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Upper Bounds for the Number of Limit Cycles for a Class of Polynomial Differential Systems

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Dedication

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To the one who carries paradise beneath her feet, whose prayers have eased my hardships, who has always dreamed of this day...
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To the light that illuminated my path, the lamp whose flame never fades in my heart
to the one who gave me all that is precious and invaluable...
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To those who strengthened my back , springs from which I have drawn my courage and resolve. To the apple of my eye...
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Abstract

One of the main open problems in the theory of ordinary differential equations is the study of the existence and number of limit cycles, due to their fundamental role in understanding the periodic behavior of dynamical systems. A limit cycle is an isolated periodic orbit of the system and plays a central role in the qualitative analysis of differential equations. This study falls within the framework of Hilbert's 16^{th} problem, specifically its second part, which concerns the existence of a uniform upper bound on the number of limit cycles in polynomial differential systems of a given degree. In this thesis of master, we conducted a comprehensive review of the concept of limit cycles, focusing on their identification within a specific class of polynomial differential systems arising from polynomial perturbations added to the linear center $\dot{x} = y, \quad \dot{y} = -x$, These perturbations, which involve small parameters ε, generate nonlinear dynamics and give rise to new limit cycles. We employed first- and second-order averaging theory to determine accurate upper bounds on the number of limit cycles bifurcating from the periodic orbits of the unperturbed system. This work is based on a detailed study of the scientific article authored by Jaume Llibre and Clàudia Valls, entitled *"On the number of limit cycles of a class of polynomial differential systems"* [1], in which we reanalyzed their theoretical results and applied them to an original example. Our main contribution lies in providing an original applied example that explicitly satisfies the conditions of the second-order averaging theory, illustrating the practical challenges involved in applying the theory and complementing the theoretical results established in the referenced article.

Keywords: Polynomial differential systems, integrability, phase portrait, limit cycles, Hilbert's 16^{th} problem, averaging theory.

ملخص

تُعد دراسة وجود وعدد الدورات الحدية من بين أبرز المشكلات المفتوحة في نظرية المعادلات التفاضلية العادية، نظرًا لدورها الأساسي في فهم السلوك الدوري للأنظمة الديناميكية. تمثل الدورة الحدية مدارًا دوريًا معزولًا للنظام، وتؤدي دورًا محوريًا في التحليل النوعي للمعادلات التفاضلية. تندرج هذه الدراسة ضمن إطار المسألة السادسة عشرة لهيلبرت، وبوجه خاص جزئها الثاني، الذي يهتم بمسألة وجود حدّ علوي موحّد لعدد الدورات الحدية في الأنظمة التفاضلية متعددة الحدود من درجة معينة. في هذه المذكرة، قمنا بمراجعة شاملة لمفهوم الدورات الحدية، مع التركيز على تحديدها ضمن فئة معينة من الأنظمة التفاضلية متعددة الحدود الناتجة عن اضطرابات متعددة الحدود مضافة إلى المركز الخطي $\dot{x} = y, \quad \dot{y} = -x$. تُحدث هذه الاضطرابات، التي تتضمن معاملات صغيرة ε، ديناميكيات غير خطية وتؤدي إلى ظهور دورات حدية جديدة. اعتمدنا على نظرية التوسيط من المرتبتين الأولى والثانية من أجل تحديد حدود عليا دقيقة لعدد الدورات الحدية المتفرعة من المدارات الدورية للنظام غير المضطرب. يعتمد هذا العمل على دراسة مفصلة للمقال العلمي الذي ألّفه جاومي ليبري وكلوديا فالس بعنوان "حول عدد الدورات الحدية في فئة من الأنظمة التفاضلية متعددة الحدود"، حيث أعدنا تحليل نتائجهما النظرية وطبقناها على مثال أصلي. وتكمن مساهمتنا الرئيسية في تقديم مثال تطبيقي أصلي يستوفي بوضوح شروط نظرية التوسيط من المرتبة الثانية، مما يوضح التحديات العملية المرتبطة بتطبيق النظرية، ويكمّل النتائج النظرية الواردة في المقال المشار إليه.

الكلمات المفتاحية:الأنظمة التفاضلية متعددة الحدود، قابلية التكامل، الصورة الطورية، الدورات الحدية، المسألة السادسة عشرة لهيلبرت، نظرية التوسيط.

Résumé

L'un des principaux problèmes ouverts dans la théorie des équations différentielles ordinaires est l'étude de l'existence et du nombre de cycles limites, en raison de leur rôle fondamental dans la compréhension du comportement périodique des systèmes dynamiques. Un cycle limite est une orbite périodique isolée du système et joue un rôle central dans l'analyse qualitative des équations différentielles. Cette étude s'inscrit dans le cadre du 16^{e} problème de Hilbert, plus précisément sa deuxième partie, qui concerne l'existence d'une borne supérieure uniforme pour le nombre de cycles limites dans les systèmes différentiels polynomiaux d'un certain degré. Dans ce mémoire, nous avons mené une revue complète du concept de cycle limite, en nous concentrant sur leur identification dans une classe spécifique de systèmes différentiels polynomiaux issus de perturbations polynomiales ajoutées au centre linéaire $\dot{x} = y, \quad \dot{y} = -x$, Ces perturbations, qui impliquent de petits paramètres ε, engendrent une dynamique non linéaire et donnent naissance à de nouveaux cycles limites. Nous avons utilisé la théorie de l'approximation (ou moyenne) d'ordre un et deux afin de déterminer des bornes supérieures précises pour le nombre de cycles limites bifurquant à partir des orbites périodiques du système non perturbé. Ce travail repose sur une étude détaillée de l'article scientifique rédigé par Jaume Llibre et Clàudia Valls, intitulé *« On the number of limit cycles of a class of polynomial differential systems »* [1], dans lequel nous avons réanalysé leurs résultats théoriques et les avons appliqués à un exemple original. Notre contribution principale réside dans la présentation d'un exemple appliqué original satisfaisant explicitement les conditions de la théorie de l'approximation du second ordre, illustrant les défis pratiques liés à l'application de cette théorie et complétant les résultats théoriques établis dans l'article de référence.

Mots-clés : Systèmes différentiels polynomiaux, intégrabilité, portrait de phase, cycles limites, 16^{e} problème de Hilbert, théorie de l'approximation.

Contents

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Introduction

Dynamical systems form a fundamental framework for modeling a wide array of phenomena encountered in disciplines such as biology, physics, engineering, and economics. They serve as essential mathematical tools for describing the temporal evolution of states within various processes. Since the early days of differential equations, these systems have attracted extensive interest from mathematicians aiming to understand their qualitative and quantitative behaviors (see [3–6]).

Many realistic models are nonlinear by nature, which makes finding explicit analytic solutions extremely difficult or often impossible. Therefore, numerical methods have become indispensable for approximating solutions. However, these approaches usually provide insight only over finite time intervals and may fail to reveal the global dynamics of the system.

At the end of the nineteenth century, the French mathematician Henri Poincaré profoundly transformed the field through his pioneering work *"Mémoire sur les courbes définies par une équation différentielle"* (see [7]). He introduced a qualitative approach to differential equations based on geometric and topological concepts, enabling the study of solutions' behavior without relying on explicit formulas. Fundamental notions such as phase portraits and return maps, introduced by Poincaré, laid the foundation of the modern qualitative theory of dynamical systems.

A central challenge in this theory is the investigation of integrability and the existence of periodic solutions, especially limit cycles, which are isolated closed trajectories. The concept of limit cycles first appeared in Poincaré's seminal works (see [8]), and since then, numerous models in physics, engineering, chemistry, biology, and economics have been formulated as planar autonomous polynomial systems exhibiting such cycles (see [9–14]).

In 1900, David Hilbert presented a list of 23 unsolved problems at the International Congress of Mathematicians (see [15]). Among them, the second part of the sixteenth problem asks about the maximum number $H(n)$ of limit cycles and their possible configurations in planar polynomial differential systems of degree n, described by

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases}$$

where P and Q are real polynomials in two variables. This problem remains open and stands as one of the most profound challenges in the theory of dynamical systems.

Given the complexity and nonlinear nature of many real-world systems, especially those lacking explicit solutions, modern research increasingly relies on analytical approximation techniques. Among these, the **averaging theory** plays a pivotal role in studying nonlinear dynamical systems subjected to small perturbations. Averaging theory facilitates the detection and approximation of periodic solutions, particularly limit cycles, by analyzing averaged functions over time (see [1], [16–20]).

This thesis of master is organized into three main chapters, each building upon the previous to provide a comprehensive study of limit cycles in planar differential systems:

- **Chapter 1** presents the fundamental theoretical background necessary for the study of polynomial differential systems. It introduces key concepts such as vector fields, critical point analysis, and integrability via invariant curves. Several original examples are included to illustrate the theoretical notions and their application in analyzing the system's behavior.

- **Chapter 2** focuses on limit cycles, beginning with their definition and basic properties. It discusses the Poincaré–Bendixson theorem and analytical techniques used to study the existence and number of limit cycles. This chapter also lays the groundwork for understanding the first- and second-order averaging theory by presenting and discussing original examples illustrating how limit cycles appear in differential systems.

- **Chapter 3** is devoted to a detailed re-study and analysis of a research article focusing on a specific type of planar polynomial differential systems [1]. This chapter centers on systems exhibiting a center at the origin and explores the application of first- and second-order averaging theories to these systems. Two families of perturbed systems will be studied

$$\begin{cases} \dot{x} = y - \varepsilon\left(g_{11}(x) + f_{11}(x)y\right), \\ \dot{y} = -x - \varepsilon\left(g_{21}(x) + f_{21}(x)y\right). \end{cases}$$

  and

$$\begin{cases} \dot{x} = y - \varepsilon\left(g_{11}(x) + f_{11}(x)y\right) - \varepsilon^2\left(g_{12}(x) + f_{12}(x)y\right), \\ \dot{y} = -x - \varepsilon\left(g_{21}(x) + f_{21}(x)y\right) - \varepsilon^2\left(g_{22}(x) + f_{22}(x)y\right). \end{cases}$$

  where $\varepsilon > 0$ is a small parameter and the functions f_{ij}, g_{ij} are polynomials.

  The chapter explains the theoretical conditions required for applying the averaging method, outlines the computational approach, and discusses the main results regarding the number and distribution of limit cycles for these systems. Original examples constructed for this work illustrate the theoretical concepts and demonstrate the practical application of averaging theory in analyzing limit cycles.

1 Preliminary concepts

1.1 Introduction

This chapter is devoted to introducing some preliminary concepts essential for the qualitative study of planar differential systems. We begin by discussing the general form of polynomial differential equations in the plane, and then we define fundamental notions such as vector fields, solutions (including periodic ones), and phase portraits, which visually represent the behavior of the system. We also give attention to equilibrium points. At the end of the chapter, we discuss the concept of invariant curves and present some approaches to studying the integrability of differential systems, such as first integrals and integrating factors. These concepts will be frequently used and further developed in the following chapters.

1.2 Planar polynomial differential systems

Definition 1.1. [21] A planar polynomial differential system is defined by two differential equations of the form

$$\begin{cases} \dot{x} = P(x(t), y(t)), \\ \dot{y} = Q(x(t), y(t)). \end{cases} \tag{1.1}$$

where $P(x(t), y(t))$ and $Q(x(t), y(t))$ are polynomial functions of the variables x and y. The system (1.1) is of degree n where $n = max(deg(P), deg(Q))$.
As usual the dot denotes derivative with respect to the independent variable t.

Definition 1.2. [22] A differential system is given by

$$\frac{dx}{dt} = f(t, x),$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$. if the function f does not depend explicitly on the time variable t (that is $f(t, x) = f(x)$), the system is called *autonomous* and can be written as

$$\dot{x} = f(x).$$

otherwise, if f depends on both t and x, the system is referred to as *non-autonomous.*

Definition 1.3. [2] A polynomial differential system in the plane is called *homogeneous* *of degree n* if it can be written in the form

$$\begin{cases} \dot{x} = P(x(t), y(t)) = \displaystyle\sum_{i+j=0}^{i+j=n} \alpha_{ij}\, x^i y^{n-j}, \\ \\ \dot{y} = Q(x(t), y(t)) = \displaystyle\sum_{i+j=0}^{i+j=n} \beta_{ij}\, x^i y^{n-j}. \end{cases}$$

1.2.1 Vector field

Drawing the vector field before beginning a deep analysis of a differential system is quite practical and can give us important information about the many types of potential solutions. It is the vector that corresponds to each point in the space shown graphically. This vector will really be tangent to the differential system's trajectory as it passes through that location. As a result, we may get a reasonably accurate sense of the potential solutions and their asymptotic behavior from the vector field.

Definition 1.4. [23] A vector field $\mathcal{X}$ is a region of the plane in which there exists at every point $A \in \Delta \subset \mathbb{R}^2$ a vector $\overrightarrow{V}(A, t)$, i.e an application:

$$\begin{aligned} \mathcal{X} : \Delta \subset \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \\ A(x, y) &\mapsto \overrightarrow{V}(A) = \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}. \end{aligned}$$

where P and Q are of class $\mathcal{C}^1$ on $\Delta \subset \mathbb{R}^2$. The vector field associated with system (1.1) can be represented by the following differential operator

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

Then for the following we consider the vector field $\mathcal{X}$ associated with the planar polynomial differential system (1.1)

$$\frac{\overrightarrow{dA}}{dt} = \overrightarrow{V}(A) \quad \Longleftrightarrow \quad \begin{cases} \dfrac{dx}{dt} = P(x(t), y(t)), \\ \dfrac{dy}{dt} = Q(x(t), y(t)). \end{cases}$$

Remark 1.1.

1. In this mémoire, we assume that the functions P and Q are of class C^1. This assumption ensures that the Cauchy–Lipschitz conditions are satisfied for the system (1.1), so that for every initial condition (x_0, y_0) , there exists a unique solution.
2. The plane formed by the variables x and y is called the phase plane.
3. On the curve $P(x, y) = 0$, known as the vertical isocline, the vector field is parallel to the y-axis; whereas on the curve $Q(x, y) = 0$, called the horizontal isocline, the vector field is parallel to the x-axis.

Example 1.1. Consider the system

$$\begin{cases} \dot{x} = \frac{1}{5}y^2 + x - 1, \\ \dot{y} = x^3 - y. \end{cases} \tag{1.2}$$

The vector field associated with system (1.2) is shown in Figure (1.1).

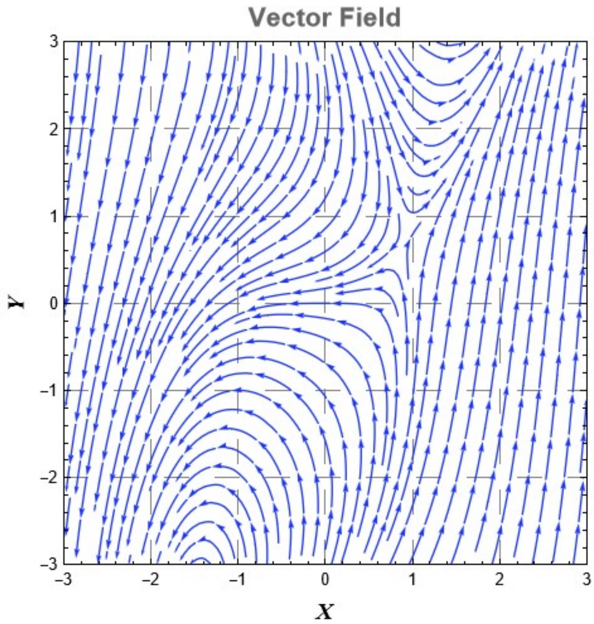

Figure 1.1: Vector Field for the system (1.2).

