$, only the positive root $r = 1$ is accepted. So for $r = 1$ we have the periodic solution $(x(t), y(t)) = (\cos(t + \theta_0), \sin(t + \theta_0))$, with $\theta(0) = \theta_0$. In the phase plane there is a single limit cycle whose equation $x^2 + y^2 = 1$ and amplitude $r = 1$.

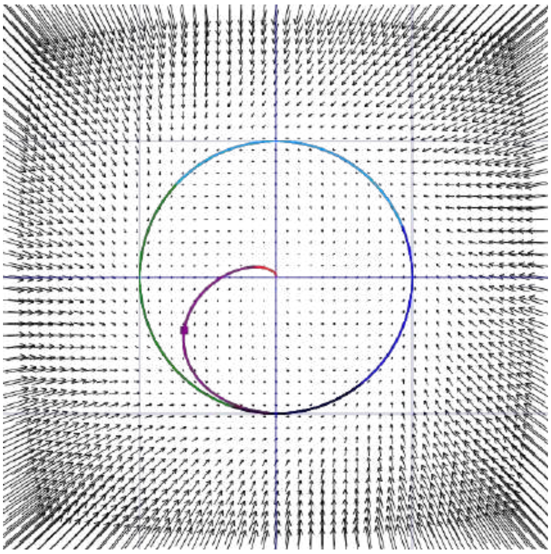


Figure 2.2: Thé phase portait of system (2.2)

Definition : 29 *[10] An algebraic limit cycle is a limit cycle which is contained in the zeroes set of an invariant algebraic curve.*

Theorem: 7 *[24] Let $f \in \mathbb{R}[x, y]$ i.e. f is a polynomial in the variables x and y. The algebraic curve $f(x, y) = 0$ is an invariant algebraic curve of the polynomial system of differential equations (1.1) if for some polynomial $K \in \mathbb{R}[x, y]$, we have*

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2.4}$$

The polynomial K is called the cofactor of the invariant algebraic curve $f = 0$.

Example: 12 *[29] Let the system be :*

$$\begin{cases} \dot{x} & = x(x^2 + y^2 - 1) - y(x^2 + y^2 + 1), \\ \dot{y} & = y(x^2 + y^2 - 1) + x(x^2 + y^2 + 1), \end{cases} \tag{2.5}$$

this system admits a single algebraic limit cycle of equation :

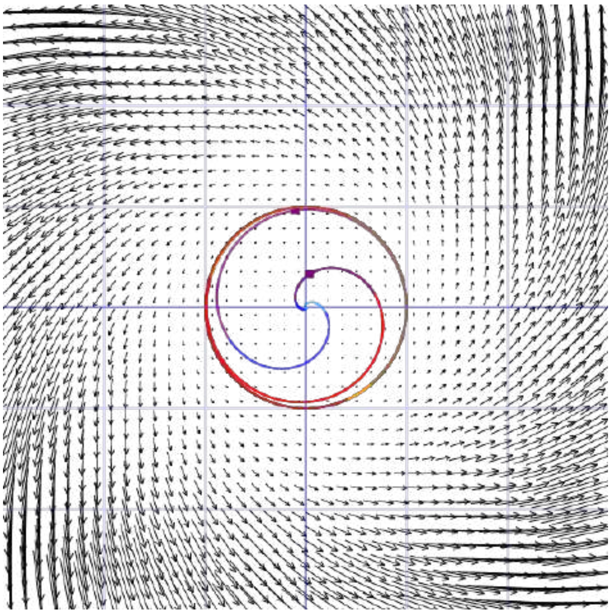


Figure 2.3: System limit cycle (2.5)

Example: 13 *[4] The differential polynomial system of degree 3*

$$\begin{cases} \dot{x} = x + (y - x)(x^2 - yx + y^2), \\ \dot{y} = y - (y + x)(x^2 - yx + y^2), \end{cases} \tag{2.6}$$

has a unique non–algebraic limit cycle.

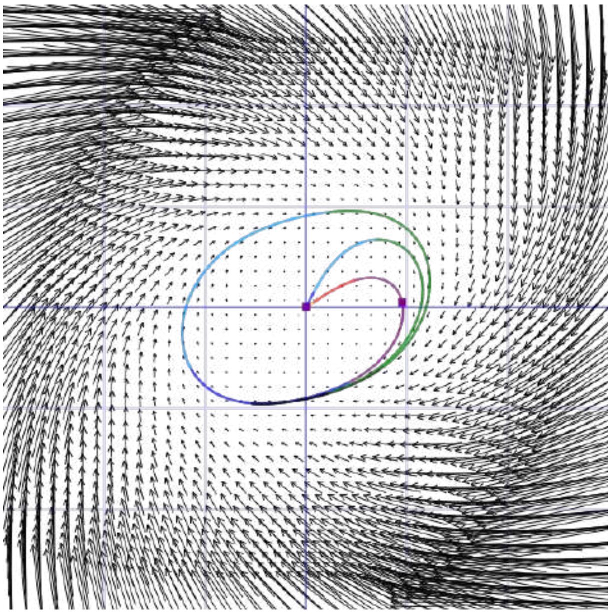


Figure 2.4: System limit cycle (2.6)

2.1 The non existence and existence of limit cycles

2.1.1 The non existence of the limit cycle

Theorem: 8 *[37] (Bendixson). If the divergence $\partial P/\partial x + \partial Q/\partial y$ of system (1.1) has constant sign in a simply connected region U , and is not identically zero on any subregion of U , then system (1.1) does not possess any limit cycle (in fact a closed trajectory) which lies entirely in U .*

Proposition: 2 *[14] Let $(P,\ Q)$ be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$ Let $V = V(x,y)$ be a C^1 solution of the linear partial differential equation . Let $\sum = (x,y) \in U : V(x,y) = 0$.If the divergence $\{\partial P/\partial x + \partial Q/\partial y\}$ of system (1.1) has constant sign in the simply connected domain of definition of $V \diagdown \sum$,then any limit cycle (in fact a closed trajectory) which lies entirely in the simply connected domain of definition of V must be contained in $\sum$.*

Proposition: 3 *[14] In the interior D of any closed trajectory of system (1.1) of a simply* *connected region we have*

$$\int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial x}\right) dxdy = 0. \tag{2.7}$$

Example: 14 *we give the following example :*

$$\begin{cases} \dot{x} = -x^3y^2 + 5, \\ \dot{y} = -x^4y^5 - 1. \end{cases} \tag{2.8}$$

We have

$$\tfrac{\partial P}{\partial x} + \tfrac{\partial Q}{\partial y} \quad = -3x^2y^2 - 5x^4y^4 < 0. \tag{2.9}$$

then there can therefore be no limit cycle in $\mathbb{R}^2$

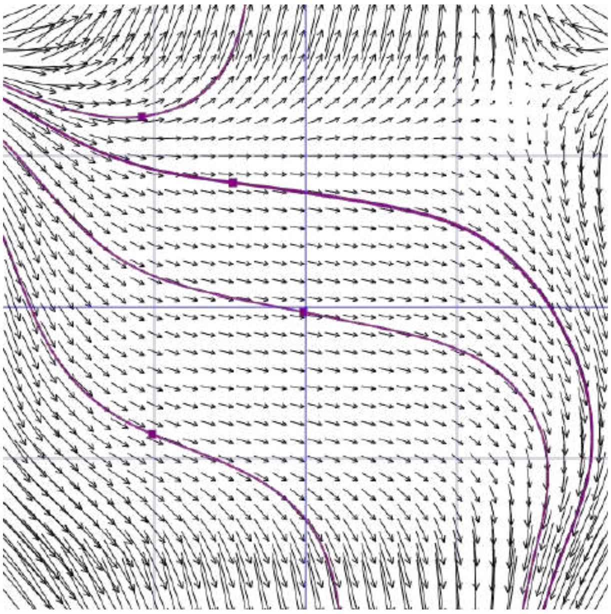


Figure 2.5: Phase portrait of the system (2.9)

Theorem: 9 *[14](Bendixon criterion) Let D be a related domain of* $\mathbb{R}^2$ *.If div(*$\frac{dP}{dx} + \frac{dQ}{dy}$*) is non-zero and of constant sign on D, then the differential system (1.1) does not admit an entirely contained periodic solution in D.*

Example: 15 *Consider the system*

$$\begin{cases} \dot{x} = 2x^5 - 3y, \\ \dot{y} = 4y + 2x^2, \end{cases} \tag{2.10}$$

$\frac{dP}{dx} + \frac{dQ}{dy} = 10x^4 + 4 > 0$ *at any point of* $\mathbb{R}^2$, *there can therefore be no limit cycle contained in* $\mathbb{R}^2$.