1.2.2 Solution and periodic solution

This section provides clear definitions of the basic terms solution and periodic solution in differential systems : solution refers to any function that, for specified initial conditions, satisfies the system's equations, whereas periodic solution refers to a function that repeats its values after a predetermined amount of time.

Definition 1.5. [2] A mapping $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}^2$ given by $\varphi(t) = (x(t), y(t))$ is called a *solution* of system (1.1) if

$$\dot{\varphi}(t) = \mathcal{X}(\varphi(t)), \quad \forall t \in I,$$

where $\mathcal{X} = (P, Q)$ is the associated vector field. if $\varphi_1(t) = (x_1(t), y_1(t))$ and $\varphi_2(t) = (x_2(t), y_2(t))$ are two solutions on I_1 and I_2 respectively, we say that $\varphi_2(t)$ is an extension of $\varphi_1(t)$ if $I_1 \subset I_2$ and $\varphi_1(t) = \varphi_2(t)$ for all $t \in I_1$. A solution is called maximal if it has no further extension.

Definition 1.6. [2] A solution $\varphi(t) = (x(t), y(t))$ of system (1.1) is called a *periodic solution* if there exists a real number $T > 0$ such that

$$\varphi(t + T) = \varphi(t), \quad \forall t \in \mathbb{R}.$$

The smallest such T is called the period of the solution.

1.3 Phase portrait

For example, the solutions of a vector field $\mathcal{X}$ are represented as trajectories or orbits, showing how the system changes over time; the phase portrait, which is the collection of these trajectories, offers important information about the qualitative behavior of the system, revealing important features like equilibrium points, stability properties, etc. The $\mathbb{R}^2$ plane is also referred to as the phase plane, where the behavior of dynamical systems is visually represented.

Definition 1.7. [23] Let $p \in \Delta$ be a point in the domain of the vector field $\mathcal{X} : \Delta \to \mathbb{R}^2$. The orbit of $\mathcal{X}$ through p, denoted by γ_p, is defined as the image of the maximal solution $\varphi_p : I_p \to \Delta$ that passes through p. In other words,

$$\gamma_p = \{\varphi_p(t) \mid t \in I_p\}.$$

Definition 1.8. [23] The phase portrait of a vector field $\mathcal{X}$ is the complete set of orbits that represent the solutions of the system in the (x, y)-plane. It provides a global view of the system's dynamics by displaying all trajectories (orbits) and equilibrium points.

Example 1.2. The phase portrait associated with system (1.2) is shown in Figure (1.2)

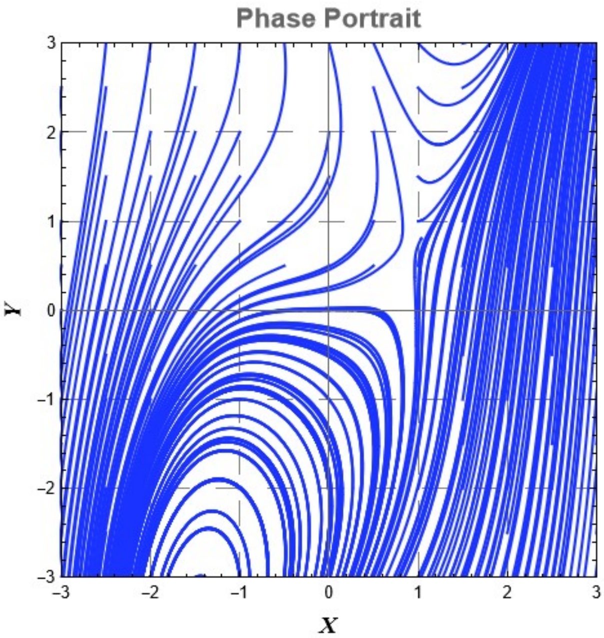

Figure 1.2: Phase Portrait for the system (1.2).

1.4 Equilibrium points

When analyzing dynamical systems, equilibrium points are essential. When describing a dynamical system with multiple variables, Henri Poincaré (1854−1912) demonstrated that it is sufficient to characterize the system without computing exact solutions. Determining the equilibrium points and evaluating their stability significantly simplifies the study of nonlinear systems near these points.

Definition 1.9. [2] A point (x_0, y_0) is called an *equilibrium point* (or a *singular point*) of the system (1.1) if

$$\begin{cases} P(x_0, y_0) = 0, \\ Q(x_0, y_0) = 0. \end{cases}$$

Remark 1.2. [23] In the context of planar systems, the terms *equilibrium point* and *singular point* are often used interchangeably. However, the term *singular point* emphasizes the local behavior of the vector field, whereas the term *equilibrium point* highlights the system's stationary solutions or trajectories.

Proposition 1.1. *[23] Every periodic solution inherently contains at least one equilibrium point.*

1.4.1 The Jacobian Matrix and Linearization

In order to analyze the behavior of trajectories near equilibrium points, it is common practice to consider the linearization of system (1.1) and then relate the trajectories of the nonlinear system to those of its linear counterpart.

Definition 1.10. [2] Let $J(x_0, y_0)$ be the Jacobian matrix of the vector field near an equilibrium point (x_0, y_0), which is defined as

$$J(x_0, y_0) = \begin{pmatrix} \frac{\partial P}{\partial x}(x_0, y_0) & \frac{\partial P}{\partial y}(x_0, y_0) \\ \frac{\partial Q}{\partial x}(x_0, y_0) & \frac{\partial Q}{\partial y}(x_0, y_0) \end{pmatrix}.$$

Then, the linearized form of system (1.1) near the equilibrium point (x_0, y_0) is given in matrix form by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.3}$$

Definition 1.11. [2] A equilibrium point (x_0, y_0) is defined as hyperbolic if the Jacobian matrix $J(x_0, y_0)$ has eigenvalues with non-zero real parts. Conversely, if at least one eigenvalue has a zero real part, the point is classified as non-hyperbolic.

Example 1.3. Consider the following nonlinear system:

$$\begin{cases} \dot{x} = x^2 + y, \\ \dot{y} = xy + 1. \end{cases} \tag{1.4}$$

To find the equilibrium points, we solve the system

$$\begin{cases} x^2 + y = 0 \\ xy + 1 = 0 \end{cases} \Rightarrow \begin{cases} y = -x^2, \\ -x^3 + 1 = 0. \end{cases}$$

Hence, the system has a unique equilibrium point at $(x_0, y_0) = (1, -1)$.
We now compute the Jacobian matrix

$$J(x, y) = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 1 \\ y & x \end{bmatrix}.$$

Evaluating the Jacobian at the equilibrium point $(x_0, y_0) = (1, -1)$, we obtain

$$J(1, -1) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Therefore, the linearization of system (1.4) at the point $(1, -1)$ is

$$\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = -x + y. \end{cases}$$

1.4.2 Topological Equivalence

Definition 1.12. [24] A function $h : \mathbb{R}^2 \to \mathbb{R}^2$ is called a *homeomorphism* if it is a continuous bijection with a continuous inverse.

Definition 1.13. [25] Two autonomous systems in the plane

$$\text{(S1):} \quad \begin{cases} \dot{x} = P_1(x(t), y(t)), \\ \dot{y} = Q_1(x(t), y(t)), \end{cases} \qquad \text{(S2):} \quad \begin{cases} \dot{x} = P_2(x(t), y(t)), \\ \dot{y} = Q_2(x(t), y(t)). \end{cases}$$

defined on two open sets V and W respectively, are said to be *topologically equivalent* if there exists a homeomorphism $h : V \to W$ such that h maps the orbits of (S1) onto the orbits of (S2) and preserves the direction of motion.

Remark 1.3. [26] Topological equivalence via a homeomorphism allows for a classification primarily based on the stability or instability of the equilibrium. Two linear systems are topologically equivalent if they have the same number of eigenvalues, with real parts of the same signs.

Remark 1.4. Consider the differential system (1.1), and let $J(x_0, y_0)$ be the Jacobian matrix associated with this system at the equilibrium point (x_0, y_0). Let λ_1 and λ_2 be the eigenvalues of this matrix.

1. A equilibrium point is said to be *elementary* if at least one eigenvalue of $J(x_0, y_0)$ is nonzero. If both eigenvalues vanish ($\lambda_1 = \lambda_2 = 0$), the point is called *non-elementary*. In this case:
   - The equilibrium point is referred to as *degenerate* if the linear part is identically zero ($J(x_0, y_0) = 0$).
   - If the linear part is nonzero, the equilibrium point is called *nilpotent* (see [23], Theorem 3.5).
2. A equilibrium point is said to be *semi-hyperbolic* if exactly one of its eigenvalues is zero while the other is nonzero. The phase portraits of such points are well known (see [23], Theorem 2.19).
3. A equilibrium point (x_0, y_0) is called a *center* if there exists a neighborhood V around it such that for every point $p \in V$ (with $P^2(p) + Q^2(p) \neq 0$), the orbits passing through p are closed and surround (x_0, y_0), indicating closed orbits and periodic dynamics.

1.4.3 Hartman-Grobman Theorem

This theorem states that a dynamical system (1.1) near a hyperbolic equilibrium point can be reduced to the study of a topologically equivalent linear system (1.3) near the origin. This theorem is a powerful tool in the analysis of dynamical systems, as it allows for the simplification of complex dynamics by examining a simpler linear model. It is particularly useful for understanding the local behavior of dynamical systems defined on an open subset of the plane.

Theorem 1.1. *[26] Suppose that the Jacobian matrix at the equilibrium point* (x_0, y_0) *has two eigenvalues such that* $\mathrm{Re}(\lambda_1) \neq 0$ *and* $\mathrm{Re}(\lambda_2) \neq 0$*. Then, the solutions of the nonlinear system* (1.1) *can be approximated by the solutions of the linearized system* (1.3) *in a neighborhood of the equilibrium point.*

In other words, the phase portrait of the linearized system (1.3) *provides a good approximation of that of the nonlinear system* (1.1) *near this equilibrium point through a continuous transformation.*

Remark 1.5. [2] In the case where $\mathrm{Re}(\lambda_1) = 0$ and $\mathrm{Re}(\lambda_2) = 0$, the linearization method does not provide sufficient information about the behavior of the nonlinear system. Specifically, if the equilibrium point (x_0, y_0) is a center for the linearized system (1.3), determining whether it remains a center or becomes a focus in the nonlinear system (1.1) requires further investigation. This is known as the center problem.

1.4.4 Stability of the equilibrium

There may be more than one equilibrium point in a nonlinear system, and these points may be unstable or stable. Ensuring the stability of an equilibrium point is crucial in various situations. The following is a definition of stability

Let (x_0, y_0) be an equilibrium point of system (1.1).
We denote $X(t) = (P(x, y), Q(x, y))$ end $X_0 = (P(x_0, y_0), Q(x_0, y_0))$

Definition 1.14. [26] We say that

1. (x_0, y_0) is *stable* if and only if

$$\forall \varepsilon > 0, \exists \eta > 0 \text{ such that } \|(x, y) - (x_0, y_0)\| < \eta \Rightarrow \forall t > 0, \|X(t) - X_0\| < \varepsilon.$$

2. (x_0, y_0) is *asymptotically stable* if and only if it is stable and

$$\lim_{t \to \infty} \|X(t) - X_0\| = 0.$$

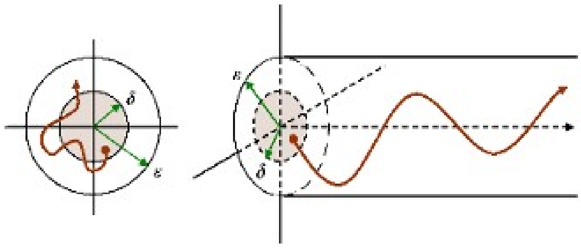

Figure 1.3: [27] Stability of an equilibrium point .

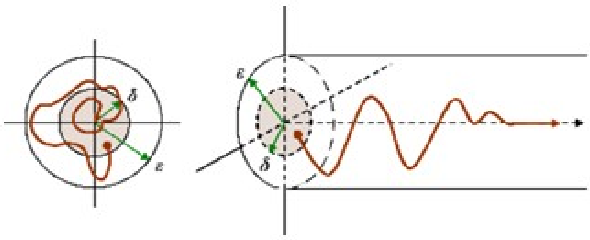

Figure 1.4: [27] Asymptotic stability of an equilibrium point .

1.4.5 Classification of equilibrium points

Definition 1.15. [2] Consider the differential system (1.1) and let $J(x_0, y_0)$ be the Jacobian matrix associated with it at the equilibrium point (x_0, y_0). Let λ_1 and λ_2 be the eigenvalues of this matrix. The classification of equilibrium points is based on the following cases:

- Node : If λ_1 and λ_2 are real and have the same sign
  - If $\lambda_1 \leq \lambda_2 < 0$, the origin is a stable node.
  - If $\lambda_1 \geq \lambda_2 > 0$, the origin is an unstable node.
- Saddle :

  If λ_1 and λ_2 are real, nonzero, and of opposite signs, the origin is a saddle. A saddle is always unstable.
- Focus : If λ_1 and λ_2 are complex conjugates with $\text{Re}(\lambda_{1,2}) \neq 0$
  - If $\text{Re}(\lambda_{1,2}) < 0$, the origin is a stable focus.
  - If $\text{Re}(\lambda_{1,2}) > 0$, the origin is an unstable focus.
- Center : If λ_1 and λ_2 are purely imaginary, the origin is a center. A center is stable but not asymptotically stable.

Example 1.4. [2] Consider the following nonlinear differential system

$$\begin{cases} \dot{x} = x + 2y + x^2 - y^2, \\ \dot{y} = 3x + 4y - 2xy. \end{cases} \tag{1.5}$$

We define the Jacobian matrix of the system (1.5) as

$$J(x, y) = \begin{pmatrix} 1 + 2x & 2 - 2y \\ 3 - 2y & 4 - 2x \end{pmatrix}.$$

At the equilibrium point $(x_0, y_0) = (0, 0)$, the Jacobian takes the form

$$J(0, 0) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

By solving $\det(J(0, 0) - \lambda I) = 0$, we obtain the characteristic equation

$$\lambda^2 - 5\lambda - 2 = 0.$$

Thus, the eigenvalues are

$$\lambda_1 = \frac{5 + \sqrt{33}}{2}, \quad \lambda_2 = \frac{5 - \sqrt{33}}{2}.$$

Since the eigenvalues are real and of opposite signs, the origin is classified as a *saddle point*, which is always unstable.

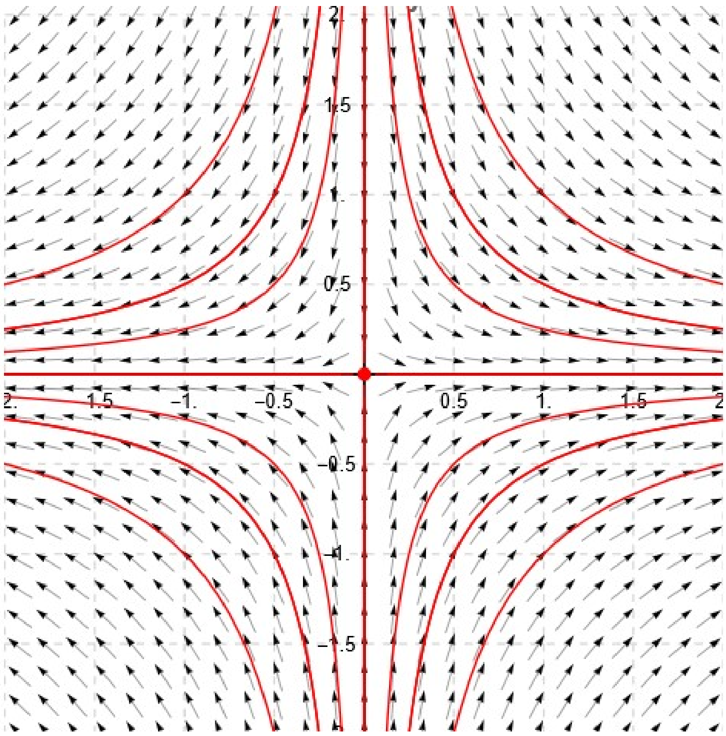

Figure 1.5: Phase portrait of the system (1.5).

Example 1.5. [2] Consider the following nonlinear differential system

$$\begin{cases} \dot{x} = -y + xy^2, \\ \dot{y} = x + y^3. \end{cases} \tag{1.6}$$

To find the equilibrium points, we solve

$$\begin{cases} -y + xy^2 = 0 \\ x + y^3 = 0 \end{cases} \Rightarrow \begin{cases} -y - y^5 = 0 \\ x = -y^3 \end{cases}$$

Thus, the only equilibrium point is $(x_0, y_0) = (0, 0)$.

The Jacobian matrix associated with the system (1.6) is

$$J(x, y) = \begin{pmatrix} y^2 & -1 + 2xy \\ 1 & 3y^2 \end{pmatrix}.$$

Evaluating at the equilibrium point $(0, 0)$, we obtain

$$J(0, 0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

To classify the equilibrium point, we compute the eigenvalues of the Jacobian by solving the characteristic equation

$$\det(J(0, 0) - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i.$$

Since the eigenvalues are purely imaginary and complex conjugates, the origin is classified as a *center*. Therefore, the system exhibits closed orbits around the equilibrium point, and the origin is *stable but not asymptotically stable*.

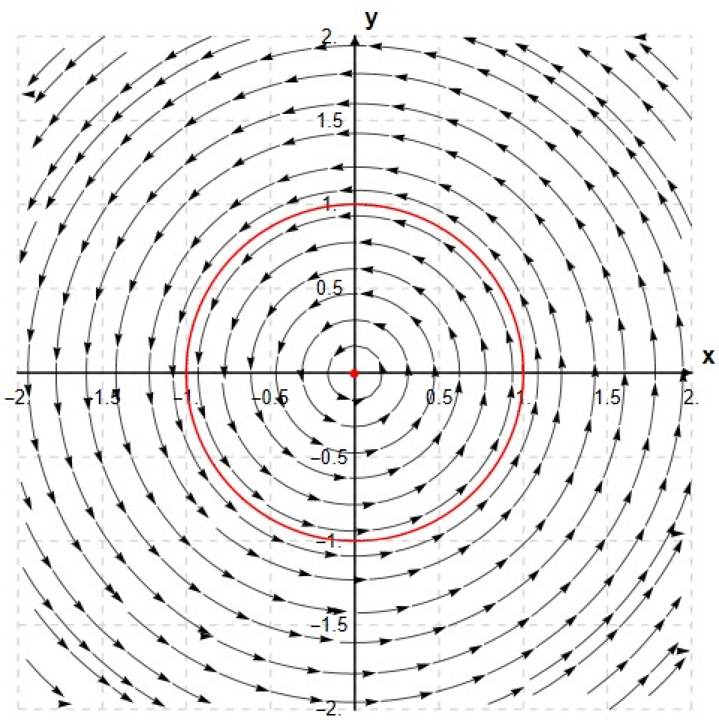

Figure 1.6: Phase portrait of a system (1.6).

Example 1.6. [2] Consider the following nonlinear differential system

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = x + y. \end{cases} \tag{1.7}$$

To find the equilibrium points, we solve

$$\begin{cases} x - y = 0, \\ x + y = 0. \end{cases}$$

Thus, the only equilibrium point is $(x_0, y_0) = (0, 0)$.

The Jacobian matrix associated with the system (1.7) is

$$J(x, y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Since the Jacobian is constant, it remains the same at the equilibrium point $(0, 0)$.

To classify the equilibrium point, we compute the eigenvalues of the Jacobian by solving the characteristic equation

$$\det(J(0,0) - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm i.$$

Since the eigenvalues have a nonzero real part ($\lambda = 1 \pm i$), the origin is classified as an *unstable focus*. Therefore, the system exhibits spiral trajectories that move away from the equilibrium point, and the origin is *unstable*.

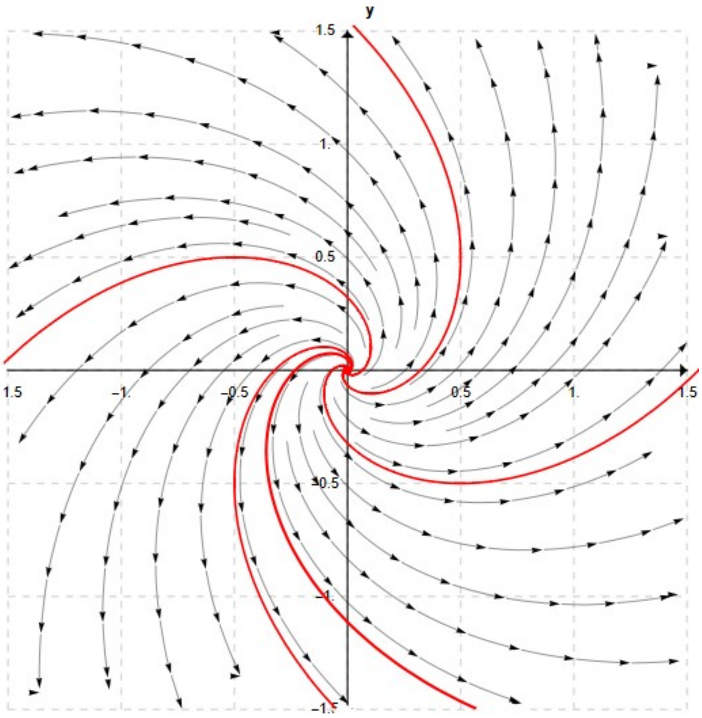

Figure 1.7: Phase portrait of a system (1.7).

Example 1.7. Consider the following nonlinear differential system

$$\begin{cases} \dot{x} = x + xy, \\ \dot{y} = 2y + y^2. \end{cases} \tag{1.8}$$

To find the equilibrium points, we solve

$$x(1+y) = 0 \quad \text{and} \quad y(2+y) = 0.$$

which yields the unique solution $(x_0, y_0) = (0, 0)$.

The Jacobian matrix associated with the system (1.8) is

$$J(x,y) = \begin{pmatrix} 1+y & x \\ 0 & 2+2y \end{pmatrix}.$$

Evaluating at the equilibrium point $(0, 0)$, we have

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = 1, \quad \lambda_2 = 2.$$

Since the eigenvalues are real and positive, the origin is classified as an *unstable node*.

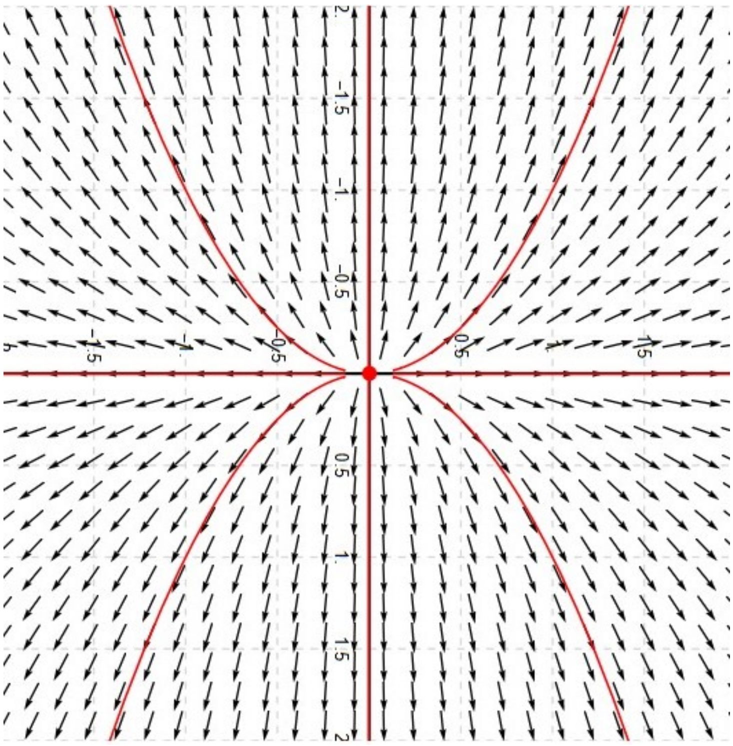

Figure 1.8: Phase portrait of a system (1.8).

1.5 Invariant curves

Invariant algebraic curves are a fundamental tool in studying the integrability of planar polynomial differential systems, as they are used to identify the existence of periodic solutions and limit cycles.

Definition 1.16. [23] We call an *invariant curve* of the system (1.1) any curve defined by the equation $U(x, y) = 0$ in the phase plane for which there exists a function $K = K(x, y)$, called the cofactor of the invariant curve $U = 0$, such that:

$$P(x, y)\frac{\partial U(x, y)}{\partial x} + Q(x, y)\frac{\partial U(x, y)}{\partial y} = K(x, y)U(x, y). \tag{1.9}$$

Equality (1.9) shows that on the invariant curve, the gradient $\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right)$ of U is orthogonal to the vector field $\mathcal{X} = (P, Q)$. This means that at every point on the invariant curve, the vector field is tangent to the curve, and consequently, the curve is formed by the solutions (or trajectories) of the vector field $\mathcal{X}$.

Example 1.8. Consider the nonlinear system

$$\begin{cases} \dot{x} = x(\pi - y) \\ \dot{y} = y(x - \pi) \end{cases}$$

Assume that $U(x, y) = xy$ is an invariant curve for the system.

$$\begin{aligned} \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} &= y \cdot x(\pi - y) + x \cdot y(x - \pi) \\ &= xy[(\pi - y) + (x - \pi)] \\ &= xy(x - y) \\ &= U(x, y)K(x, y). \end{aligned}$$

Thus, the curve $xy = 0$ is an invariant curve of the system, and its associated cofactor is given by $K(x, y) = x - y$.

Definition 1.17. [28] An invariant curve $U(x, y) = 0$ is called algebraic of degree m if $U(x, y)$ is a polynomial of degree m. Otherwise, it is called non-algebraic.

Definition 1.18. [28] An algebraic curve $U(x, y) = 0$ is said to be irreducible if $U(x, y)$ is a polynomial that cannot be factored into polynomials of lower degrees in the ring $\mathbb{R}[x, y]$.

Remark 1.6. [23] When the cofactor $k(x, y)$ is a polynomial, the invariant curve defined by $U(x, y) = 0$ is said to have a polynomial cofactor. This allows us to apply algebraic techniques specific to polynomials in its analysis.

Theorem 1.2. *[29] Consider the system* (1.1) *and let* $\Gamma(t)$ *be a periodic orbit with period* $T > 0$. *Suppose that* $U : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *is an invariant curve such that:*

$$\Gamma(t) = \{(x, y) \in \Delta \mid U(x, y) = 0\},$$

and let $K(x, y) \in C^1$ *be the cofactor associated with the invariant curve* $U(x, y) = 0$ *as given in equation* (1.9) . *If there exists a point* $p \in \Delta$ *such that* $U(p) = 0$ *and* $\nabla U(p) \neq 0$, *then the following holds:*

$$\int_0^T div(\Gamma(t))dt = \int_0^T K(\Gamma(t))dt.$$

Remark 1.7. The condition $\nabla U(p) \neq 0$ ensures that the invariant curve $U(x, y) = 0$ does not contain singular points, meaning that the periodic orbit does not pass through critical points of the system.

1.6 Integrability of polynomial differential systems

In the qualitative analysis of polynomial differential systems, the concept of *integrability* plays a fundamental role. A polynomial differential system is said to be *integrable* if it admits a *first integral*, as defined below. However, determining a first integral for a given differential system is often a challenging task. The significance of the existence of a first integral lies in the fact that it completely characterizes the phase portrait of the system, providing a comprehensive understanding of its global dynamics.

1.6.1 First integral

Definition 1.19. [23] Let $H : \Delta \to \mathbb{R}$ be a C^1 function that is not locally constant. We say that H is a *first integral* of the differential system (1.1) in Δ if it remains constant along every trajectory of the system that is contained in Δ. In other words, H is a first integral if

$$\frac{dH(x,y)}{dt} = P(x,y)\frac{\partial H(x,y)}{\partial x} + Q(x,y)\frac{\partial H(x,y)}{\partial y} \equiv 0.$$

The general solution of this equation is given by $H(x, y) = k$, where k is an arbitrary constant. Therefore, the system (1.1) is said to be *integrable* in Δ if it possesses a first integral H in Δ.

1.6.2 Darboux integrability

Definition 1.20. [30] A Darboux function is a function of the form

$$f(x,y) = f_1(x,y)^{\lambda_1} f_2(x,y)^{\lambda_2} \dots f_p(x,y)^{\lambda_p} \exp\left(\frac{g(x,y)}{h(x,y)}\right),$$

where $f_i(x, y)$ for $i = 1, \dots, p$, $g(x, y)$, and $h(x, y)$ are polynomials in $\mathbb{C}[x, y]$ and the λ_i for $i = 1, \dots, p$ are complex numbers.

Definition 1.21. System (1.1) is called Darboux integrable if it has a first integral which is a Darboux function.

Definition 1.22. [31] A function that can be represented by quadratures of elementary functions is known as a Liouvillian function.

Determining whether a given class of functions has an integrating factor or an inverse integrating factor is another aspect of studying the integrability problem.

1.6.3 Integrating factors

Definition 1.23. [2] On the open subset $\Delta \subseteq \mathbb{R}^2$, the function $R(x, y)$ is an integrating factor of differential system (1.1).

$$\operatorname{div}(RP, RQ) = 0 \quad \text{or} \quad P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -R\operatorname{div}(P,Q)$$

if $R \in C^1(U)$, $R \neq 0$ on U and

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}.$$

As is customary,

$$\operatorname{div}(X) = \operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

defines the divergence of the vector field X.

It is evident that the function H that satisfies

$$\frac{\partial H}{\partial x} = RQ, \quad \frac{\partial H}{\partial y} = -RP,$$

is a first integral, then the first integral H associated to the integrating factor R is given by

$$H(x, y) = -\int R(x, y)P(x, y)\, dy + h(x).$$

$$H(x, y) = \int R(x, y)Q(x, y)\, dx + h(y).$$

Inverse integrating factor

Definition 1.24. [13] If a nonzero function $V : \Delta \to \mathbb{R}$ of class $C^1(\Delta)$ satisfies the following linear partial differential equation and is not locally null:

$$Q\frac{\partial V}{\partial y} + P\frac{\partial V}{\partial x} = V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right),$$

then V is called an *inverse integrating factor* of system (1.1).

It is simple to confirm that an integrating factor in $\Delta \setminus \{V = 0\}$ of the system is defined by the function:

$$R = \frac{1}{V}.$$

One of the instruments used to investigate whether limit cycles exist or not is the inverse integrating factor. An important relationship between a limit cycle and an inverse integrating factor is provided by the following theorem.