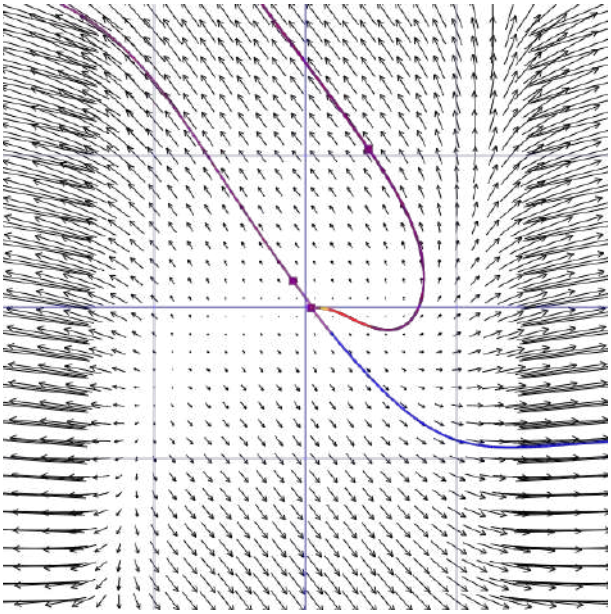


Figure 2.6: Phase portrait of the system (2.10)

Theorem: 10 *[37] (Bendixson-Dulac). If there exists a continuously differentiable function B(x, y) in a simply connected region U such that* $\partial(BP)/\partial x + \partial(BQ)/\partial y$ *has constant sign and is not identically zero in any subregion, then system (1.1) does not possess any limit cycle (in fact a closed trajectory) which lies entirely in U .*

Proposition: 4 *[14] Let (P, Q) be a* C^1 *vector field defined in the open subset U of* $\mathbb{R}^2$ *Let* $V = V(x, y)$ *be a* C^1 *solution of the linear partial differential equation (1.2). Let B(x, y) a continuously differentiable function in U such that* $\partial(BP)/\partial x + \partial(BQ)/\partial y$ *has constant sign, then any limit cycle (in fact a closed trajectory) which lies entirely in the simply* *connected domain of definition of V must be contained in* $\{\sum = (x,y) \in U : V(x,y) = 0\}$.

Theorem: 11 *[14] (Cherkas).the vecteur field $\mathcal{X}(x,y)$ defined in the in the open subset U of $\mathbb{R}^2$ Suppose that in a simply connected domain $U \subset \mathbb{R}^2$, there exists a function $\Psi(x,y)$ f class C^1 and a number $k > 0$ such that*

$$k\Psi div\mathcal{X} + \mathcal{X}\Psi > 0,$$

then the domain U contains no limit cycles of system (1.1).

Corollaire: 1 *A closed trajectory has a critical point in its interior.If we turn this statement around, we see that it is really a criterion for non-existence: it says that if a region $\mathbb{R}$ is simply-connected (i.e., without holes) and has no critical points, then it cannot contain any limit cycles.For if it did, the Critical-point Criterion says there would be a critical point inside the limit cycle, and this point would also lie in $\mathbb{R}$ since $\mathbb{R}$ has no holes.*

2.1.2 The existences of limit cycles

Theorem: 12 *[13] (First criterion). Let (P , Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$, $(u(t), v(t)$ a periodic solution of (P , Q) of period $T, R : U \longrightarrow \mathbb{R}$ a C^1 map such that $\int_0^T R(u(t), v(t))dt \neq 0$ and $V = V(x,y)$ a C^1 solution of the linear partial differential equation (1). Then the closed trajectory*

$$y = ((u(t), v(t)) \in U : t \in [0,T]),$$

is contained in

$$\left\{\sum = (x,y) \in U : V(x,y) = 0\right\}.$$

and γ is not contained in a period annulus of $(P\ ,\ Q)$. Moreover, if the vector field $(P\ ,\ Q)$ is analytic, then γ is a limit cycle.

Remark: 10 *[13] We note that theorem 12 shows that when we know explicitly the trajectories $V(x,y) = 0$ of a vector field $(P\ ,\ Q)$ through equation (1.1) for a given $\mathbb{R}$, we additionally have information on the periodic solutions of $(P\ ,\ Q)$, because if γ is a closed trajectory it must satisfy that either γ is contained in $(x,y) \in U : V(x,y) = 0$, or $\int_\gamma Rdt = 0$.*

The Poincare-Bendixson Theorem

Suppose $\mathbb{R}$ is the finite region of the plane lying between two simple closed curves D_1 and D_2, and fis the velocity vector field for the system (1.1). If

1. at each point of D_1 and D_1, the field points toward the interior of $\mathbb{R}$.

2. $\mathbb{R}$ contains no critical points,

then the system (1.1) has a closed trajectory lying inside $\mathbb{R}$.

Example: 16 *[28] Consider the system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2), \end{cases} \tag{2.11}$$

had a limit cycle Γ represented by $\gamma(t) = (\cos t, \sin t)$,the Poincaré map for Γ can be found by solving this system written in polar coordinates on pose

$$\begin{cases} x = rcos(\theta), \\ y = rsin(\theta), \end{cases} \tag{2.12}$$

the formulation of the system allows you to go from Cartesian coordinates (x,y) to polar coordinates (r,θ)

$$\begin{cases} \dot{r}cos(\theta) - r\dot{\theta}sin(\theta) = -rsin(\theta) + rcos(\theta)(1-r^2), \\ \dot{r}sin(\theta) + r\dot{\theta}cos(\theta) = rrcos(\theta) + rsin(\theta)(1-r^2). \end{cases} \tag{2.13}$$

This means that :

$$\begin{aligned} cos(\theta)\dot{x} + sin(\theta)\dot{y} &= \dot{r} \\ &= -rsin(\theta)cos(\theta) + rcos^2(\theta)(1-r^2) + rsin(\theta)cos(\theta) + rsin^2(\theta)(1-r^2) \\ &= r(1-r^2). \end{aligned}$$

and we replace the value of $\dot{r}$ in (2.13)

$$sin(\theta)(r(1-r^2) - r(1-r^2)) = rcos(\theta)(1-\dot{\theta}),$$

$$\dot{\theta} = 1.$$

We obtain the system:

$$\begin{cases} \dot{r} = r(1-r^2), \\ \dot{\theta} = 1, \end{cases}$$

with $r(0) = r_0$ and $\theta(0) = \theta_0$.the first equation can be solved either as a separable differential equation or as a Bernoulli equation,the solution is given by

$$r(t, r_0) = \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}, \tag{2.14}$$

and

$$\theta(t, \theta_0) = t + \theta_0$$

If $\sum$ is the ray $\theta = \theta_0$ through the origin,then $\sum$ is perpendicular to Γ and the trajectory *through the point* $(r_0, \theta_0) \in \sum \cap \Gamma$ *at* $t = 0$. *intersects the ray* $\theta = \theta_0$ *again at* $t = 2\pi$. *It follows that the Poincare map is given by*

$$P(r_0) = \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}, \tag{2.15}$$

Clearly $P(1) = 1$ *corresponding to the cycle* Γ *and we see that*

$$P'(r_0) = e^{-4\pi} {r_0}^{-3} \left[1 + \left(\frac{1}{r^2} - 1\right) e^{-4\pi}\right]^{-1/2}, \tag{2.16}$$

and that $P'(1) = e^{-4\pi} < 1$.

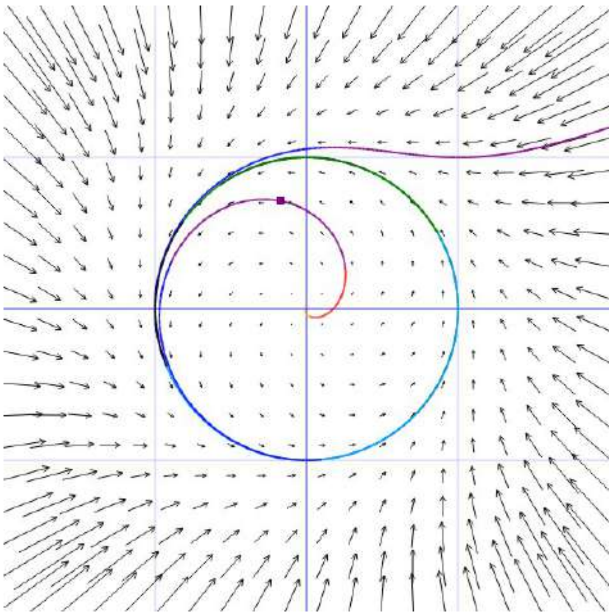


Figure 2.7: Phase portrait of the system (2.11)

Example: 17 *[27] Let be the following dynamic system*

$$\begin{cases} \dot{x} = y + x(1 - x^2 - y^2), \\ \dot{y} = -x + y(1 - x^2 - y^2). \end{cases} \tag{2.17}$$

The function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ *is a Lyaponov function for this System*

$$
\begin{aligned}
V &= x\dot{x} + y\dot{y} \\
&= (x^2+y^2)(1-x^2-y^2).
\end{aligned}
$$

If we consider the Domain D defined by the circle of radius $\frac{1}{2}$, and the circle of radius 2,

$$D = \left\{ \ (x,y) \backslash \tfrac{1}{4} \leq x^2 + y^2 \leq 4 \ \right\} \tag{2.18}$$

we get an attractive domain for our system.The domain of therefore, we can conclude from the corollary of the Poincare-Bendixon theorem that there is at least one limit cycle completely contained in D. We consider the domain D_1 defined by

$$D_1 = \left\{ \ (x,y) \backslash x^2 + y^2 \leq 2 \ \right\} \tag{2.19}$$

the domain D contains a single unstable equilibrium point (the origin), so we can conclude from the Poincare-Bendixon theorems that there exists at least one limit cycle