Theorem 1.3. *[13] Let $V : \Delta \to \mathbb{R}$ be one of the inverse integrating factors of system (1.1). If Γ is a limit cycle of (1.1), then*

$$\Gamma \subset \{(x, y) \in \Delta : V(x, y) = 0\}.$$

2 Limit cycles of differential systems

2.1 Introduction

Limit cycles are dynamic phenomena that appear in nonlinear planar differential systems, representing regularly repeating patterns within the system's phase space. The concept was first extensively studied by Henri Poincaré in the late 19th century, notably in his seminal memoir *"On curves defined by a differential equation"* see[7]. These phenomena play a central role in explaining recurring behaviors observed in natural and engineering contexts, such as self-sustained oscillations in chemical reactions, biological rhythms, and electrical circuits. This chapter aims to review the fundamental mathematical principles regarding the existence of limit cycles, their stability properties, and essential characteristics, supported by practical examples illustrating their application in various systems.

2.2 Limit cycles

Definition 2.1. [28] A limit cycle is an isolated periodic solution of a planar differential system (1.1). It represents a closed trajectory in the phase space such that neighboring trajectories either approach or diverge from it, depending on its stability

Remark 2.1. The limit cycle is stable if every neighboring trajectory approaches it; if not, it is unstable.

Definition 2.2. [28] A periodic solution of system (1.1) is called an algebraic limit cycle if it is a limit cycle and contained within an irreducible algebraic invariant curve $U(x, y) = 0$ of system (1.1); otherwise, it is referred to as a non-algebraic limit cycle.

Example 2.1. In [8] Chapter VII of his foundational work, Henri Poincaré presented the first known example of a limit cycle. The studied system is a planar polynomial differential system of degree three:

$$\begin{cases} \dot{x} = x(x^2 + y^2 - 1) - y(x^2 + y^2 + 1), \\ \dot{y} = y(x^2 + y^2 - 1) + x(x^2 + y^2 + 1). \end{cases} \tag{2.1}$$

This system has a unique singular point at the origin, which is a focus. There are no singular points on the circle $x^2 + y^2 = 1$, which acts as a characteristic trajectory and therefore constitutes an isolated limit cycle. Hence, the unit circle:

$$x^2 + y^2 = 1$$

is the only limit cycle in the system, as observed by Poincaré.

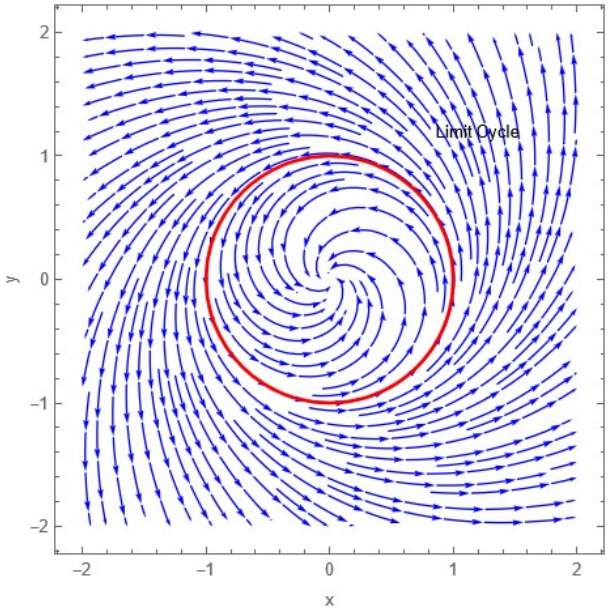

Figure 2.1: The Algebraic Limit Cycle in Poincaré's Example for System (2.1).

2.3 Existence and non existence of limit cycles in the plane

We provide some conclusions in this section that allow us to demonstrate the existence or non-existence of limit cycles for a polynomial differential system (1.1).

Remark 2.2. Limit cycles appear only in nonlinear differential systems.

Theorem 2.1. *[32] Let (P, Q) be a C^1 vector field defined in the open subset $\Delta \subset \mathbb{R}^2$, $(u(t), v(t))$ a periodic solution of period T of the system (1.1), and $R : \Delta \to \mathbb{R}$ a C^1 map such that*

$$\int_0^T R(u(t), v(t))\, dt \neq 0,$$

and let $U = U(x, y)$ be a C^1 solution of the linear partial differential equation (1.9). Then the closed trajectory

$$\gamma = \{(u(t), v(t)) \in \Delta : t \in [0, T]\}$$

is contained in the set

$$\Sigma = \{(x, y) \in \Delta : U(x, y) = 0\},$$

and γ is not contained in a periodic annulus of the vector field (P, Q). Moreover, if the vector field (P, Q) and the functions R and U is analytic, then γ is a limit cycle.

Theorem 2.2. *[33] Let C and C' be two simple closed curves in the plane, with C' entirely enclosed by C. If the vector field (P, Q) along every point of C points strictly outward and along every point of C' points strictly inward, then there is at least one limit cycle lying in the region between C' and C.*

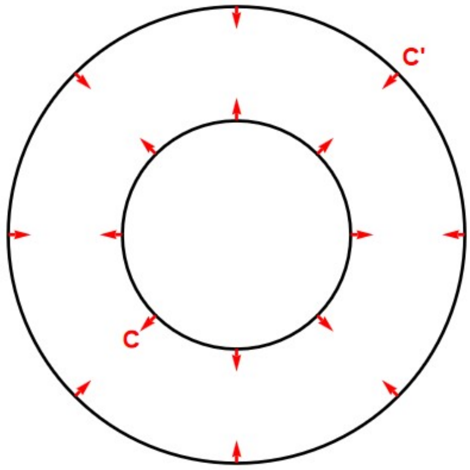

Figure 2.2: Existence of a limit cycle located between C and C'

Theorem 2.3. *[34] (**Bendixson Criterion**) Let Δ be a simply connected domain in $\mathbb{R}^2$. If the quantity $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is not identically zero and has a constant sign throughout Δ, then the vector field $\mathcal{X} = (P, Q)$ does not admit any limit cycle entirely contained in Δ.*

Theorem 2.4. *[35] If the system (1.1) has no singular point, then it has no limit cycles*

2.4 Stability of limit cycles

Let γ represent the trajectory associated with the limit cycle of the system (1.1). The neighboring trajectories, although not closed, should resemble γ. Depending on whether these nearby trajectories spiral towards or away from γ, the behavior of γ as a limit cycle can be categorized as stable, semi-stable, or unstable. This classification depends on whether the surrounding trajectories approach γ, diverge from it, or exhibit both behaviors.

Theorem 2.5. *[33] Consider a closed trajectory γ representing a limit cycle in a nonlinear dynamical system. The stability characteristics of γ can be described as follows:*

a. Stable (attractive): *If all trajectories in the vicinity of γ, both inside and outside, spiral towards γ as $t \to +\infty$.*

b. Unstable (repulsive): *If all neighboring trajectories spiral towards γ as $t \to -\infty$.*

c. Semi-Stable: *If trajectories inside γ approach it as $t \to +\infty$ while those outside approach it as $t \to -\infty$, or vice versa.*

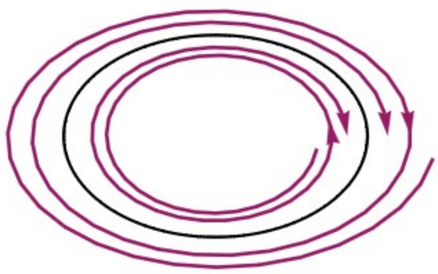

Figure 2.3: Stable limit cycle.

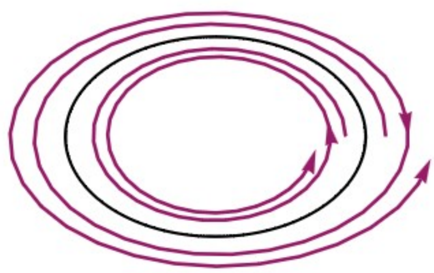

Figure 2.4: Unstable limit cycle.

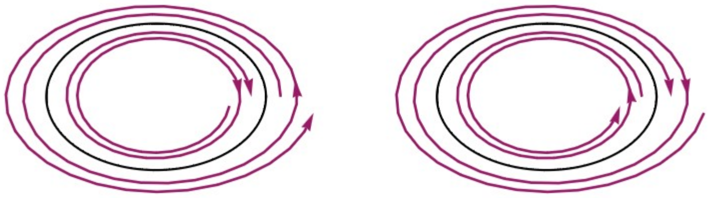

Figure 2.5: Semi-stable limit cycle.

2.5 Poincaré map

The Poincaré map,introduced by Henri Poincaré in 1881 [7], is one of the fundamental tools used to study the stability of periodic orbits in dynamical systems, particularly when analyzing the stability of isolated periodic trajectories. The basic idea behind this map is as follows: Suppose we have a periodic orbit of system (1.1) that passes through the point $X_0 = (x_0, y_0)$, and let Σ be a section (a local surface) transverse to the orbit at the point X_0. Then, the surface Σ intersects the orbit at that point.
Now, if we take any point $X = (x, y) \in \Sigma$ sufficiently close to X_0, the solution of system (1.1) starting from X at $t = 0$ will evolve and intersect the surface Σ again at a new point $\Pi(X)$ located near the original point X_0. This transformation $X \mapsto \Pi(X)$ (Figure 2.6) is known as the Poincaré first return map.
The following theorem establishes the existence and continuity of this map, and it also guarantees the continuity of its first derivative near X_0. In other words, the theorem ensures not only that the map is well-defined, but also that it behaves smoothly in a neighborhood of the periodic orbit.

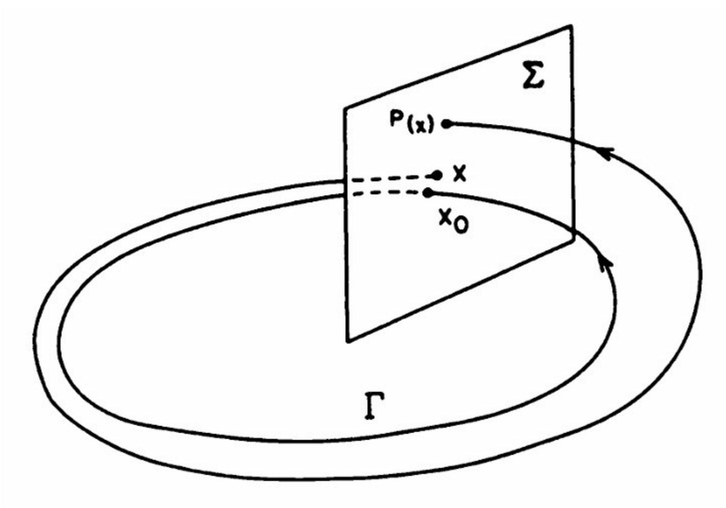

Figure 2.6: [2] The Poincaré map .

Theorem 2.6. *[2] Let Δ be an open subset of $\mathbb{R}^n$ and Let the vector field of system (1.1) . Suppose that $\varphi_t(X_0)$ is a periodic solution of (1.1) of period T, and that the cycle*

$$\gamma = \{X \in \mathbb{R}^n \mid X = \varphi_t(X_0), 0 \leq t \leq T\}$$

is contained in Δ .Let Σ be the hyperplane orthogonal to γ at X_0; i.e., let

$$\Sigma = \{X \in \mathbb{R}^n \mid (X - X_0) \cdot (P(X_0), Q(X_0)) = 0\}.$$

Then there exists $\delta > 0$ and a unique function $\tau(X)$, defined and continuously differentiable for $X \in N_\delta(X_0)$, such that

$$\tau(X_0) = T \quad \textit{and} \quad \varphi_t(X) \in \Sigma \quad \textit{for all } X \in N_\delta(X_0).$$

Definition 2.3. [2] Let γ, Σ, δ and $\tau(X)$ be defined as in Theorem (2.6). Then for $X \in N_\delta(X_0) \cap \Sigma$, the function

$$\Pi(X) = \varphi_{\tau(X)}(X)$$

is called the *Poincaré map* for γ at X_0.

Theorem 2.7. *[2] Let $\gamma(t)$ be a periodic solution of (1.1) of period T. Then the derivative of the Poincaré map $\Pi(s)$ along a straight line Σ normal to $\gamma = \{X \in \mathbb{R}^2 \mid X = \gamma(t) - \gamma(0) \quad 0 \le t \le T\}$ at $X = (0,0)$ is given by*

$$\Pi'(0,0) = \exp \int_0^T div \cdot (P(\gamma(t), Q(\gamma(t))dt.$$

Corollary 2.1. *[2] Under the hypotheses of Theorem (2.7), the periodic solution $\gamma(t)$ is a stable limit cycle if*

$$\int_0^T div \cdot (P(\gamma(t), Q(\gamma(t))\, dt < 0,$$

and it is an unstable limit cycle if

$$\int_0^T div \cdot (P(\gamma(t), Q(\gamma(t))\, dt > 0.$$

It may be a stable, unstable, or semi-stable limit cycle, or it may belong to a continuous band of cycles if this quantity is zero.

2.6 Averaging Theory

Determining limit cycles remains one of the most fundamental open problems in the qualitative theory of differential systems [36]. Various analytical techniques have been developed to investigate the number of limit cycles that can bifurcate from the periodic orbits of a center. Among these, the averaging method stands out as one of the most powerful perturbation techniques used in the study of nonlinear dynamical systems. Other notable methods include the Poincaré map, Melnikov's method, and bifurcation theory.

The averaging method simplifies the analysis of periodic solutions by transforming a non-autonomous, time-periodic system into an associated autonomous averaged system. Consider the perturbed differential equation

$$\dot{x} = \varepsilon f(x, t, \varepsilon),$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$, ε is a small parameter, and the function f is T-periodic in t. The corresponding first-order averaged system is given by

$$\dot{x} = \varepsilon f^0(x), \quad \text{where} \quad f^0(x) = \frac{1}{T} \int_0^T f(x, t, 0)\, dt.$$

The zeros of the averaged function $f^0(x)$ correspond to approximate periodic solutions of the original system, which may give rise to limit cycles.

This technique provides an effective framework for identifying and analyzing limit cycles in complex polynomial differential systems. For this reason, averaging theory is central to the approach adopted in this work.

2.6.1 Perturbation of Differential Systems

Definition 2.4. [16] A widely used strategy for studying limit cycles in planar polynomial differential systems is to start from an integrable system with a center-type singularity. This system is then perturbed by adding small polynomial terms of a prescribed degree. The perturbed system takes the form:

$$\begin{cases} \dot{x} = P(x,y) + \varepsilon f(x,y), \\ \dot{y} = Q(x,y) + \varepsilon g(x,y), \end{cases}$$

where ε is a small parameter, and $f(x,y)$, $g(x,y)$ are polynomial perturbations. The aim is to destroy the continuum of periodic orbits of the integrable system in such a way that some isolated closed orbits persist, corresponding to limit cycles.

2.6.2 First-Order Averaging Method

Theorem 2.8. *[37] Consider the differential equation*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon),$$

where $F_1 : D \to \mathbb{R}$ and $R : D(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions that are T-periodic in t, and $D \subset \mathbb{R}$ is an open set.

Assume that

- *$F_1(t,\cdot) \in \mathcal{C}^1(D)$ for all $t \in \mathbb{R}$,*
- *F_1 and R are locally Lipschitz continuous with respect to x, and R is differentiable with respect to ε.*

Define the averaged function

$$F_{10}(x) = \frac{1}{T}\int_0^T F_1(s,x)\,ds.$$

Suppose that there exists an open bounded set $V \subset D$ and a family of points $a_\varepsilon \in V$ such that for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, we have:

$$F_{10}(a_\varepsilon) = 0 \quad \text{and} \quad d_B(F_{10}, V, a_\varepsilon) \neq 0,$$

where d_B denotes the Brouwer degree.

Then, for sufficiently small $|\varepsilon|$, the system admits a T-periodic solution $x(t,\varepsilon)$ such that

$$x(0,\varepsilon) \to a_\varepsilon \quad \text{as} \quad \varepsilon \to 0.$$

2.6.3 Second-Order Averaging Method

Theorem 2.9. *[37] Consider the differential equation*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon),$$

where $F_1, F_2 : D \to \mathbb{R}$ and $R : D(-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous, T-periodic in t, and $D \subset \mathbb{R}$ is an open domain.

Assume

- *$F_1(t,\cdot) \in \mathcal{C}^2(D)$, and $F_2(t,\cdot) \in \mathcal{C}^1(D)$,*
- *F_1, F_2, R are locally Lipschitz with respect to x, and R is twice differentiable in ε.*

Define the first and second-order averaged functions

$$F_{10}(x) = \frac{1}{T}\int_0^T F_1(s,x)\,ds, \quad F_{20}(x) = \frac{1}{T}\int_0^T \left[\frac{\partial F_1(s,x)}{\partial x} y_1(s,x) + F_2(s,x)\right] ds,$$

where

$$y_1(s,x) = \int_0^s F_1(t,x)\,dt.$$

Suppose there exists an open bounded set $V \subset D$ and a family of points $a_\varepsilon \in V$ such that for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$

$$F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0, \quad \text{and} \quad d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0.$$

Then, for sufficiently small $|\varepsilon|$, the system admits a T-periodic solution $x(t,\varepsilon)$ satisfying

$$x(0,\varepsilon) \to a_\varepsilon \quad \text{as} \quad \varepsilon \to 0.$$

Here, $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon)$ denotes the Brouwer degree of the function $F_{10} + \varepsilon F_{20}$ at the point a_ε relative to the open set V. The condition $d_B \neq 0$ guarantees, in a topological sense, the existence of a zero of the function inside V. In other words, this ensures that the averaged function vanishes at some point in V, which implies the existence of a periodic solution of the original system.

The reader may refer to the following references for a more in-depth treatment [16],[17], [38],[39].

3 Estimating the Maximum Number of Limit Cycles in Polynomial Liénard Systems

In this chapter, we primarily rely on the study presented in the article "On the number of limit cycles of a class of polynomial differential systems" by Jaume Llibre and Clàudia Valls [1]. We conducted a detailed analysis of the methodology and main steps outlined in their work. Rather than merely restating their results, we supplemented the study with illustrative examples of our own creation. The construction of these examples proved to be challenging due to the precision and depth of analysis required to capture the system's properties accurately. These additions aim to enrich the discussion and clarify practical applications of the employed methodology, thereby providing a more comprehensive perspective.

3.1 Introduction

The analysis of limit cycles in polynomial differential systems is one of the fundamental challenges in nonlinear dynamics. This topic is closely related to the celebrated 16^{th} Hilbert problem [15],[40], which seeks to determine the maximum number of limit cycles that can occur in polynomial vector fields of a given degree. In this context, Liénard systems represent an important class that has attracted significant interest, beginning with the classical work of [41], who established sufficient conditions for the existence and uniqueness of a limit cycle.

In [1], we consider a generalized Liénard-type system given by

$$\begin{cases} \dot{x} = y - g_1(x) - f_1(x)y, \\ \dot{y} = -x - g_2(x) - f_2(x)y, \end{cases} \tag{3.1}$$

where g_1, f_1, g_2, and f_2 are polynomial functions of specified degrees. When $g_1(x) = f_1(x) = 0$, the system reduces to the classical Liénard equation

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - f(x)y, \end{cases} \tag{3.2}$$

In this work, [42] proposed that the maximum number of limit cycles is given by $\left[\frac{n}{2}\right]$, where $f(x)$ is a polynomial of degree n. This bound has been confirmed for $n = 1, 2, 3$ [43],[44], but counterexamples were found for $n \geq 5$ [45].

This work focuses on analyzing the linear center system

$$\dot{x} = y, \quad \dot{y} = -x,$$

which possesses an infinite number of periodic orbits surrounding the origin. The goal is to study the effect of polynomial perturbations on this system using the Averaging Theory of first and second order [17],[16].

We begin by analyzing the first-order perturbed system

$$\begin{cases} \dot{x} = y - \varepsilon \left(g_{11}(x) + f_{11}(x)y \right), \\ \dot{y} = -x - \varepsilon \left(g_{21}(x) + f_{21}(x)y \right), \end{cases} \tag{3.3}$$

where $g_{11}, f_{11}, g_{21}, f_{21}$ are polynomials of degrees k, l, m, n, respectively, and ε is a small parameter.

We then extend the analysis to a second-order perturbed system

$$\begin{cases} \dot{x} = y - \varepsilon \left(g_{11}(x) + f_{11}(x)y \right) - \varepsilon^2 \left(g_{12}(x) + f_{12}(x)y \right), \\ \dot{y} = -x - \varepsilon \left(g_{21}(x) + f_{21}(x)y \right) - \varepsilon^2 \left(g_{22}(x) + f_{22}(x)y \right), \end{cases} \tag{3.4}$$

where the functions g_{11}, g_{12} are of degree k, f_{11}, f_{12} are of degree l, g_{21}, g_{22} are of degree m, and f_{21}, f_{22} are of degree n. Introducing second-order perturbations in ε increases the system's nonlinearity and allows for a more detailed study of the bifurcating periodic behavior.

The objective of this work is to apply the tools of Averaging Theory to provide a detailed analysis of these two perturbed systems and to estimate the number of limit cycles that bifurcate from the periodic orbits of the unperturbed linear system.

3.2 First-Order Averaging Analysis of Perturbed Systems

Theorem 3.1. *[1] For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the generalized Liénard polynomial differential system (3.3) bifurcating from the periodic orbits of the linear centre $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first order is*

$$\lambda_1 = \max \left\{ \left[\frac{n}{2} \right], \left[\frac{k-1}{2} \right] \right\}. \tag{3.5}$$

Proof. We shall need the first-order averaging theory to prove Theorem 3.1. We write system (3.3) in polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$, $r > 0$. Time Derivatives of x and y

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial x}{\partial r} \cdot \dfrac{dr}{dt} + \dfrac{\partial x}{\partial \theta} \cdot \dfrac{d\theta}{dt} \\[2ex] \dfrac{dy}{dt} = \dfrac{\partial y}{\partial r} \cdot \dfrac{dr}{dt} + \dfrac{\partial y}{\partial \theta} \cdot \dfrac{d\theta}{dt} \end{cases} \implies \begin{cases} \dfrac{dx}{dt} = \cos\theta \cdot \dfrac{dr}{dt} - r\sin\theta \cdot \dfrac{d\theta}{dt} \\[2ex] \dfrac{dy}{dt} = \sin\theta \cdot \dfrac{dr}{dt} + r\cos\theta \cdot \dfrac{d\theta}{dt} \end{cases}$$

From this, we obtain the simplified form

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \\[1ex] \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \end{cases}$$

After transforming to polar coordinates, the system (3.3) takes the following form

$$\begin{cases} \dot{r}\cos\theta - r\dot{\theta}\sin\theta = r\sin\theta - \varepsilon\left(g_{11}(r\cos\theta) + f_{11}(r\cos\theta)\cdot r\sin\theta\right), \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -r\cos\theta - \varepsilon\left(g_{21}(r\cos\theta) + f_{21}(r\cos\theta)\cdot r\sin\theta\right). \end{cases}$$

Start with polar definitions $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$.Thus, differentiating with respect to time gives the following result

$$r\dot{r} = x\dot{x} + y\dot{y} \quad \text{and} \quad r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

Upon substitution, we find

$$\begin{aligned}
\dot{r} &= \frac{1}{r}\Bigg[r\cos\theta\Big[r\sin\theta - \varepsilon\big(g_{11}(r\cos\theta) + f_{11}(r\cos\theta)r\sin\theta\big)\Big] \\
&\qquad + r\sin\theta\Big[-r\cos\theta - \varepsilon\big(g_{21}(r\cos\theta) + f_{21}(r\cos\theta)r\sin\theta\big)\Big]\Bigg] \\
&= -\varepsilon\Big(g_{11}(r\cos\theta)\cos\theta + f_{11}(r\cos\theta)r\sin\theta\cos\theta + g_{21}(r\cos\theta)\sin\theta + f_{21}(r\cos\theta)r\sin^2\theta\Big)
\end{aligned}$$

$$\begin{aligned}
\dot{\theta} &= \frac{1}{r^2}\Bigg[r\cos\theta\Big[-r\cos\theta - \varepsilon\big(g_{21}(r\cos\theta) + f_{21}(r\cos\theta)r\sin\theta\big)\Big] \\
&\qquad - r\sin\theta\Big[r\sin\theta - \varepsilon\big(g_{11}(r\cos\theta) + f_{11}(r\cos\theta)r\sin\theta\big)\Big]\Bigg] \\
&= \frac{1}{r}\Bigg[-r(\cos^2\theta + \sin^2\theta) - \varepsilon\big(g_{21}(r\cos\theta)\cos\theta + f_{21}(r\cos\theta)r\sin\theta\cos\theta\big) \\
&\qquad + \varepsilon\big(g_{11}(r\cos\theta)\sin\theta + f_{11}(r\cos\theta)r\sin^2\theta\big)\Bigg] \\
&= -1 - \frac{\varepsilon}{r}\Big(g_{21}(r\cos\theta)\cos\theta + f_{21}(r\cos\theta)r\sin\theta\cos\theta\big) - g_{11}(r\cos\theta)\sin\theta - f_{11}(r\cos\theta)r\sin^2\theta\Big)
\end{aligned}$$

Thus, system (3.3) can be written in the standard form suitable for applying the averaging theory. If we define the following polynomials

$$f_{11}(x) = \sum_{i=0}^{l} a_{i,1}x^i, \quad f_{21}(x) = \sum_{i=0}^{n} a_{i,2}x^i, \quad g_{11}(x) = \sum_{i=0}^{k} b_{i,1}x^i \quad \text{and} \quad g_{21}(x) = \sum_{i=0}^{m} b_{i,2}x^i, \tag{3.1}$$

then, system (3.3) becomes

$$\begin{cases} \dot{r} = -\varepsilon\Bigg(\displaystyle\sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta \\ \qquad + \displaystyle\sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{k} b_{i,1} r^i \cos^{i+1}\theta\Bigg), \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r}\Bigg(\displaystyle\sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^{i+1}\theta \\ \qquad - \displaystyle\sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^i\theta \sin^2\theta - \sum_{i=0}^{k} b_{i,1} r^i \cos^i\theta \sin\theta\Bigg). \end{cases} \tag{3.6}$$

We now denote the expression appearing in $\dot{r}$ by $A(r,\theta)$, and the expression inside the perturbation term in $\dot{\theta}$ (excluding the factor -1) by $B(r,\theta)$, that is:

$$A(r,\theta) = \sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta + \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{k} b_{i,1} r^i \cos^{i+1}\theta,$$

and

$$B(r,\theta) = \sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^{i+1}\theta - \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^i\theta \sin^2\theta - \sum_{i=0}^{k} b_{i,1} r^i \cos^i\theta \sin\theta.$$

Hence, system (3.6) becomes:

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \frac{-\varepsilon A(r,\theta)}{-1 - \frac{\varepsilon}{r} B(r,\theta)}.$$

Since ε is small, we apply a first-order approximation and expand the denominator as follows:

$$\frac{1}{-1 - \frac{\varepsilon}{r} B(r,\theta)} = -1 + \frac{\varepsilon}{r} B(r,\theta) + O(\varepsilon^2).$$

Multiplying this expansion by $-\varepsilon A(r,\theta)$, we obtain:

$$\frac{dr}{d\theta} = \varepsilon A(r,\theta) + O(\varepsilon^2).$$

Finally, defining

$$F_1(r,\theta) = A(r,\theta),$$

we write the differential equation in the simplified form:

$$\begin{aligned} \frac{dr}{d\theta} = \varepsilon\Bigg(&\sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta \\ &+ \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{k} b_{i,1} r^i \cos^{i+1}\theta\Bigg) + O(\varepsilon^2). \end{aligned}$$

Let us take

$$F_1(r,\theta) = \sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta \tag{3.7}$$
$$+ \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{k} b_{i,1} r^i \cos^{i+1}\theta.$$

We compute $F_{10}(r)$ by applying its definition as the average of $F_1(r,\theta)$ over one period

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta)\, d\theta.$$

Thus

$$\begin{aligned}
F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} \Bigg(&\sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta \\
&+ \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{k} b_{i,1} r^i \cos^{i+1}\theta \Bigg) d\theta \\
= \frac{1}{2\pi}\Bigg(&\sum_{i=0}^{n} a_{i,2} r^{i+1} \int_0^{2\pi} \cos^i\theta \sin^2\theta d\theta + \sum_{i=0}^{m} b_{i,2} r^i \int_0^{2\pi} \cos^i\theta \sin\theta d\theta \\
&+ \sum_{i=0}^{l} a_{i,1} r^{i+1} \int_0^{2\pi} \cos^{i+1}\theta \sin\theta d\theta + \sum_{i=0}^{k} b_{i,1} r^i \int_0^{2\pi} \cos^{i+1}\theta d\theta \Bigg).
\end{aligned}$$

In order to simplify the expression and eliminate the integrals that involve odd powers of $\sin\theta$ or $\cos\theta$, which vanish when integrated over the interval $[0, 2\pi]$, the sums are split according to whether the index $i \in \mathbb{N}$ is even or odd.

$$\begin{aligned}
F_{10}(r) = \frac{1}{2\pi}\Bigg[&\sum_{i=0}^{[\frac{n}{2}]} a_{2i,2} r^{2i+1} \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta d\theta + \sum_{i=0}^{[\frac{n-1}{2}]} a_{2i+1,2} r^{2i+2} \int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta d\theta \\
&+ \sum_{i=0}^{m} b_{i,2} r^i \int_0^{2\pi} \cos^i\theta \sin\theta d\theta + \sum_{i=0}^{l} a_{i,1} r^{i+1} \int_0^{2\pi} \cos^{i+1}\theta \sin\theta d\theta \\
&+ \sum_{i=0}^{[\frac{k}{2}]} b_{2i,1} r^{2i} \int_0^{2\pi} \cos^{2i+1}\theta d\theta + \sum_{i=0}^{[\frac{k-1}{2}]} b_{2i+1,1} r^{2i+1} \int_0^{2\pi} \cos^{2i+2}\theta d\theta \Bigg].
\end{aligned}$$

Now, using the expressions for the integrals given in Appendix A 54, and noting that $\alpha_{i+1} = (2i+1)\alpha_i$, we obtain the following

$$\begin{aligned}
F_{10}(r) &= \frac{1}{2\pi}\sum_{i=0}^{[n/2]} a_{2i,2} r^{2i+1} \frac{\pi\alpha_i}{2^i(i+1)!} + \frac{1}{2\pi}\sum_{i=0}^{[(k-1)/2]} b_{2i+1,1} r^{2i+1} \frac{2\pi\alpha_{i+1}}{2^{i+1}(i+1)!} \\
&= r\sum_{i=0}^{[n/2]} \frac{a_{2i,2}\alpha_i}{2^{i+1}(i+1)!} r^{2i} + r\sum_{i=0}^{[(k-1)/2]} \frac{b_{2i+1,1}\alpha_{i+1}}{2^{i+1}(i+1)!} r^{2i} \\
&= r\sum_{i=0}^{[n/2]} \frac{a_{2i,2}\alpha_i}{2^{i+1}(i+1)!} r^{2i} + r\sum_{i=0}^{[(k-1)/2]} \frac{b_{2i+1,1}(2i+1)\alpha_i}{2^{i+1}(i+1)!} r^{2i}. \qquad (3.8)
\end{aligned}$$

Thus, the polynomial $F_{10}(r)$ has at most λ_1 positive roots. By appropriately choosing the coefficients $a_{2i,2}$ and $b_{2i+1,1}$, we can ensure that $F_{10}(r)$ has exactly λ_1 simple positive roots, where

$$\lambda_1 = \max\left\{\left[\frac{n}{2}\right], \left[\frac{k-1}{2}\right]\right\}.$$

Hence, Theorem 3.1 is proved.