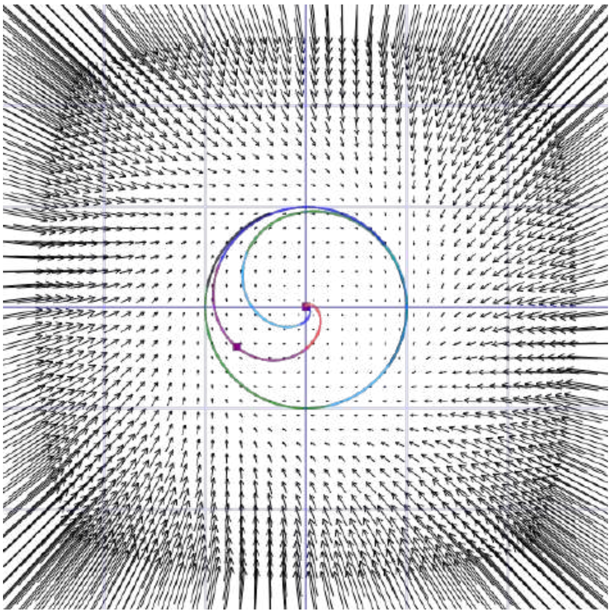


Figure 2.8: The phase portait of system (2.17)

Theorem: 13 *[34] Let (P, Q) be a C^1 vector field defined in the open subset U of $\mathbb{R}^2$.*

Let V (x, y) be an inverse integrating factor If γ is a limit cycle of the vector field (P, Q) in the domain of definition of V , then γ is contained in $\{\sum = (x, y) \in U : V(x, y) = 0\}$.

2.2 Stability of limit cycles

Theorem: 14 *[7] Let E be an open subset of $\mathbb{R}^2$ and suppose that $f \in C^1(E)$. Let $\gamma(t)$ be a periodic solution of (1) of period T,then the derivative of the Poincaré map P(s) along a straight line $\sum$ normal to $\Gamma = x \in \mathbb{R}^2 | x = \gamma(t) - \gamma(0), 0 \leq t \leq T$ at $x = 0$ is given by*

$$P'(0) = exp \int_0^T \nabla . f(\gamma(t))dt$$

.

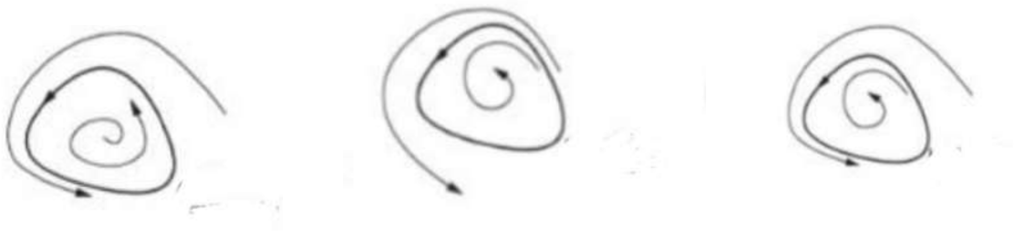
Corollaire: 2 *[18] Under the hypotheses of Theorem 14, the periodic solution $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla . f(\gamma(t))dt < 0$$

and it is an unstable limit cycle if

$$\int_0^T \nabla . f(\gamma(t))dt > 0$$

. It may be a stable.unstable or semi-stable limit cycle or it may belong to a continuous band of cycles if this quantity is zero.



Stable limite cycle   Unstable limite cycle   half stable limite cycle

Figure 2.9: Classification of limit cycles [35]

For Example2.11 above , we have $\gamma(t) = (cost, sint)^T, \nabla.f(x, y) = 2 - 4x^2 - 4y^2$ and

$$\int_0^{2x} \nabla.f(\gamma(t))dt = \int_0^{2x} (2 - 4cos^2 t - 4sin^2 t)dt = -4\pi.$$

Thus, with $s = r - 1$,it follows from Theorem 14 that

$$p'(0) = e^{-4\pi}$$

, which agrees with the result found in Example 2.11 by direct computation. Since $P'(0) < 1$ the cycle $\gamma(t)$ is a stable limit cycle in this example.

Definition : 30 *[18] Let P(s) be the Poincare map for a cycle Γ of a planar analytic system(1.1)and let*

$$d(s) = P(s) - s$$

be the displacement function,then if

$$d(0) = d'(0) = \cdots = d^{(k-1)}(0) = 0 \quad \textit{and} \quad d^{(k)}(0) \neq 0,$$

Γ is called a multiple limit cycle of multiplicity k If $k = 1$ then Γ is called a simple limit

cycle. We note that $\Gamma = \{x \in \mathbb{R}^2 | x = \gamma(t), 0 \leq t \leq T\}$ *is a simple limit cycle of (1) if*

$$\int_0^T \nabla . f(\gamma(t)) dt \neq 0$$

It can be shown that if k is even then Γ *is a semi-stable limit cycle and if k is odd then* Γ *is a stable limit cycle if* $d^{(k)}(0) < 0$ *and* Γ *is an unstable limit cycle if* $d^{(k)}(0) > 0$.

Example: 18 *[31]*

$$\begin{cases} \dot{x} = 3x - y - 3x(x^2 + y^2), \\ \dot{y} = x + 3y - 3y(x^2 + y^2), \end{cases} \tag{2.20}$$

the periodic solution is a stable limit cycle.

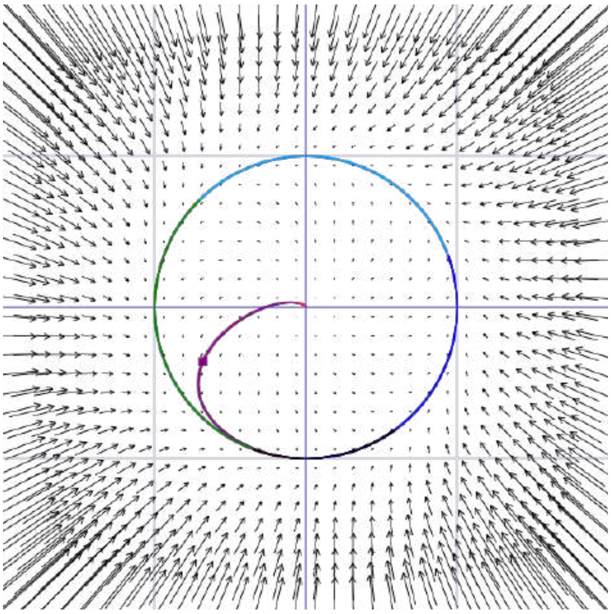


Figure 2.10: Stable limit cycle of system (2.20)

$$\begin{cases} \dot{x} = -5x - y + 5x(x^2 + y^2), \\ \dot{y} = x - 5y + 5y(x^2 + y^2), \end{cases} \tag{2.21}$$

the periodic solution is a unstable limit cycle

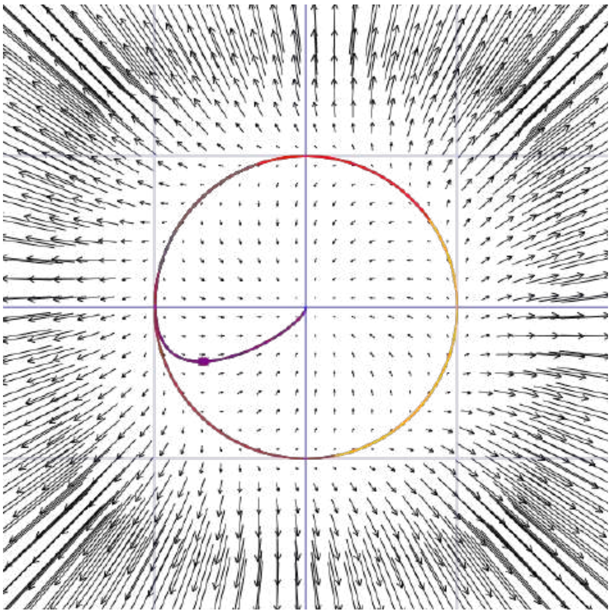


Figure 2.11: Unstable limit cycle of system (2.21)

Example: 19 *[36] The system*

$$\begin{cases} \dot{x} = -y + \eta x(1 - x^2 - y^2), \\ \dot{y} = x + \eta y(1 - x^2 - y^2), \end{cases} \tag{2.22}$$

has a limit cycle Γ *represented by*

$$\Gamma(\theta) = (\cos\theta, \sin\theta),$$

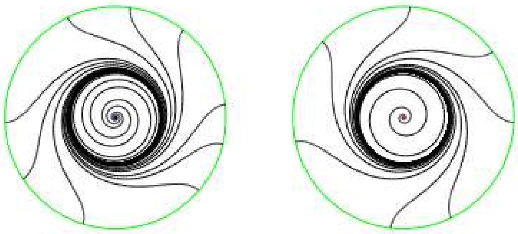
because we have

$$div(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \eta(2 - 4x^2 - 4y^2),$$

where $P = -y + \eta x(1 - x^2 - y^2)$ *and* $Q = x + \eta y(1 - x^2 - y^2)$.

$$\begin{aligned}\int_0^T div(\Gamma(\theta))\,dt &= \int_0^T div(\cos\theta,\sin\theta)\,dt\\ &= \eta\int_0^{2\pi}(4(\cos\theta)^2+4(\sin\theta)^2-2)\,d\theta,\\ &= \eta\int_0^{2\pi} 2\,d\theta.\end{aligned}$$

So the cycle $\Gamma(\theta) = (\cos(\theta), \sin(\theta))$ is an unstable limit cycle if $\eta > 0$ and is a stable limit cycle if $\eta > 0$



(a) limit cyle for $\eta = -0.2$. (b) limit cyle for $\eta = 0.2$.