□

3.2.1 Application example

By choosing the polynomial functions The degrees $k = 3, \quad l = 2, \quad m = 1, \quad n = 4$ as

$$\begin{aligned}
g_{11}(x) &= -\frac{31}{5}x + 5x^3 && \text{(degree 3)},\\
f_{11}(x) &= x^2 && \text{(degree 2)},\\
g_{21}(x) &= x + 1 && \text{(degree 1)},\\
f_{21}(x) &= \frac{6}{5} + 2x^2 - x^4 && \text{(degree 4)},
\end{aligned}$$

the system (3.3) becomes

$$\begin{cases}
\dot{x} = y - \varepsilon\left(-\frac{31}{5}x + 5x^3 + x^2 y\right),\\
\dot{y} = -x - \varepsilon\left(x + 1 + \left(\frac{6}{5} + 2x^2 - x^4\right)y\right).
\end{cases} \tag{3.9}$$

To apply the Averaging Theory, we first convert the system into polar coordinates using the transformation $x = r\cos\theta$, $y = r\sin\theta$. This yields the following system in terms of r and θ

$$\begin{cases}
\dot{r} = -\varepsilon\Bigg(\frac{6}{5}r\sin^2\theta + 2r^3\cos^2\theta\sin^2\theta - r^5\cos^4\theta\sin^2\theta\\
\qquad + r\cos\theta\sin\theta + \sin\theta + r^3\cos^4\theta\sin\theta - \frac{31}{5}r\cos^2 + 5r^3\cos^4\theta\Bigg),\\
\dot{\theta} = -1 - \frac{\varepsilon}{r}\Bigg(\frac{6}{5}r\cos\theta\sin\theta + 2r^3\cos^3\theta\sin\theta - r^5\cos^5\theta\sin\theta\\
\qquad - r\cos^2\theta + \cos\theta - r^3\cos^2\sin^2\theta + \frac{31}{5}r\cos^2\theta - 5r^3\cos^4\theta\Bigg).
\end{cases}$$

Next, we derive the first-order differential equation $\frac{dr}{d\theta}$ by dividing $\dot{r}$ by $\dot{\theta}$, and expand the expression to first order in ε. we obtain

$$\begin{aligned}
\frac{dr}{d\theta} = \varepsilon\Bigg(&\frac{6}{5}r\sin^2\theta + 2r^3\cos^2\theta\sin^2\theta - r^5\cos^4\theta\sin^2\theta + r\cos\theta\sin\theta + \sin\theta\\
&+ r^3\cos^4\theta\sin\theta - \frac{31}{5}r\cos^2 + 5r^3\cos^4\theta\Bigg) + O(\varepsilon^2).
\end{aligned}$$

We then compute the first-order averaged function $F_{10}(r)$ using the formula

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta)\,d\theta.$$

Substituting in the expression, we get

$$\begin{aligned} F_{10}(r) &= \frac{1}{2\pi}\int_0^{2\pi}\Bigg(\frac{6}{5}r\sin^2\theta + 2r^3\cos^2\theta\sin^2\theta - r^5\cos^4\theta\sin^2\theta + r\cos\theta\sin\theta + \sin\theta \\ &\quad + r^3\cos^4\theta\sin\theta - \frac{31}{5}r\cos^2 + 5r^3\cos^4\theta\Bigg)\,d\theta \\ &= \frac{1}{16}r\left(-r^4 + 34r^2 - 40\right). \end{aligned}$$

This equation has two positive roots approximately $r_1 \approx 1$ and $r_2 \approx 6$. According to the theory (3.1), system admits the maximum possible number of small-amplitude limit cycles emerging from the origin, which corresponds to the number of positive nonzero roots of the averaged function. Therefore, the system (3.9)admits two limit cycles.

3.3 Second-Order Averaging Analysis of Perturbed Systems

Theorem 3.2. *[1] For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the generalized Liénard polynomial differential systems (3.4) bifurcating from the periodic orbits of the linear centre $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order is $\lambda_3 = \max\{\lambda_1, \lambda_2\}$, where*

$$\begin{aligned} \lambda_2 = \max\Bigg\{&\mu + \left[\frac{m-1}{2}\right],\ \mu + \left[\frac{l}{2}\right],\ \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right], \\ &\left[\frac{k}{2}\right] + \left[\frac{m}{2}\right] - 1,\ \left[\frac{n-1}{2}\right] + \left[\frac{l-1}{2}\right] + 1,\ \left[\frac{k}{2}\right] + \left[\frac{l-1}{2}\right]\Bigg\}. \end{aligned} \tag{3.10}$$

with $\mu = \min\{[n/2], [(k-1)/2]\}$.

Proof. We shall need the second-order averaging theory to prove Theorem 3.2. We write system (3.4) in polar coordinates (r,θ) where $x = r\cos\theta$ and $y = r\sin\theta$, $r > 0$. After transforming system (3.4), we obtain

$$\begin{cases} \dot{r}\cos\theta - r\dot{\theta}\sin\theta = r\sin\theta - \varepsilon\big(g_{11}(r\cos\theta) + f_{11}(r\cos\theta)r\sin\theta\big) \\ \qquad\qquad -\varepsilon^2\big(g_{12}(r\cos\theta) + f_{12}(r\cos\theta)r\sin\theta\big), \\ \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta = -r\cos\theta - \varepsilon\big(g_{21}(r\cos\theta) + f_{21}(r\cos\theta)r\sin\theta\big) \\ \qquad\qquad -\varepsilon^2\big(g_{22}(r\cos\theta) + f_{22}(r\cos\theta)r\sin\theta\big). \end{cases}$$

Thus, the system (3.4) can be written in polar coordinates in the following form

$$\dot{r} = \frac{1}{r}\Bigg[r\cos\theta\Big[r\sin\theta - \varepsilon\big(g_{11}(r\cos\theta) + f_{11}(r\cos\theta)r\sin\theta\big) - \varepsilon^2\big(g_{12}(r\cos\theta) + f_{12}(r\cos\theta)r\sin\theta\big)\Big]$$

$$+ r\sin\theta\Big[-r\cos\theta - \varepsilon\big(g_{21}(r\cos\theta) + f_{21}(r\cos\theta)r\sin\theta\big) - \varepsilon^2\big(g_{22}(r\cos\theta) + f_{22}(r\cos\theta)r\sin\theta\big)\Big]\Bigg]$$

$$= -\varepsilon\Bigg(g_{11}(r\cos\theta)\cos\theta + f_{11}(r\cos\theta)r\sin\theta\cos\theta + g_{21}(r\cos\theta)\sin\theta + f_{21}(r\cos\theta)r\sin^2\theta\Bigg)$$

$$- \varepsilon^2\Bigg(g_{12}(r\cos\theta)\cos\theta + f_{12}(r\cos\theta)r\sin\theta\cos\theta g_{22}(r\cos\theta)\sin\theta + f_{22}(r\cos\theta)r\sin^2\theta\Bigg)$$

$$\dot{\theta} = \frac{1}{r^2}\Bigg[r\cos\theta\Big[-r\cos\theta - \varepsilon\big(g_{21}(r\cos\theta) + f_{21}(r\cos\theta)r\sin\theta\big) - \varepsilon^2\big(g_{22}(r\cos\theta) + f_{22}(r\cos\theta)r\sin\theta\big)\Big]$$

$$- r\sin\theta\Big[r\sin\theta - \varepsilon\big(g_{11}(r\cos\theta) + f_{11}(r\cos\theta)r\sin\theta\big) - \varepsilon^2\big(g_{12}(r\cos\theta) + f_{12}(r\cos\theta)r\sin\theta\big)\Big]\Bigg].$$

$$= -1 - \frac{\varepsilon}{r}\Bigg(g_{21}(r\cos\theta)\cos\theta + f_{21}(r\cos\theta)r\cos\theta\sin\theta - g_{11}(r\cos\theta)\sin\theta - f_{11}(r\cos\theta)r\sin^2\theta\Bigg)$$

$$- \frac{\varepsilon^2}{r}\Bigg(g_{22}(r\cos\theta)\cos\theta + f_{22}(r\cos\theta)r\cos\theta\sin\theta - g_{12}(r\cos\theta)\sin\theta - f_{12}(r\cos\theta)r\sin^2\theta\Bigg).$$

We write f_{11}, f_{21}, g_{11} and g_{21} as in (3.1), and

$$f_{12}(x) = \sum_{i=0}^{l} c_{i,1}x^i, \quad f_{22}(x) = \sum_{i=0}^{n} c_{i,2}x^i, \quad g_{12}(x) = \sum_{i=0}^{k} d_{i,1}x^i \quad \text{and} \quad g_{22}(x) = \sum_{i=0}^{m} d_{i,2}x^i.$$

The system becomes

$$
\begin{cases}
\dot{r} = -\varepsilon\Bigg(\displaystyle\sum_{i=0}^{n} a_{i,2} r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta\sin\theta \\
\qquad + \displaystyle\sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} b_{i,1} r^i\cos^{i+1}\theta\Bigg) \\
\qquad - \varepsilon^2\Bigg(\displaystyle\sum_{i=0}^{n} c_{i,2} r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} d_{i,2} r^i\cos^i\theta\sin\theta \\
\qquad + \displaystyle\sum_{i=0}^{l} c_{i,1} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} d_{i,1} r^i\cos^{i+1}\theta\Bigg), \\
\dot{\theta} = -1 - \dfrac{\varepsilon}{r}\Bigg(\displaystyle\sum_{i=0}^{n} a_{i,2} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{m} b_{i,2} r^i\cos^{i+1}\theta \\
\qquad - \displaystyle\sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^i\theta\sin^2\theta - \sum_{i=0}^{k} b_{i,1} r^i\cos^i\theta\sin\theta\Bigg) \\
\qquad - \dfrac{\varepsilon^2}{r}\Bigg(\displaystyle\sum_{i=0}^{n} c_{i,2} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{m} d_{i,2} r^i\cos^{i+1}\theta \\
\qquad - \displaystyle\sum_{i=0}^{l} c_{i,1} r^{i+1}\cos^i\theta\sin^2\theta - \sum_{i=0}^{k} d_{i,1} r^i\cos^i\theta\sin\theta\Bigg).
\end{cases}
\tag{3.11}
$$

Taking θ as the new independent variable, system (3.11) becomes

$$
\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + O(\varepsilon^3),
$$

where

$$
\begin{aligned}
F_1(\theta, r) = &\sum_{i=0}^{n} a_{i,2} r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i\cos^i\theta\sin\theta \\
&+ \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} b_{i,1} r^i\cos^{i+1}\theta
\end{aligned}
\tag{3.12}
$$

and

$$
F_2(\theta, r) = \mathcal{I}(r, \theta) + r\mathcal{II}(r, \theta),
$$

where

$$
\begin{aligned}
\mathcal{I}(r, \theta) = &\sum_{i=0}^{n} c_{i,2} r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} d_{i,2} r^i\cos^i\theta\sin\theta \\
&+ \sum_{i=0}^{l} c_{i,1} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} d_{i,1} r^i\cos^{i+1}\theta
\end{aligned}
\tag{3.13}
$$

and

$$\mathcal{II}(r,\theta) = -\left(\sum_{i=0}^{n} a_{i,2} r^i \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^{i-1}\cos^i\theta\sin\theta \right. \tag{3.14}$$

$$\left. + \sum_{i=0}^{l} a_{i,1} r^i \cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} b_{i,1} r^{i-1}\cos^{i+1}\theta \right)$$

$$\left(\sum_{i=0}^{n} a_{i,2} r^i \cos^{i+1}\theta\sin\theta + \sum_{i=0}^{m} b_{i,2} r^{i-1}\cos^{i+1}\theta \right. \tag{3.15}$$

$$\left. - \sum_{i=0}^{l} a_{i,1} r^i \cos^i\theta\sin^2\theta - \sum_{i=0}^{k} b_{i,1} r^{i-1}\cos^i\theta\sin\theta \right).$$

In order to compute $F_{20}(r)$,weneedthat F_{10} be identically zero.Then from(3.8), all coefficients of r^{2i+1} must vanish. This gives.

$$\frac{a_{2i,2}\alpha_i}{2^{i+1}(i+1)!} + \frac{b_{2i+1,1}(2i+1)\alpha_i}{2^{i+1}(i+1)!} = 0$$

By simplifying the expression, we get

$$a_{2i,2} + b_{2i+1,1}(2i+1) = 0$$

Therefore, we choose

$$\begin{cases} b_{2i+1,1} = -\dfrac{a_{2i,2}}{2i+1}, & \text{for } i = 0, 1, \ldots, \mu, \\ b_{2i+1,1} = a_{2i,2} = 0, & \text{for } i = \mu+1, \ldots, \lambda_1. \end{cases} \tag{3.16}$$

Let us define the following indices $\mu = \min([\frac{n}{2}], [\frac{k-1}{2}]), \lambda_1 = \max([\frac{n}{2}], [\frac{k-1}{2}])$
We compute $F_{20}(r)$ by applying its definition as the average

$$F_{20}(r) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\partial F_1}{\partial r}(\theta,r)\left(\int_0^{\theta} F_1(\psi,r)\,d\psi\right) + F_2(\theta,r)\right)d\theta.$$

We split the computation of the function F_{20} into three parts, i.e., we define

$$2\pi F_{20} = r\big(P_1(r^2) + P_2(r^2) + P_3(r^2)\big).$$

where

$$rP_1(r^2) = \int_0^{2\pi} \mathcal{III}(r,\theta)\,d\theta = \int_0^{2\pi} \frac{d}{dr}F_1(\theta,r)\,y_1(\theta,r)\,d\theta,$$

$$P_2(r^2) + P_3(r^2) = \int_0^{2\pi} \mathcal{I}(r,\theta)\,d\theta + \int_0^{2\pi} \mathcal{II}(r,\theta)\,d\theta$$

where

$$y_1(\theta,r) = \int_0^{\theta} F_1(\psi,r)\,d\psi \qquad \textit{and} \qquad \int_0^{2\pi} \mathcal{I}(r,\theta) + \mathcal{II}(r,\theta)\,d\theta = \int_0^{2\pi} F_2(\theta,r)\,d\theta.$$

We now compute the partial derivative of $F_1(\theta,r)$ as given in equation (3.12). Differentiating with respect to r, we obtain

$$\frac{\partial F_1}{\partial r}(\theta,r) = \sum_{i=0}^{n}(i+1)a_{i,2}r^i\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} ib_{i,2}r^{i-1}\cos^i\theta\sin\theta$$
$$+\sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} ib_{i,1}r^{i-1}\cos^{i+1}\theta. \qquad (3.17)$$

For $i \in \mathbb{N}$, where i is odd or even, we find

$$= \sum_{i=0}^{[n/2]}(2i+1)a_{2i,2}r^{2i}\cos^{2i}\theta\sin^2\theta + \sum_{i=0}^{[(n-1)/2]}(2i+2)a_{2i+1,2}r^{2i+1}\cos^{2i+1}\theta\sin^2\theta$$
$$+\sum_{i=0}^{m} ib_{i,2}r^{i-1}\cos^i\theta\sin\theta + \sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta$$
$$+\sum_{i=0}^{[k/2]} 2ib_{2i,1}r^{2i-1}\cos^{2i+1}\theta + \sum_{i=0}^{[(k-1)/2]}(2i+1)b_{2i+1,1}r^{2i}\cos^{2i+2}\theta$$