Figure 2.12: Limit cycle of system (2.22) [36]

2.3 The first return map

Probably the most basic tool for studying the stability of periodic orbits is the Poincaré map or first return map, defined by Henri Poincaré in 1881. The idea of the Poincaré map is quite simple: If Γ is a periodic orbit of system (1.1), through the point (x_0, y_0) and Σ is a hyperplane perpendicular to Γ at (x_0, y_0), then for any point (x, y) in Σ sufficiently near (x_0, y_0), the solution of (1.1) through (x, y) at $t = 0$, $\Phi_t(x, y)$ will cross Σ again at a point $\Pi(x, y)$ near (x_0, y_0). The mapping $(x, y) \to \Pi(x, y)$ is called the Poincaré map.

The next theorem establishes the existence and continuity of the Poincaré map $\Pi(x, y)$ and of its first derivative $D\Pi(x, y)$.

Theorem: 15 *[18] Let E be an open subset of $\mathbb{R}^2$ and let $(P, Q) \in C^1(E)$. Suppose that*

$\Phi_t(x_0, y_0)$ *is a periodic solution of* ([1.1](#)) *of period* T *and that the cycle*

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = \Phi_t(x_0, y_0), 0 \leq t \leq T\}$$

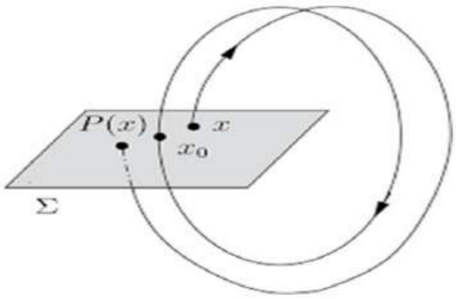


Figure 2.13: The poincare map [18]

is contained in E. *Let* Σ *be the hyperplane orthogonal to* Γ *at* (x_0, y_0), *i.e.; let*

$$\Sigma = \{(x, y) \in \mathbb{R}^2 \mid (x - x_0, y - y_0) \cdot (P(x_0, y_0), Q(x_0, y_0)) = 0\}.$$

Then there is a $\delta > 0$ *and a unique function* $\tau(x, y)$, *defined and continuously differentiable for* $(x, y) \in N_\delta(x_0, y_0)$, *such that*

$$\tau(x_0, y_0) = T,$$

and

$$\Phi_{\tau(x,y)}(x, y) \in \Sigma,$$

for all $(x, y) \in N_\delta(x_0, y_0)$.

Definition : 31 *[28] Let* Γ, Σ, δ, *and* $\tau(x, y)$ *be defined as in Theorem (2.5). Then for* $(x, y) \in N_\delta(x_0, y_0) \cap \Sigma$, *the function*

$$\Pi(x, y) = \Phi_{\tau(x,y)}(x, y)$$

is called the Poincaré map for Γ at (x_0, y_0).

The following theorem gives the formula of $\Pi_0(0,0)$.

Example: 20 *[18] Let the following system of equations:*

$$\begin{cases} \dot{x} = -y + x(4 - x^2 - y^2), \\ \dot{y} = x + y(4 - x^2 - y^2). \end{cases} \tag{2.23}$$

Note that $(0,0)$ *is the only critical point of (2.23). In polar coordinates, the system (2.23) becomes:*

$$\begin{cases} \dot{r} = r(4 - r^2), \\ \dot{\theta} = 1. \end{cases} \tag{2.24}$$

The differential system (2.24) is equivalent to the following Bernoulli differential equation

$$\frac{dr}{d\theta} = 4r - r^3.$$

The general solution is

$$r(\theta) = \left[\frac{1}{4} + ce^{-8\theta}\right]^{-\frac{1}{2}}.$$

A solution of (2.24) satisfying the initial condition $r(0) = r_0$ *is given by*

$$r(\theta, r_0) = \left[\frac{1}{4} + \left[\frac{1}{r_0^2} - \frac{1}{4}\right] e^{-8\theta}\right]^{-\frac{1}{2}}.$$

For $\theta = 2\pi$, *it follows that the Poincaré first return map is given by:*

$$\Pi(r_0) = r(2\pi, r_0) = \left[\frac{1}{4} + \left[\frac{1}{r_0^2} - \frac{1}{4}\right] e^{-16\pi}\right]^{-\frac{1}{2}}.$$

Definition : 32 *[18] A fixed point of the application* Π *is a point* (x, y) *such that*

$\Pi(x, y) = (x, y)$. *It corresponds to a periodic orbit of the system (1.1).*

2.4 Stability of poincare map

Theorem: 16 *[28] Let $\Gamma(t)$ be a periodic solution of (1.1) of period T. Then the derivative of the Poincaré map $\Pi(s)$ along a straight line Σ normal to $\Gamma = \{(x, y) \in \mathbb{R}^2 : (x, y) = \Phi_t(x_0, y_0), 0 \leq t \leq T\}$ at $(x, y) = (0, 0)$ is given by*

$$\Pi'(0) = \exp \int_0^T \nabla \cdot (P(\Gamma(t)), Q(\Gamma(t)))\, dt.$$

Corollaire: 3 *[12] Under the hypotheses of (1.10.2), the periodic solution $\Gamma(t)$ is a stable limit cycle if*

$$\int_0^T \nabla \cdot (P(\Gamma(t)), Q(\Gamma(t)))\, dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T \nabla \cdot (P(\Gamma(t)), Q(\Gamma(t)))\, dt > 0.$$

Example: 21 *[18] The system (1.12) has a limit cycle Γ_1 represented by*

$$\Gamma_1(\theta) = (2\cos\theta, 2\sin\theta),$$

we have $\nabla \cdot (P(x, y), Q(x, y)) = 8 - 4x^2 - 4y^2$ and

$$\int_0^{2\pi} \nabla \cdot (P(\Gamma_1(\theta)), Q(\Gamma_1(\theta)))\, d\theta = \int_0^{2\pi} 8 - 4((2\cos(\theta))^2 + (2\sin(\theta))^2)\, d\theta.$$

$$= -8\pi < 0.$$

Then, the cycle Γ_1 is a stable limit cycle.

2.5 Hyperbolic limit cycle

Definition : 33 *[28] A limit cycle $\Gamma = (x(t), y(t)), t \in [0, T]$ is a T-periodic solution isolated with respect to all other possible periodic solutions of the system.*

A T-periodic solution Γ is a hyperbolic limit cycle if $\int_0^T div(\Gamma)dt$ is different from zero.

Example: 22 *[28] The system*

$$\begin{cases} \dot{x} = -y + x(1 - x^2 - y^2), \\ \dot{y} = x + y(1 - x^2 - y^2), \end{cases} \tag{2.25}$$

polar coordinates

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned} \tag{2.26}$$

coordinates the previous system becomes

$$\begin{cases} \dot{r} = r(1 - r^2), \\ \dot{\theta} = 1. \end{cases} \tag{2.27}$$

So

$$\dot{r} = 0 \Leftrightarrow r = 0 \ \ or \ \ r = 1$$

, for

$$\gamma(t) = (\cos(t), \sin(t))$$

$$\begin{aligned}
\int_0^{2\pi} div(\gamma(t))dt &= \int_0^{2\pi}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)(\cos(t),\sin(t))\,dt,\\
&= \int_0^{2\pi}(1-3\cos^2(t)-\sin^2(t))+(1-\cos^2(t)-3\sin^2(t)),dt\\
&= \int_0^{2\pi}(1-4\cos^2(t)-4\sin^2(t))dt,\\
&= \int_0^{2\pi}-2dt=-4\pi\neq 0.
\end{aligned} \tag{2.28}$$

So, system (1.12) has a hyperbolic limit cycle $\gamma(t)=(\cos(t),\sin(t))$,*which is stable*

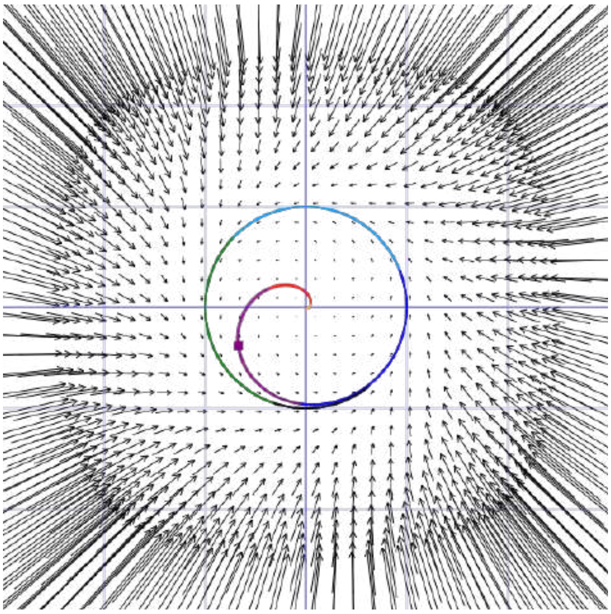


Figure 2.14: Hyperbolic limit cycle stable of system (2.25) .

CHAPTER 3

ON THE RATIONAL LIMIT CYCLES OF ABEL EQUATIONS

In this chapter, we will detail the work of Changjian Liu et al [21] . The Abel equations, which are $\frac{dt}{dy} = A(t)y^2 + B(t)y^3$, are the subject of this work. $A(t)$ and $B(t)$ are real polynomials. They call a solution to the preceding equations, $y = \varphi(t)$, periodic if it fulfills the formula $\varphi(0) = \varphi(1)$. A periodic solution is referred to as a limit cycle if it is isolated. They refer to a limit cycle $y = \varphi(t)$ as nontrivial rational if it is a rational function rather than a polynomial.

Firstly, They investigate the existence of nontrivial rational limit cycles. They demonstrate that there are Abel equations with at least two nontrivial rational limit cycles and others with at least one nontrivial rational limit cycle and one non-rational limit cycle. Secondly, we examine the relationship between the existence of nontrivial rational limit cycles and the degrees of $A(t)$ and $B(t)$. Lastly, They show that the multiplicity of a nontrivial rational limit cycle can be unbounded.

Definition : 34 *[23] we consider Abel equations,*

$$\frac{dy}{dt} = A(t)y^2 + B(t)y^3. \tag{3.1}$$

where t $[0,1]$ *and* y *is real variables and* $A(t)$ *and* $B(t)$ *are real polynomials.*

Definition : 35 *[23] We say that a limit cycle is a periodic solution isolated in the set of periodic solutions of a differential equation (3.1) is a solution* $y(t)$ *defined in* $[0,1]$ *such as* $y(0) = y(1)$.

Remark: 11 *We are interested in the rational limit cycles of equation (3.1) when the functions* $A(t)$ *and* $B(t)$ *are polynomials.*

Definition : 36 *[6] Let* $y = \psi(t, y_0)$ *be the solution of equation (3.1) such that* $\psi(0, y_0) = y_0$*. We say that the solution* $y = \psi(t, y_0)$ *of equation (3.1) is periodic if* $\psi(1, y_0) = y_0$*. If* $\psi(t, y_0)$ *is well defined on* $[0,1]$*, then we can define a map*$\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $\Phi(y_0) = \psi(1, y_0) - y_0$*. Clearly,* $\Phi(y_0) = 0$ *if and only if system (3.1) has a periodic solution starting at* y_0.

Theorem: 17 *[21] For equation (3.1), the following statements hold:*

(a) Any polynomial limit cycle of equation (3.1) has the form $y = c$*. Furthermore, if there exists a polynomial limit cycle* $y = c$*,* $c \neq 0$*, then except for the polynomial solution* $y = 0$*, there is no other polynomial solution, thus no other polynomial limit cycle.*

(b) For any given integer k *with* $k \geq 2$*, there exist polynomials* $A(t)$ *and* $B(t)$ *such that* $y = 0$ *is a polynomial limit cycle of multiplicity* k *of equation (3.1).*

Proof: 1 *It is worth pointing out that if a polynomial limit cycle* $y = c, c \neq 0$*, exists, then it is hyperbolic. The proof will be found in the next section. Recall the a function* $y = \phi(t)$

is called algebraic if and only if there exists a polynomial $f(t,y)$ such that $f(t,\phi(t)) = 0$. Is a limit cycle $y = \phi(t,y_0)$ of system (1.1)is an algebraic limit cycle if $\phi(t,y_0)$ is algebraic.

To solve the problem of algebraic limit cycles, it is natural for us to start from the simple case; rational case, in other words, the limit cycle $y = \phi(t,y_0)$ can be written as $y = \frac{q(t)}{p(t)}$, where p(t), q(t) are polynomials and $(p(t),q(t)) = 1$. Since we are only interested in rational limit cycles, which are not polynomial limit cycles, is a polynomial limit cycles trivial rational limit cycles and other rational limit cycles nontrivial rational limit cycles.

The following lemma gives the sufficient and necessary conditions that a rational function can be a periodic solution of equation (3.1).

Lemma: 1 *[21] The nonzero rational function $y = \frac{q(t)}{p(t)}$ is a periodic solution of equation (3.1) if and only if all the following three conditions hold:*

(a) $q(t) = c_1$, *where* c_1 *is a nonzero constant.*

(b) $c_1 B(t) + \frac{p(t)p'(t)}{c_1 + p(t)A(t)} = 0$.

(c) $p(0) = p(1)$ *and* $p(t)$ *has no zeros in* $[0,1]$.

Proof: 2 *If $y = \frac{q(t)}{p(t)}$ is a periodic solution of equation (3.1), then obviously $p(t)$ has no zero in $[0,1]$. Denote by $F(t,y) = p(t)y - q(t)$, then along the curve $\{F(t,y) = 0\}$, the derivative of $F(t,y)$ in t is zero, in other words,*

$$\frac{dF}{dt} = p'(t)y - q'(t) + p(t)(A(t)y^2 + B(t)y^3) = 0.$$

Notice that $F(x,y)$ is irreducible, so there exists a polynomial $k(t,y)$ so that

$$p'(t)y - q'(t) + p(t)(A(t)y^2 + B(t)y^3) = k(t,y)F(t,y). \tag{3.2}$$

The left-hand side of formula (3.2) is a polynomial of degree 3 in y and $F(t,y)$ is a polynomial of degree 1 in y, so we can suppose that $k(t,y) = k_0(t) + k_1(t)y + k_2(t)y^2$, where k_0, k_1, and k_2 are polynomials in t.

$$p'(t)y - q'(t) + p(t)(A(t)y^2 + B(t)y^3) = K(t,y)F(t,y)$$

Let $K(t,y) = K_0(t) + K_1(t)y + K_2(t)y^2$ where $K_0(t), K_1(t)$ and $K_2(t)$ are polynomials in t .

$$\begin{aligned} p'(t)y - q'(t) + p(t)(A(t)y^2 + B(t)y^3) &= [K_0(t) + K_1(t)y + K_2(t)y^2]\,[p(t)y - q(t)] \\ &= -K_0q(t) + [K_0(t)p(t) - K_1(t)p(t) - K_1(t)q(t)]\,y \\ &+ [K_1(t)p(t) - K_2(t)q(t)]\,y^2 + K_2(t)p(t)y^3. \end{aligned}$$

By comparing the coefficients of y^j, $j = 0,1,2,3$, in formula (3.2), we obtain

(i) $-q'(t) = -k_0(t)q(t)$.

(ii) $p'(t) = k_0(t)p(t) - k_1(t)q(t)$.

(iii) $p(t)A(t) = k_1(t)p(t) - k_2(t)q(t)$.

(iv) $p(t)B(t) = k_2(t)p(t)$.

From (i), $q(t)$ is a factor of $q'(t)$, that is, $q(t)/q'(t)$.This fact implies that $q(t)$ is a constant and we denote it by $q(t) = c_1 \in \mathbb{R}$. If $c_1 = 0$, then $y = \frac{q(t)}{p(t)} = 0$. This is impossible, so we have that $c_1 \neq 0$, and condition (a) holds. Furthermore, $y = \frac{q(t)}{p(t)} = \frac{c_1}{p(t)}$ is a periodic solution, thus $p(0) = p(1)$, and condition (c) holds.

From (ii), $k_1(t) = -\frac{p'(t)}{c_1}$; from (iv), $k_2(t) = B(t)$. Substituting them into (iii), we *have*

$$c_1 B(t) + p(t)\left(\frac{p'(t)}{c_1} + p(t)A(t)\right) = 0.$$

Condition (b) holds.

On the contrary, if all three conditions (a), (b), (c) hold, then one can easily check that the rational function $y = \frac{c_1}{p(t)}$ *is a periodic solution of equation (3.1).*

Since $q(t) = c_1$ *is a nonzero constant, without loss of generality, in the rest of this paper, we will suppose that* $q(t) = 1$. *To decide if a periodic solution of equation (3.1) is a hyperbolic limit cycle,*

Lemma: 2 *[21] Consider the differential equation* $\frac{dy}{dt} = F(t, y)$*, where* $F(t, y)$ *is a function of class* C^2 *in* $\mathbb{R}^2$. *Assume that* $y = \varphi(t)$ *is a periodic orbit of this equation, then it is a hyperbolic limit cycle if and only if* $D_1(1) \neq 0$*, where*

$$D_1(t) = \int_0^t \frac{\partial F}{\partial y}(t, \varphi(t))\, dt.$$

We firstly use Lemma 3 to deal with polynomial limit cycles:

Corollaire: 4 *[21] If* $y = c$, $c \neq 0$*, is a limit cycle of equation (3.1), then it must be a hyperbolic limit cycle.*

Proof: 3 *Simce* $y = c$ *is solution of equation (3.1) by lemme 2 we have that*

$$A(x) = \frac{-q(t)B(t)}{p(t)} - \frac{p(t)p'(t)}{q(t)p(t)}.$$

and as $y = \frac{q(t)}{p(t)} = c \Longrightarrow q(t) = cp(t)$*, so*

$$A(t) = \frac{-cp(t)B(t)}{p(t)} - 0$$

$$= -cB(t)$$

and system (3.1) becomes

$$\frac{dy}{dt} = -cB(t)y^2 + B(y)y^3 \quad = B(t)y^2(y-c).$$

If $\int_0^1 B(t)dt \neq 0$,

$$\begin{aligned} D_1 &= \int_0^1 \frac{\partial B(t)y^2(y-c)}{\partial y}(t,c)dt, \\ &= \int_0^1 3y^2B(t) - 2ycB(t)dt, \\ &= \int_0^1 3c^2B(t) - 2c^2B(t), \\ &= c^2\int_0^1 B(t)dt \neq 0. \end{aligned}$$

By lemma 2 the priodic solution y=c is a hyperbolic limit cycle.