From the condition (3.16) for $F_{10} = 0$, it follows that

$$= \sum_{i=0}^{\mu}(2i+1)a_{2i,2}r^{2i}\cos^{2i}\theta\sin^2\theta - (2i+1)\frac{a_{2i,2}}{2i+1}r^{2i}\cos^{2i+2}\theta$$
$$+\sum_{i=0}^{[(n-1)/2]}(2i+2)a_{2i+1,2}r^{2i+1}\cos^{2i+1}\theta\sin^2\theta$$
$$+\sum_{i=0}^{m} ib_{i,2}r^{i-1}\cos^i\theta\sin\theta + \sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta$$
$$+\sum_{i=0}^{[k/2]} 2ib_{2i,1}r^{2i-1}\cos^{2i+1}\theta$$
$$= \sum_{i=0}^{\mu} a_{2i,2}r^{2i}\cos^{2i}\theta\left((2i+1)\sin^2\theta - \cos^2\theta\right)$$
$$+\sum_{i=0}^{[(n-1)/2]}(2i+2)a_{2i+1,2}r^{2i+1}\cos^{2i+1}\theta\sin^2\theta$$
$$+\sum_{i=0}^{m} ib_{i,2}r^{i-1}\cos^i\theta\sin\theta + \sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta$$
$$+\sum_{i=0}^{[k/2]} 2ib_{2i,1}r^{2i-1}\cos^{2i+1}\theta$$

We now compute the integral $y_1(\theta,r) = \int_0^\theta F_1(\psi,r)\,d\psi$ by rewriting $F_1(\theta,r)$ (3.12) in a suitable form

Using the trigonometric identity $\sin^2\theta = 1-\cos^2\theta$, we obtain the following expression.

$$\begin{aligned}
F_1(\theta,r) = &\sum_{i=0}^{[n/2]}(2i+1)a_{2i,2}r^{2i}\cos^{2i}\theta(1-\cos^2\theta)\\
&+\sum_{i=0}^{[(n-1)/2]}(2i+2)a_{2i+1,2}r^{2i+1}\cos^{2i+1}\theta(1-\cos^2\theta)\\
&+\sum_{i=0}^{m} ib_{i,2}r^{i-1}\cos^i\theta\sin\theta+\sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta\\
&+\sum_{i=0}^{[k/2]}2ib_{2i,1}r^{2i-1}\cos^{2i+1}\theta+\sum_{i=0}^{[(k-1)/2]}(2i+1)b_{2i+1,1}r^{2i}\cos^{2i+2}\theta
\end{aligned}$$

After applying the condition $F_{10}\equiv 0$, the expression simplifies to

$$\begin{aligned}
F_1(\theta,r) = &\sum_{i=0}^{[(n-1)/2]}a_{2i+1,2}r^{2i+2}\cos^{2i+1}\theta-\sum_{i=0}^{[(n-1)/2]}a_{2i+1,2}r^{2i+2}\cos^{2i+3}\theta\\
&+\sum_{i=0}^{[n/2]}a_{2i,2}r^{2i+1}\cos^{2i}\theta-\sum_{i=0}^{[n/2]}a_{2i,2}r^{2i+1}\cos^{2i+2}\theta\\
&+\sum_{i=0}^{m}b_{i,2}r^i\cos^i\theta\sin\theta+\sum_{i=0}^{l}a_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta\\
&+\sum_{i=0}^{[(k-1)/2]}b_{2i+1,1}r^{2i+1}\cos^{2i+2}\theta+\sum_{i=0}^{[k/2]}b_{2i,1}r^{2i}\cos^{2i+1}\theta
\end{aligned}$$

Finally, we obtain the final form after applying the condition (3.16)

$$\begin{aligned}
F_1(\theta,r) = &\sum_{i=0}^{[(n-1)/2]}a_{2i+1,2}r^{2i+2}\cos^{2i+1}\theta-\sum_{i=0}^{[(n-1)/2]}a_{2i+1,2}r^{2i+2}\cos^{2i+3}\theta\\
&+\sum_{i=0}^{\mu}a_{2i,2}r^{2i+1}\cos^{2i}\theta-\sum_{i=0}^{\mu}\frac{2i+2}{2i+1}a_{2i,2}r^{2i+1}\cos^{2i+2}\theta\\
&+\sum_{i=0}^{m}b_{i,2}r^i\cos^i\theta\sin\theta+\sum_{i=0}^{l}a_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta\\
&+\sum_{i=0}^{[k/2]}b_{2i,1}r^{2i}\cos^{2i+1}\theta.
\end{aligned}$$

The integral can now be computed using Appendix A (54)

$$
\begin{aligned}
y_1(\theta, r) = & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2} r^{2i+2} \int_0^\theta \cos^{2i+1}\psi \, d\psi - \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2} r^{2i+2} \int_0^\theta \cos^{2i+3}\psi \, d\psi \\
& + \sum_{i=0}^{\mu} a_{2i,2} r^{2i+1} \int_0^\theta \cos^{2i}\psi \, d\psi - \sum_{i=0}^{\mu} \frac{2i+2}{2i+1} a_{2i,2} r^{2i+1} \int_0^\theta \cos^{2i+2}\psi \, d\psi \\
& + \sum_{i=0}^{m} b_{i,2} r^i \int_0^\theta \cos^i\psi \sin\psi \, d\psi + \sum_{i=0}^{l} a_{i,1} r^{i+1} \int_0^\theta \cos^{i+1}\psi \sin\psi \, d\psi \\
& + \sum_{i=0}^{[k/2]} b_{2i,1} r^{2i} \int_0^\theta \cos^{2i+1}\psi \, d\psi.
\end{aligned}
$$

Using the above, we obtain

$$
\begin{aligned}
y_1(\theta, r) = & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2} r^{2i+2} \sum_{l=0}^{i} \gamma_{i,l} \sin\big((2l+1)\theta\big) - \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2} r^{2i+2} \sum_{l=0}^{i+1} \gamma_{i+1,l} \sin\big((2l+1)\theta\big) \\
& + \sum_{i=0}^{\mu} a_{2i,2} r^{2i+1} \frac{1}{2^{2i}} \binom{2i}{i} \theta + \sum_{l=1}^{i} \beta_{i,l} \sin(2l\theta) \\
& - \sum_{i=0}^{\mu} \frac{2i+2}{2i+1} a_{2i,2} r^{2i+1} \frac{1}{2^{2i+2}} \binom{2i+2}{i+1} \theta + \sum_{l=1}^{i+1} \beta_{i+1,l} \sin(2l\theta) \\
& + \sum_{i=0}^{m} b_{i,2} r^i \frac{1}{i+1} \left(1 - \cos^{i+1}\theta\right) + \sum_{i=0}^{l} a_{i,1} r^{i+1} \frac{1}{i+2} \left(1 - \cos^{i+2}\theta\right) \\
& + \sum_{i=0}^{[k/2]} b_{2i,1} r^{2i} \sum_{l=0}^{i} \gamma_{i,l} \sin\big((2l+1)\theta\big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y_1(\theta, r) = & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}\, r^{2i+2} \sum_{l=0}^{i+1} \tilde{\gamma}_{i,l} \sin\big((2l+1)\theta\big) \\
& + \sum_{i=0}^{\mu} a_{2i,2}\, r^{2i+1} \sum_{l=1}^{i+1} \tilde{\beta}_{i,l} \sin(2l\theta) \\
& + \sum_{i=0}^{m} \frac{b_{i,2}}{i+1} r^i \left(1 - \cos^{i+1}\theta\right) + \sum_{i=0}^{l} \frac{a_{i,1}}{i+2} r^{i+1} \left(1 - \cos^{i+2}\theta\right) \\
& + \sum_{i=0}^{[k/2]} b_{2i,1}\, r^{2i} \sum_{l=0}^{i} \gamma_{i,l} \sin\big((2l+1)\theta\big)
\end{aligned}
$$

where

$$
\tilde{\gamma}_{i,l} = \begin{cases} \gamma_{i,l} - \gamma_{i+1,l}, & \text{if } 0 \le l \le i, \\ -\gamma_{i+1,i+1}, & \text{if } l = i+1. \end{cases}
\qquad
\tilde{\beta}_{i,l} = \begin{cases} \dfrac{\beta_{i,l} - 2(i+1)\beta_{i+1,l}}{2i+1}, & \text{if } 0 \le l \le i, \\ \dfrac{-2(i+1)\beta_{i+1,i+1}}{2i+1}, & \text{if } l = i+1. \end{cases}
$$

We now turn to the computation of the following integral

$$\int_0^{2\pi} \mathcal{III}(r,\theta)d\theta = \int_0^{2\pi} \frac{d}{dr}F_1(\theta,r)\, y_1(\theta,r)\, d\theta = rP_1(r^2).$$

$$\begin{aligned}
\int_0^{2\pi} \mathcal{III}(r,\theta)d\theta = \int_0^{2\pi} \Bigg[\Bigg(& \sum_{i=0}^{\mu} a_{2i,2}\, r^{2i} \cos^{2i}\theta \left((2i+1)\sin^2\theta - \cos^2\theta\right) \\
& + \sum_{i=0}^{[(n-1)/2]} (2i+2)a_{2i+1,2}\, r^{2i+1} \cos^{2i+1}\theta \sin^2\theta + \sum_{i=0}^{m} i\, b_{i,2}\, r^{i-1}\cos^i\theta\sin\theta \\
& + \sum_{i=0}^{l}(i+1)a_{i,1}\, r^i \cos^{i+1}\theta\sin\theta + \sum_{i=0}^{[k/2]} 2i\, b_{2i,1}\, r^{2i-1}\cos^{2i+1}\theta \Bigg) \\
\Bigg(& \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}\, r^{2i+2} \sum_{l=0}^{i+1} \tilde{\gamma}_{i,l} \sin\left((2l+1)\theta\right) + \sum_{i=0}^{\mu} a_{2i,2}\, r^{2i+1} \sum_{l=1}^{i+1} \tilde{\beta}_{i,l}\sin(2l\theta) \\
& + \sum_{i=0}^{m} \frac{b_{i,2}}{i+1} r^i \left(1-\cos^{i+1}\theta\right) + \sum_{i=0}^{l} \frac{a_{i,1}}{i+2} r^{i+1}\left(1-\cos^{i+2}\theta\right) \\
& + \sum_{i=0}^{[k/2]} b_{2i,1}\, r^{2i} \sum_{l=0}^{i} \gamma_{i,l}\sin\left((2l+1)\theta\right) \Bigg) \Bigg] d\theta,
\end{aligned}$$

After simplification, and by eliminating the terms whose integrals vanish due to the orthogonality of sine and cosine functions over the interval $[0, 2\pi]$, we obtain the expression

$$\begin{aligned}
\int_0^{2\pi} \mathcal{III}(r,\theta)d\theta = & \sum_{i=0}^{\mu}\sum_{j=0}^{[(m-1)/2]} A_{i,j}(r^2)^{i+j} + \sum_{i=0}^{\mu}\sum_{j=0}^{[l/2]} B_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]} C_{i,j}(r^2)^{i+j} \\
& + \sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(l-1)/2]} D_{i,j}(r^2)^{i+j+1} + \sum_{i=0}^{[k/2]}\sum_{j=0}^{[m/2]} E_{i,j}(r^2)^{i+j-1} + \sum_{i=0}^{[k/2]}\sum_{j=0}^{[(l-1)/2]} F_{i,j}(r^2)^{i+j} \\
& + \sum_{i=0}^{[m/2]}\sum_{j=0}^{[(n-1)/2]} G_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{\mu} H_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]} I_{i,j}(r^2)^{i+j-1} \\
& + \sum_{i=0}^{[(l-1)/2]}\sum_{j=0}^{[(n-1)/2]} J_{i,j}(r^2)^{i+j+1} + \sum_{i=0}^{[l/2]}\sum_{j=0}^{\mu} K_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(l-1)/2]}\sum_{j=0}^{[k/2]} L_{i,j}(r^2)^{i+j} \\
& = rP_1(r^2).
\end{aligned} \tag{3.18}$$

with

$$A_{i,j} = \frac{\pi a_{2i,2} b_{2j+1,2} \alpha_{i+j+1}}{2^{i+j+1}(i+j+2)!}, \qquad B_{i,j} = \frac{\pi a_{2i,2} a_{2j,1} \alpha_{i+j+1}}{2^{i+j+1}(i+j+2)!},$$

$$C_{i,j} = -\frac{\pi(2i+2) a_{2i+1,2} b_{2j,2} \alpha_{i+j+1}}{(2j+1)2^{i+j+1}(i+j+2)!}, \qquad D_{i,j} = -\frac{\pi(2i+2) a_{2i+1,2} a_{2j+1,1} \alpha_{i+j+2}}{(2j+3)2^{i+j+2}(i+j+3)!},$$

$$E_{i,j} = \frac{\pi 2i b_{2i,1} b_{2j,2} \alpha_{i+j+1}}{(2j+1)2^{i+j}(i+j+1)!}, \qquad F_{i,j} = \frac{\pi 2i b_{2i,1} a_{2j+1,1} \alpha_{i+j+2}}{(2j+3)2^{i+j+1}(i+j+2)!},$$

$$G_{i,j} = \pi \sum_{s=0}^{j+1} 2i b_{2i,2} a_{2j+1,2} \tilde{\gamma}_{j,s} C_{i,s}, \qquad H_{i,j} = \pi \sum_{s=1}^{j+1} (2i+1) b_{2i+1,2} a_{2j,2} \tilde{\beta}_{j,s} K_{i,s},$$

$$I_{i,j} = \pi \sum_{s=0}^{j} 2i b_{2i,2} b_{2j,1} \gamma_{j,s} C_{i,s}, \qquad J_{i,j} = \pi \sum_{s=0}^{j+1} (2i+2) a_{2i+1,1} a_{2j+1,2} \tilde{\gamma}_{j,s} C_{i,s},$$

$$K_{i,j} = \pi \sum_{s=1}^{j+1} (2i+1) a_{2i,1} a_{2j,2} \tilde{\beta}_{j,s} K_{i,s}, \qquad L_{i,j} = \pi \sum_{s=0}^{j} (2i+2) a_{2i+1,1} b_{2j,1} \gamma_{j,s} C_{i,s}.$$

Then, $P_1(r^2)$ is a polynomial in the variable r^2 of degree λ_2,

$$\lambda_2 = \max\left\{ \mu + \left[\frac{m-1}{2}\right], \mu + \left[\frac{l}{2}\right], \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right], \left[\frac{k}{2}\right] + \left[\frac{m}{2}\right] - 1, \left[\frac{n-1}{2}\right] + \left[\frac{l-1}{2}\right] + 1, \left[\frac{k}{2}\right] + \left[\frac{l-1}{2}\right] \right\}.$$

We now turn to the computation of $\int_0^{2\pi} F_2(\theta, r)\, d\theta = \int_0^{2\pi} \mathcal{I}(r,\theta) d\theta + \int_0^{2\pi} \mathcal{II}(r,\theta)\, d\theta$ We begin with the first part of the integral used in (3.13), and we have that

$$\int_0^{2\pi} \mathcal{I}(r,\theta) d\theta = r P_2(r^2).$$

Thus, we obtain the following

$$\begin{aligned}
\int_0^{2\pi} \mathcal{I}(r,\theta)\, d\theta &= \sum_{i=0}^{[n/2]} c_{2i,2} r^{2i+1} \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta\, d\theta + \sum_{i=0}^{[(n-1)/2]} c_{2i+1,2} r^{2i+2} \int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta\, d\theta \\
&+ \sum_{i=0}^{m} d_{i,2} r^i \int_0^{2\pi} \cos^i\theta \sin\theta\, d\theta + \sum_{i=0}^{l} c_{i,1} r^{i+1} \int_0^{2\pi} \cos^{i+1}\theta \sin\theta\, d\theta \\
&+ \sum_{i=0}^{[k/2]} d_{2i,1} r^{2i} \int_0^{2\pi} \cos^{2i+1}\theta\, d\theta + \sum_{i=0}^{[(k-1)/2]} d_{2i+1,1} r^{2i+1} \int_0^{2\pi} \cos^{2i+2}\theta\, d\theta \\
&= \sum_{i=0}^{[n/2]} c_{2i,2} r^{2i+1} \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta\, d\theta + \sum_{i=0}^{[(k-1)/2]} d_{2i+1,1} r^{2i+1} \int_0^{2\pi} \cos^{2i+2}\theta\, d\theta \\
&= \pi r \sum_{i=0}^{[n/2]} \frac{c_{2i,2}\alpha_i}{2^i(i+1)!} r^{2i} + \pi r \sum_{i=0}^{[(k-1)/2]} \frac{d_{2i+1,1}\alpha_i(2i+1)}{2^i(i+1)!} r^{2i} \\
&= r P_2(r^2).
\end{aligned} \tag{3.19}$$

where P_2 is a polynomial in the variable r^2 of degree λ_1.