Corollaire: 5 *[21] . If* $y = \frac{1}{p(t)}$ *is a periodic solution of equation (3.1), then it is a hyperbolic limit cycle if and only if* $\int_0^1 \frac{B(t)}{P^2(t)}dt \neq 0$

Proof: 4

$$\begin{aligned} D_1 &= \int_0^1 \frac{\partial A(t)y^2 + B(t)y^3}{\partial y}(t + \frac{1}{p(t)})dt, \\ &= \int_0^1 2A(t)y + 3B(t)y^2(t + \frac{1}{p(t)})dt, \\ &= \int_0^1 2A(t)\frac{1}{p(t)} + 3B(t)\frac{1}{p^2(t)}dt, \\ &= \int_0^1 \frac{2A(t)p(t) + 3B(t)}{p^2(t)}dt. \end{aligned}$$

By lemme 2, $p(0) = p(1)$ *and* $p(t)A(t) = p(t)p'(t) - B(t)$,

$$\begin{aligned} D_1 &= \int_0^1 \frac{B(t)}{p^2(t)} - 2\frac{p'(t)}{p(t)}dt, \\ &= \int_0^1 \frac{B(t)}{p^2(t)}dt - 2\left[\ln p(t)\right]_0^1, \\ &= \int_0^1 \frac{B(t)}{p^2(t)}dt. \end{aligned}$$

Which finishes the proof .

Theorem: 18 *[26] Let $v_1, v_2, \ldots, v_n$ be elements of a vector space E endowed with an inner product $\langle \cdot, \cdot \rangle$. Then*

$$G(v_1, v_2, \ldots, v_n) := \begin{vmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \langle v_n, v_n \rangle \end{vmatrix} \geq 0,$$

and it is zero if and only if the vectors $v_1, v_2, \ldots, v_n$ are linearly dependent. The determinant $G(v_1, v_2, \ldots, v_n)$ is usually called the Gram determinant.

In fact, we will use the above result when E is the space of continuous functions on the interval $[0, 1]$ and the inner product is $\langle u, v \rangle = \int_0^1 u(t)v(t)\, dt$. In this context, it is also called the integral Gram determinant .

3.1 The existence and the number of nontrivial rational limit cycles

Firstly, we are interested in the existence of nontrivial rational limit cycles and the degrees of $A(t)$ and $B(t)$ in equation (3.1). Suppose that $\deg A(t) = m$ and $\deg B(t) = n$, then we have:

Theorem: 19 *[21] If (m, n) satisfies one of the following conditions:*

(a) $n \geq 2m + 3$ *and* n. *is odd;*

(b) $m + 2 \leq n \leq 2m + 1$.

(c) $n = m + 1 \geq 2$.

where $deg\,A(t) = m$ *and* $deg\,B(t) = n$*, then we can find suitable* $A(t)$ *and* $B(t)$ *so that equation (3.1) has a nontrivial rational limit cycle; else* (m, n) *does not satisfy any these conditions, then equation (3.1) has no nontrivial rational limit cycle.*

Proof: 5 *If equation (3.1) has a nontrivial limit cycle* $y = \frac{1}{p(t)}$ *(meaning* $P(t)$ *is not a constant polynomial) then by lemma 2* $P(t)$ *must satisfies the equation*

$$B(t) + P(t)(p'(t) + A(t)) = 0.$$

Here , A (t) and B(t) are polynomials where

$$deg(A(t)) = m \quad and \quad deg(B(t)) = n.$$

For $y = \frac{1}{p(t)}$ *to be a limit cycle,it is required that* $p(0) = p(1)$*,this implies that* $p(t)$ *must have a degree of least if* $deg(p(t)) = 0$*,then* $p(t)$ *would be a contradict the fact that* $y = \frac{1}{p(t)}$ *is a nontrivial limit cycle because a constant* $p(t)$ *would imply a trivial solution*

$B(t) = -p(t)(p'(t) + A(t))$

$$\begin{aligned} deg(B(t)) &= deg\ p(t) + deg\ p'(t) \quad (if\ deg\ p(t) > m+1), \\ &= deg\ p(t) + deg\ A(t) \quad (if\ deg\ p(t) > m+1). \end{aligned}$$

So we have

- *if* $deg\ p(t) = \ell > m + 1$ *then* $deg\ p'(t) > m = deg\ A(t)$

$$\begin{aligned} n = deg(B(t)) &= deg\ p(t) + deg\ p'(t) \\ &= 2\ell - 1 > 2m + 1 \geq 2m. \end{aligned}$$

  thus n *is an add integer and* $n \geq 2m + 3$ *the condition (a) holds;*

- *if deg* $p(t) = \ell = m+1$ *then* $m \geq 1$

$$\begin{aligned} m+1 = deg(p(t)) &\leq n = deg(B(t)), \\ &\leq deg\ p(t) + deg\ p'(t), \\ &\leq m+1+m, \\ &\leq m+2. \end{aligned}$$

so $m+1 \leq n \leq 2m+1$ *the condition (b) holds; or* $n = m+1 \geq 2$ *the conditio (c) holds;*

if $deg(p(t)) = \ell < m+1$ *then* $deg\ p'(t) < m = deg(A(t))$

$n = deg(B(t)) = deg\ p(t) + degA(t) = m + \ell$ *which implies that* $2+m \leq n \leq 2m$ *the condition (b) holds; In any case , on of the condition (a) (b) (c) holds so if (m,n) does not satisfies these conditions, equation (3.1) has no nontivial rationale limit cycles.*

On the contrary,

If the condition (a) holds : $n \geq 2m+3$ *, then we let.*

We Correct the degree of p(t)

$$\begin{aligned} p(t) &= t^{\frac{n-1}{2}}(1-t)+1. \\ A(t) &= t^m. \\ B(t) &= p(t)(A(t)-p'(t)). \end{aligned}$$

we need to check that $deg(A(t)) = m$ *and* $deg(B(t)) = n$

  - $A(t) = t^m$ *cleary, the degree of A(t) is m*
  - *Degree of B(t) is a degree of the product* $P(t)(A(t) - p'(t))$ *involve* $deg(p(t)) = \frac{n-1}{2} + 1 = \frac{n+1}{2}$

Degree $A(t) - p'(t)$ *having degree* m *and* $\frac{n+1}{2} - 1$

So, the highest degree term dominates is $\frac{n-1}{2}$, *thas*

$deg(B(t)) = \frac{n+1}{2} + \frac{n-1}{2} = n$

$y = \frac{1}{p(t)}$ *and* $p(t)$ *is not constant polynomial (its degree is at least* $\frac{n+1}{2}$ *),* $y = \frac{1}{p(t)}$ *represents a nontrivial rational limit cycle*

Additionally , because $p(t)$ *is a polynomial and not a constant , this limit cycle is hyperbolic*

If the condition (b) holds $m + 2 \leq n \leq 2m + 1$ *then we let*

$$\begin{aligned} p(t) &= t^{n-m-1}(1-t) + 1 \\ A(t) &= t^m \\ B(t) &= p(x)(A(t) - p'(t)) \end{aligned}$$

- *The degree of* $A(t)$ *is* m
- *The higest degree term in* $B(t)$ *will come from the highest degree Term in* $p(t) \times t^m$

$$deg(p(t)) = n - m$$

so : $deg(p(t) \times t^m) = n - m + m = n$ *there fore* $deg(B(t)) = n$

The rational function $y = \frac{1}{p(t)}$ *is a hyperbolic nontrivial rational limit cycle of the equation*

Application : [21]

There exist examples of equation (3.1) with at least two nontrivial rational limit cycles; there also exist examples of equation (3.1) with at least one nontrivial rational limit cycle coexisting with one nontrivial irrational limit cycle. Firstly we construct the following equations

$$\frac{dy}{dt} = A(t)y^2 + B(t)y^3,$$

where

$$\begin{aligned} A(t) &= (t-1)(8t^3 - 35t^2 + 24t - 3). \\ B(t) &= (2+t^2-t)(t^2-t+1)(t-4)(t+1)(t-3)(5-3t^2-12t). \end{aligned} \tag{3.3}$$

Then the equation (3.1) has two rational solutions. $\frac{1}{p_1(t)}$ and $\frac{1}{p_2(t)}$ which are both periodic, where

$$\begin{aligned} P_1(t) &= (t-3)(t+1)(t-4)(t^2-t+1), \\ P_2(t) &= (t-3)(t+1)(t-4)(t^2-t+2). \end{aligned} \tag{3.4}$$

we have:

$$\begin{aligned} \frac{B(t)}{p_1^2(t)} &= \frac{(2+t^2-t)(5-3t^2-12t)}{(t-3)(t+1)(t-4)(t^2-t+1)} \\ \int_0^1 \frac{B(t)}{p_1^2(t)} &= \int_0^1 \frac{(2+t^2-t)(5-3t^2-12t)}{(t-3)(t+1)(t-4)(t^2-t+1)} \\ &= \frac{1}{546}[-46\sqrt{3}\arctan\left[\frac{-1+2t}{\sqrt{3}}\right] + 588\ln(-4+t) \\ &+624\ln(-3+t) + 728\ln(1+t) - 151\ln(1-t+t^2)]|_0^1 \\ &= \frac{1}{819}\left(-23\sqrt{3}\pi + 264\ln 2 - 54\ln 3\right) \neq 0 \end{aligned}$$

$$\begin{aligned} \frac{B(t)}{p_2^2(t)} &= \frac{(1+t^2-t)(5+3t^2-12t)}{(t-3)(t+1)(t-4)(t^2-t+2)} \\ \int_0^1 \frac{B(t)}{p_2^2(t)} &= \int_0^1 \frac{(1+t^2-t)(5+3t^2-12t)}{(t-3)(t+1)(t-4)(t^2-t+2)} \\ &= \frac{1}{112}[6\sqrt{7}\arctan\left(\frac{-1+2t}{\sqrt{7}}\right) + 104\ln(-4+t) \\ &+98\ln(-3+t) + 84\ln(1+t) + 25\ln 2 - t - t^2]|_0^1 \\ &= \frac{1}{56}\left(6\sqrt{7}\arctan\left(\frac{1}{\sqrt{7}}\right) - 13\ln 2 + \ln 27\right) \neq 0 \end{aligned}$$

Thus they are hyperbolic limit cycles. The system that we construct has at least two nontrivial limit cycles.

Next we consider the perturbation of the above system

$$\tfrac{dy}{dt} = (A(t) + \varepsilon)\, y^2 + (B(t) - \varepsilon P_1(t))\, y^3. \tag{3.5}$$

Where A(t), B(t) and $P_1(t)$ are defined in 3.4 and 3.3 is a small parament if $y = \frac{1}{p_1(t)}$ So

$$\tfrac{dy}{dt} = (A(t) + \varepsilon)\, \tfrac{1}{p_1^2(t)} + (B(t) - \varepsilon P_1(+))\, \tfrac{1}{p_1^3(t)}.$$

$y = \frac{1}{p_1(t)}$ remains a hyperbolique nontrivial rational limit cycle.

Additionally, equation (3.5) has a hyperbolic limit cycle ant tends to $y = \frac{1}{p_2(t)}$ when $\varepsilon \longrightarrow 0$, since the aforementioned equation has another hyperbolique limit cycle Γ_ε when tends to $y = \frac{1}{p_2(t)}$, we argue that if is rational, then $p(t) = P_2(t) + \varepsilon p_3(t) + O(\varepsilon)$ will exist as a polynomial, where $P_3(0) = P_3(1)$ and $y = \frac{1}{p(t)}$ is a limit cycle of equation. (3.5) Through Lemma 2

$$(p'(t) + A(t))p_3(t) + p_2(t)p_3'(t) + p_2(t) = 0$$

$$B(t) - \varepsilon P_1(t) + (p_2'(t) + \varepsilon P_3'(t))(P_2(t) + \varepsilon P_3(t)) = -(A(t) + \varepsilon)(P_2(t) + \varepsilon P_3(t))$$

$$B(t) - \varepsilon P_1(t) + P_2'(t)P_2(t) + \varepsilon P_3'(t)P_2(t) + P_2'(t)\varepsilon P_3(t) + \varepsilon^2 P_3'(t)p_3(t)$$

$$= -A(t)P_2(t) - A(t)\varepsilon P_3(t) + P(1) + \varepsilon^2 P_3'(t).$$

comparing the coefficients of ε of the above equation, we have

$$\begin{aligned} \varepsilon(-P_1(t) + P_3'(t)p_2(t) + P_2'(t)P_3(t)) &= -\varepsilon(A(t)P_3(t))(P_2'(t) + A(t))p_3(t) \\ +P_2(t)p_3'(t) - P_1(t) + p_2(t) & \\ &= 0 \end{aligned}$$

So the polynomial. $P_3(t)$ satisfies the following equation:

$$(t-3)(t+1)(t-4)(2+t^2-t)p_3'(t)-(t^2-t+1)(5+3t^2-12t)p_3(t)+(t-3)(t+1)(t-4)=0.$$

Given that $P_3(t)$ is a fraction of $(t-3)(t+1)(t-4)$, $degP_3(x)\geq 3$. Assume that, for any $n\geq 3$, $P_3(t)=a_0+a_1t+...+a_nt^n$. where $a_n\neq 0$ Then equation (3.1) becomes

$$(n-3)a_nt^{4+n}+\ell.O.t=0.$$

where $\ell.O.t$ stands for the degree - less elements of equation (3.1) that imply $n=3$. For the equation to hold, the coefficient of t^n+4 must be zero:

$$(n-3)a_n=0\Rightarrow n-3=0.\Rightarrow n=3$$

There exists a constant c so that $P_3(t)=c(t-3)(t+1)(t-4)$ Substituting it to equation (3.1), we obtain $3ct^2-12ct+5c+1=0$

$$(2+t^2-t)p_3'(t)-\tfrac{(t^2-t+1)(5+3t^2-12t)}{(t-3)(t+1)(t-4)}P_3(t)+1=0,$$
$$(2+t^2-t)(c(-4+t)(-3+t)+c(-4+t)(1+t)+c(-3+t)(1+t)),$$
$$-(t^2-t+1)(5+3t^2-12t)c+1=0,$$
$$1+c(5-12t+3t^2)=0.$$

Thus $3c=12c=5c+1$, which is impossible.

3.2 The multiplicity of nontrivial rational limit cycles

In this section, we will show that, unlike the polynomial limit cycle $y=c$, $c\neq 0$, the multiplicity of the nontrivial rational limit cycle is unbounded.

Theorem: 20 *[21] For any integer number k with $k \geq 2$, there exist polynomials $A(t)$ and $B(t)$ such that equation (3.1) has a nontrivial rational limit cycle, whose multiplicity is k.*

Proof: 6 *[2] Firstly, we consider the following Abel differential equation*

$$\frac{dy}{dt} = -p'(t)y^2, \tag{3.6}$$

where $p(t) = t^2 - v + 1$, which satisfies that $p(0) = p(1) = 1$ and $p(v) > 0$ in $[0, 1]$.

It is the solutions of equation (3.6) have the form

$$y = \frac{1}{p(t) + c},$$

thus any solution of equation (3.6), which is well-defined in $[0, 1]$, is periodic, so equation (3.6) has no limit cycle. Now we consider a perturbation of equation (3.6)

$$\frac{dy}{dt} = -p'(t)y^2 + \epsilon A(t, \epsilon)y^2(1 - p(t)y), \tag{3.7}$$

where ϵ is a small parameter, $A = A(t, \epsilon)$ is a polynomial in t and only analytic in ϵ.

Notice that $y = \frac{1}{p(t)}$ remains a periodic solution, but in general, the other solutions are not periodic solutions anymore, thus $y = \frac{1}{p(t)}$ will be a limit cycle.

We need to show that, for any integer number k with $k \geq 2$, there exists a polynomial A in t such that $y = \frac{1}{p(t)}$ is a rational limit cycle of multiplicity k of the Abel differential Eq. (3.7). Denote by $y = \psi(t, y_0, \epsilon)$, the solution of Eq. (3.7), with initial condition $\psi(0, y_0, \epsilon) = y_0$. Since we need to consider the multiplicity of $y = \frac{1}{p(t)}$, we expand the *function* $\Delta(y_0, \epsilon) = \psi(1, y_0, \epsilon) - y_0$ *as a power series of* $z = y_0 - 1$*:*

$$\Psi(y_0, \epsilon) = \sum_{i=1}^{\infty} \varphi_i(\epsilon) z^i,$$

where $\varphi_i(\epsilon)$ *is analytic in* ϵ. *To get the desired multiplicity* k*, one needs to find suitable* A *so that* $\varphi_1(\epsilon) = \varphi_2(\epsilon) = \cdots = \varphi_{k-1}(\epsilon) = 0$*, but* $\varphi_k(\epsilon) \neq 0$. *The main difficulty of this problem comes from not knowing* $\varphi_i(\epsilon)$ *explicitly. For any integer* k *with* $k \geq 2$*, we choose* A *having the following form:*

$$A(t, \epsilon) = p^{2k+1}(t) \frac{1}{\epsilon} \sum_{i=0}^{k-1} \tilde{b}_i(\epsilon) T^i(t),$$

where $T(x) = \frac{1-p(t)}{p(t)}$ *and the functions* $\tilde{b}_i(\epsilon)$ *are analytic in a neighborhood of* $\epsilon = 0$*, to be chosen in order to get the desired multiplicity. Obviously,* A *is a polynomial in* x *of degree* $2k+1$. *In order to simplify our notations, we will denote by* $b_i = \tilde{b}_i(0)$.

We expand the function $\psi(t, y_0, \epsilon)$ *in a Taylor series in a neighborhood of* $\epsilon = 0$*:*

$$\psi(t, y_0, \epsilon) = \psi_0(t, y_0) + \psi_1(t, y_0)\epsilon + o(\epsilon), \tag{3.8}$$

then $\psi_0(0, y_0) = y_0$ *and* $\psi_1(0, y_0) = 0$. *Expanding the equality*

We Correct the functions:

$$\frac{\partial \psi(t, y_0, \epsilon)}{\partial t} = -p(t)\psi(t, y_0, \epsilon)^2 + \epsilon \left(A(t)\psi(t, y_0, \epsilon)^2 - p(t)A(t)\psi(t, y_0, \epsilon)^3 \right)$$

as a power series of ϵ *and equating the coefficients of the powers* ϵ^0 *and* ϵ^1*, we deduce that*

$$\psi_0(t, y_0) = \frac{y_0}{1 + (p(t) - 1)y_0}.$$

and that

$$\psi_1(t, y_0) = \psi_0(t, y_0)^2 \int_0^t A(s, 0) \left(1 - p(s)\psi_0(s, y_0)\right) \, ds.$$

We evaluate $\psi_1(t, y_0)$ *at* $t = 1$ *and recall that* $\psi_0(1, y_0) = y_0$*, then we have*

$$\psi_1(1, y_0) = (1 - y_0)y_0^2 \int_0^1 A(s, 0) \frac{1}{1 - y_0 + p(s)y_0} \, ds.$$

Assume that $y_0 - 1 = z$ *and use that* $T(s) = \frac{1-p(s)}{p(s)}$*, then we can write*

$$\Psi_1(y_0) = \psi_1(1, y_0) - \psi_1(0, y_0) = -z(z^2 + 2z + 1)\kappa(z),$$

where $\kappa(z) = \int_0^1 \frac{A(s,0)}{p(s)(1-T(s)z)} \, ds$*. For any integer* i *with* $i > 0$*, we denote by* $I_i = \int_0^1 p^{2k}(s)T_i(s) \, ds$*. Since*

$$A(s, 0) = p^{2k+1}(s)(b_0 + b_1T(s) + b_2T^2(s) + \cdots + b_{k-1}T^{k-1}(s)),$$

we can write

$$\kappa(z) = \int_0^1 p^{2k}(s) \left(\sum_{j=0}^{k-1} b_jT^j(s)\right) \left(\sum_{i=0}^{k-1} T^i(s)z^i + o(z^{k-1})\right) ds$$

$$= \sum_{i=0}^{k-1} \left(\int_0^1 \sum_{j=0}^{k-1} b_jp^{2k}(s)T^{i+j}(s) \, ds\right) z_i + o(z^{k-1})$$

$$= \sum_{i=0}^{k-1} \left(\int_0^1 \sum_{j=0}^{k-1} b_jI_{i+j} \, ds\right) z^i + o(z^{k-1}).$$

We consider the system of k *linear equations with the* $k \times k$ *matrix* $M = (k_{ij}) = (I_{i+j})$*, for* $0 \leq i, j \leq k - 1$*, the vector of* k *unknowns* $(b_0, b_1, b_2, \ldots, b_{k-1})$ *and the independent vector* $(0, 0, \ldots, 0, 1)$*. If we show that the determinant of* M *is different from zero, we have that this system always has a unique solution and, thus, we can always find values of*

$\mathbf{b}^* = (b_0, b_1, b_2, \ldots, b_{k-1})$ *for which* $\kappa(z) = z^{k-1} + o(z^{k-1})$*. We consider the vector space* S *of linear combinations of* $p^k(s)T^i(s)$*, with* $i = 0, 1, \ldots, k-1$*, and where we recall that* $T(s) = \frac{1-p(s)}{p(s)}$*. The space*

$$S = \sum_{i=0}^{m-1} c_i p^k(s)T^i(s) : c_i \in \mathbb{R}$$

is a real vector space of dimension k*. For given* $f(x)$*,* $g(t)$ *in* S*, we can define the inner product of* $f(t)$ *and* $g(t)$ *by*

$$\langle f(t), g(t) \rangle = \int_0^1 f(t)g(t)\, dt.$$

It is clear that $p^k(s)T^i(s)$*, with* $i = 0, 1, \ldots, k-1$*, forms a basis of* S*, and that* M *is the symmetric matrix of the metrics corresponding to this inner product. Therefore, by Theorem 20,*

det(M*)* $\neq 0$*. Thus, we can find values of* $\mathbf{b}^* = (b_0, b_1, b_2, \ldots, b_{k-1})$ *so that*

$$\frac{\Psi_1(y_0 - \epsilon)}{1 + 2z + z^2} = -\kappa(z) = -z^k + o(z^k). \tag{3.9}$$

Let us reconsider the function $\Psi(y_0, \epsilon)$*. Since the multiplicity of* $z = 0$ *of* $\Psi(y_0, \epsilon)$ *is the same as that of* $\frac{\Psi_1(y_0-\epsilon)}{1+2z+z^2}$*, we can use* $\frac{\Psi_1(y_0-\epsilon)}{1+2z+z^2}$ *instead of* $\Psi(y_0, \epsilon)$*. Suppose that*

$$\frac{\Psi_1(y_0 - \epsilon)}{1 + 2z + z^2} = \sum_{i=1}^{\infty} \phi_i(\epsilon, \mathbf{b}) z^i, \tag{3.10}$$

where $\mathbf{b}$ *is the vector* $(b_0, b_1, b_2, \ldots, b_{k-1})$ *and* $\phi_i(\epsilon, \mathbf{b})$ *is analytic in* ϵ *and* $\mathbf{b}$*. By (3.9),we have*

$$\Psi(y_0, \epsilon)\left(1 + 2z + z^2\right) = \Psi_1(y_0)\left(1 + 2z + z^2\right)\epsilon + o(\epsilon) \tag{3.11}$$

$$= -z \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} b_j I_{i+j} z^i + o(z^{k-1})\epsilon + o(\epsilon).$$

From (3.10) and (3.11), we know that for i satisfying $1 \leq i \leq k$,

$$\phi_i(\epsilon, \mathbf{b}) = -\epsilon \sum_{j=0}^{k-1} b_j I_{i-1+j} + o(\epsilon).$$

For i satisfying $1 \leq i \leq k$, we define the function $\tilde{\phi}_i(\epsilon, \mathbf{b})$ so that

$$\phi_i(\epsilon, \mathbf{b}) = -\epsilon \tilde{\phi}_i(\epsilon, \mathbf{b}),$$

then $\tilde{\phi}_i(\epsilon, \mathbf{b}) = \sum_{j=0}^{k-1} b_j I_{i-1+j} + o(1)$.

At last, we consider the function $\mathbf{v}(\epsilon, \mathbf{b}) : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$, defined by

$$\mathbf{v}(\epsilon, \mathbf{b}) = (\tilde{\phi}_1(\epsilon, \mathbf{b}), \tilde{\phi}_2(\epsilon, \mathbf{b}), \ldots, \tilde{\phi}_k(\epsilon, \mathbf{b})).$$

We have known we can find $\mathbf{b}^*$ so that $\mathbf{v}(0, \mathbf{b}^*) = (0, 0, \ldots, 0, 1)$. On the other hand, the Jacobian matrix $\mathbf{D_b}\Delta(\epsilon, \mathbf{b})$ corresponding to the derivation only with respect to the variables $\mathbf{b}$ satisfies that $\mathbf{D_b}\Delta(\epsilon, \mathbf{b}) = M$. Thus, the determinant of M is nonzero. By the Implicit Function Theorem, there exist k functions $\tilde{b}_0(\epsilon), \tilde{b}_1(\epsilon), \ldots, \tilde{b}_{k-1}(\epsilon)$ analytically near $\epsilon = 0$, such that $\tilde{b}_i(0) = b_i$ for $i = 0, 1, \ldots, k-1$ and

$$\mathbf{v}(\epsilon, \tilde{b}_0(\epsilon), \tilde{b}_1(\epsilon), \ldots, \tilde{b}_{k-1}(\epsilon)) \equiv (0, 0, \ldots, 0, 1).$$

Therefore, if we let $b_i = \tilde{b}_i(\epsilon)$ for $i = 0, 1, \ldots, k-1$, then by (3.10), we have

$$\Delta(y_0, \epsilon)\left(1 + 2z + z^2\right) = -\epsilon z^k + o(z^k),$$

which has a zero $z = 0$ with the multiplicity of k.

CONCLUSION

This work has undertaken a comprehensive qualitative study of polynomial differential systems, with a specific focus on their stability, limit cycles, and applications in Abel equations. Through the detailed analysis, several significant findings have been made. This work has significantly advanced the understanding of polynomial differential systems, particularly in the context of Abel equations. The findings not only enrich the theoretical framework but also have practical implications that extend across multiple scientific disciplines. The work sets a foundation for future research, promising further advancements in the analysis and application of differential systems.

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قسم الرياضيات و الاعلام الالي

جامعة غرداية

Université de GHARDAIA

Faculté des Sciences et de la Technologie
Département des Mathématiques et de l'Informatique

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الاسم : شلاوة

اللقب : حورية

الرتبة : أستاذ محاضر ب

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اسم و امضاء الاستاذ المشرف(ة)

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