We now turn to the second part of the integral used in (3.14),we have that

$$\int_0^{2\pi} \mathcal{II}(r,\theta)\, d\theta = rP_3(r^2).$$

we have

$$\begin{aligned}
\int_0^{2\pi} \mathcal{II}(r,\theta) = &-\Bigg(\sum_{i=0}^{n} a_{i,2} r^i \int_0^{2\pi} \cos^i\theta \sin^2\theta\, d\theta + \sum_{i=0}^{m} b_{i,2} r^{i-1} \int_0^{2\pi} \cos^i\theta \sin\theta\, d\theta \\
&+ \sum_{i=0}^{l} a_{i,1} r^i \int_0^{2\pi} \cos^{i+1}\theta \sin\theta\, d\theta + \sum_{i=0}^{k} b_{i,1} r^{i-1} \int_0^{2\pi} \cos^{i+1}\theta\, d\theta\Bigg) \\
&\Bigg(\sum_{j=0}^{n} a_{j,2} r^j \int_0^{2\pi} \cos^{j+1}\theta \sin\theta\, d\theta + \sum_{j=0}^{m} b_{j,2} r^{j-1} \int_0^{2\pi} \cos^{j+1}\theta\, d\theta \\
&- \sum_{j=0}^{l} a_{j,1} r^j \int_0^{2\pi} \cos^j\theta \sin^2\theta\, d\theta - \sum_{j=0}^{k} b_{j,1} r^{j-1} \int_0^{2\pi} \cos^j\theta \sin\theta\, d\theta\Bigg).
\end{aligned}$$

After simplification, all terms whose integrals vanish due to the orthogonality properties of sine and cosine functions on the interval $[0, 2\pi]$ have been excluded,we find

$$\begin{aligned}
\int_0^{2\pi} \mathcal{II}(r,\theta)\, d\theta = &-2\sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,2} b_{j,2} r^{i+j-1} \int_0^{2\pi} \cos^{i+j+1}\theta \sin^2\theta\, d\theta \\
&+ \sum_{i=0}^{n}\sum_{j=0}^{l} a_{i,2} a_{j,1} r^{i+j} \int_0^{2\pi} \cos^{i+j}\theta \sin^4\theta\, d\theta \\
&+ \sum_{i=0}^{m}\sum_{j=0}^{k} b_{i,2} b_{j,1} r^{i+j-2} \int_0^{2\pi} \cos^{i+j}\theta \sin^2\theta\, d\theta \\
&- \sum_{i=0}^{l}\sum_{j=0}^{n} a_{i,1} a_{j,2} r^{i+j} \int_0^{2\pi} \cos^{i+j+2}\theta \sin^2\theta\, d\theta \\
&+ 2\sum_{i=0}^{l}\sum_{j=0}^{k} a_{i,1} b_{j,1} r^{i+j-1} \int_0^{2\pi} \cos^{i+j+1}\theta \sin^2\theta\, d\theta \\
&- \sum_{i=0}^{k}\sum_{j=0}^{m} b_{i,1} b_{j,2} r^{i+j-2} \int_0^{2\pi} \cos^{i+j+2}\theta\, d\theta.
\end{aligned}$$

We split the double sums according to the even and odd indices of both i and j, which allows us to apply the condition $F_{10} \equiv 0$

$$
\begin{aligned}
\int_0^{2\pi} \mathcal{II}(r,\theta)\, d\theta =& -2 \sum_{i=0}^{[n/2]} \sum_{j=0}^{[(m-1)/2]} a_{2i,2} b_{2j+1,2} r^{2i+2j} \int_0^{2\pi} \cos^{2i+2j+2}\theta \sin^2\theta \, d\theta \\
& -2 \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} a_{2i+1,2} b_{2j,2} r^{2i+2j} \int_0^{2\pi} \cos^{2i+2j+2}\theta \sin^2\theta \, d\theta \\
& + \sum_{i=0}^{[n/2]} \sum_{j=0}^{[l/2]} a_{2i,2} a_{2j,1} r^{2i+2j} \int_0^{2\pi} \cos^{2i+2j}\theta \sin^4\theta \, d\theta \\
& + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} a_{2i+1,2} a_{2j+1,1} r^{2i+2j+2} \int_0^{2\pi} \cos^{2i+2j+2}\theta \sin^4\theta \, d\theta \\
& + \sum_{i=0}^{[m/2]} \sum_{j=0}^{[k/2]} b_{2i,2} b_{2j,1} r^{2i+2j-2} \int_0^{2\pi} \cos^{2i+2j}\theta \sin^2\theta \, d\theta \\
& - \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{[(k-1)/2]} \frac{a_{2j,2}}{2j+1} b_{2i+1,2} r^{2i+2j} \int_0^{2\pi} \cos^{2i+2j+2}\theta \sin^2\theta \, d\theta \\
& - \sum_{i=0}^{[l/2]} \sum_{j=0}^{[n/2]} a_{2i,1} a_{2j,2} r^{2i+2j} \int_0^{2\pi} \cos^{2i+2j+2}\theta \sin^2\theta \, d\theta \\
& - \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} a_{2i+1,1} a_{2j+1,2} r^{2i+2j+2} \int_0^{2\pi} \cos^{2i+2j+4}\theta \sin^2\theta \, d\theta \\
& + 2 \sum_{i=0}^{[l/2]} \sum_{j=0}^{[k/2]} a_{2i,1} b_{2j,1} r^{2i+2j-1} \int_0^{2\pi} \cos^{2i+2j+1}\theta \sin^2\theta \, d\theta \\
& + 2 \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(k-1)/2]} a_{2i+1,1} \left(-\frac{a_{2j,2}}{2j+1}\right) r^{2i+2j+1} \int_0^{2\pi} \cos^{2i+2j+2}\theta \sin^2\theta \, d\theta \\
& - \sum_{i=0}^{[k/2]} \sum_{j=0}^{[m/2]} b_{2i,1} b_{2j,2} r^{2i+2j-2} \int_0^{2\pi} \cos^{2i+2j+2}\theta \, d\theta \\
& - \sum_{i=0}^{[(k-1)/2]} \sum_{j=0}^{[(m-1)/2]} \left(-\frac{a_{2i,2}}{2i+1}\right) b_{2j+1,2} r^{2i+2j} \int_0^{2\pi} \cos^{2i+2j+3}\theta \, d\theta
\end{aligned}
$$

Using relation (3.16) , we get

$$\begin{aligned}\int_0^{2\pi} \mathcal{II}(r,\theta)\,d\theta = & -2\sum_{i=0}^{\mu}\sum_{j=0}^{[(m-1)/2]} a_{2i,2}b_{2j+1,2}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta\\
& -2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]} a_{2i+1,2}b_{2j,2}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta\\
& -\sum_{i=0}^{\mu}\sum_{j=0}^{[l/2]} a_{2i,2}a_{2j,1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j}\theta\sin^2\theta\cos(2\theta)\,d\theta\\
& -\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(l-1)/2]} a_{2i+1,2}a_{2j+1,1}r^{2i+2j+2}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\cos(2\theta)\,d\theta\\
& -\sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]} b_{2i,2}b_{2j,1}r^{2i+2j-2}\int_0^{2\pi}\cos^{2i+2j}\theta\cos(2\theta)\,d\theta\\
& -\sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[(k-1)/2]} b_{2i+1,2}b_{2j+1,1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\cos(2\theta)\,d\theta\\
& -2\sum_{i=0}^{[l/2]}\sum_{j=0}^{\mu} a_{2i,1}\frac{a_{2j,2}}{2j+1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta\\
& +2\sum_{i=0}^{[(l-1)/2]}\sum_{j=0}^{[k/2]} a_{2i+1,1}b_{2j,1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta.\end{aligned}$$

we have

$$\begin{aligned}\int_0^{2\pi} \mathcal{II}(r,\theta)\,d\theta = & \sum_{i=0}^{\mu}\sum_{j=0}^{\lfloor (m-1)/2\rfloor}\tilde{A}_{i,j}(r^2)^{i+j} + \sum_{i=0}^{\lfloor (n-1)/2\rfloor}\sum_{j=0}^{\lfloor m/2\rfloor}\tilde{B}_{i,j}(r^2)^{i+j}\\
& +\sum_{i=0}^{\mu}\sum_{j=0}^{\lfloor l/2\rfloor}\tilde{C}_{i,j}(r^2)^{i+j} + \sum_{i=0}^{\lfloor (n-1)/2\rfloor}\sum_{j=0}^{\lfloor (l-1)/2\rfloor}\tilde{D}_{i,j}(r^2)^{i+j+1}\\
& +\sum_{i=0}^{\lfloor m/2\rfloor}\sum_{j=0}^{\lfloor k/2\rfloor}\tilde{E}_{i,j}(r^2)^{i+j-1} + \sum_{i=0}^{\lfloor (l-1)/2\rfloor}\sum_{j=0}^{\lfloor k/2\rfloor}\tilde{F}_{i,j}(r^2)^{i+j}\\
= & \, rP_3(r^2).\end{aligned} \tag{3.20}$$

where

$$\tilde{A}_{i,j} = \frac{\pi a_{2i,2}b_{2j+1,2}\alpha_{i+j+1}}{2^{i+j}(i+j+2)!}\left(-1+\frac{i+j+1}{2i+1}\right) \qquad \tilde{B}_{i,j} = -\frac{\pi a_{2i+1,2}b_{2j,2}\alpha_{i+j+1}}{2^{i+j}(i+j+2)!}$$

$$\tilde{C}_{i,j} = -\frac{\pi a_{2i,2}a_{2j,1}\alpha_{i+j}}{2^{i+j}(i+j+2)!}\left(i+j-1+\frac{2(i+j)+1}{2i+1}\right) \qquad \tilde{D}_{i,j} = -\frac{\pi(i+j)a_{2i+1,2}a_{2j+1,1}\alpha_{i+j+1}}{2^{i+j+1}(i+j+3)!}$$

$$\tilde{E}_{i,j} = -\frac{\pi(i+j)b_{2i,2}b_{2j,1}\alpha_{i+j}}{2^{i+j-1}(i+j+1)!} \qquad \tilde{F}_{i,j} = \frac{\pi a_{2i+1,1}b_{2j,1}\alpha_{i+j+1}}{2^{i+j}(i+j+2)!}$$

Then , P_3 is a polynomial in the variable r^2 of degree λ_2,
Then

$$2\pi F_{20} = r\left(P_1(r^2) + P_2(r^2) + P_3(r^2)\right).$$

To find the real positive roots of F_{20}, we must find the zeros of a polynomial in r^2 of degree λ_3.
This implies that F_{20} has at most λ_3 real positive roots. Moreover, we can choose the coefficients $a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}, c_{i,1}, c_{i,2}, d_{i,1}, d_{i,2}$ in such a way that F_{20} has exactly λ_3 real positive roots.
where

$$\lambda_3 = \max\{\lambda_1, \lambda_2\}.$$

Hence , the theorem (3.2) is proved. □

3.3.1 Application example

By choosing the polynomial functions the degrees $n = l = 3$ and $k = m = 1$ as

$$\begin{aligned}
g_{11}(x) &= -\frac{7}{5} + x, && \text{(degree 1)}\\
f_{11}(x) &= \frac{1}{10} + \frac{11}{10}x, && \text{(degree 1)}\\
g_{12}(x) &= \frac{7}{10}x, && \text{(degree 1)}\\
f_{12}(x) &= 1 + x^3, && \text{(degree 3)}\\
g_{21}(x) &= -\frac{7}{10} - \frac{7}{10}x, && \text{(degree 1)}\\
f_{21}(x) &= \frac{1}{5} - \frac{4}{5}x + \frac{4}{5}x^2 + \frac{1}{2}x^3, && \text{(degree 3)}\\
g_{22}(x) &= x, && \text{(degree 1)}\\
f_{22}(x) &= \frac{11}{5} - \frac{9}{5}x^2, && \text{(degree 2)},
\end{aligned}$$

the system (3.4) becomes

$$\begin{cases}
\dot{x} = y - \varepsilon\left(-\frac{7}{5} + x + \frac{1}{10}y + \frac{11}{10}xy\right) - \varepsilon^2\left(\frac{7}{10}x + x^3y + y\right),\\
\dot{y} = -x - \varepsilon\left(-\frac{7}{10} - \frac{7}{10}x + \frac{1}{5}y - \frac{4}{5}xy + \frac{4}{5}x^2y + \frac{1}{2}x^3y\right) - \varepsilon^2\left(x + \frac{11}{5}y - \frac{9}{5}x^2y\right).
\end{cases} \tag{3.21}$$

Transforming the system into polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\begin{cases} \dot{r} = -\varepsilon\Bigg(\dfrac{1}{5}r\sin^2\theta - \dfrac{4}{5}r^2\cos\theta\sin^2\theta + \dfrac{4}{5}r^3\cos^2\theta\sin^2\theta + \dfrac{1}{2}r^4\cos^3\theta\sin^2\theta - \dfrac{7}{10}\sin\theta \\ \quad - \dfrac{6}{10}r\cos\theta\sin\theta + \dfrac{11}{10}r^2\cos^2\theta\sin\theta - \dfrac{7}{5}\cos\theta + r\cos^2\theta\Bigg) \\ \quad - \varepsilon^2\left(\dfrac{11}{5}r\sin^2\theta - \dfrac{9}{5}r^3\cos^2\theta\sin^2\theta + 2r\cos\theta\sin\theta + r^4\cos^4\theta\sin\theta + \dfrac{7}{10}r\cos^2\theta\right), \\ \dot{\theta} = -1 - \dfrac{\varepsilon}{r}\Bigg(\dfrac{1}{5}r\cos\theta\sin\theta - \dfrac{4}{5}r^2\cos^2\theta\sin\theta + \dfrac{4}{5}r^3\cos^3\theta\sin\theta + \dfrac{1}{2}r^4\cos^4\theta\sin\theta \\ \quad - \dfrac{7}{10}\cos\theta - \dfrac{7}{10}\cos^2\theta - \dfrac{1}{10}r\sin^2\theta - \dfrac{11}{10}r^2\cos\theta\sin^2\theta + \dfrac{7}{5}\sin\theta - r\cos\theta\sin\theta\Bigg) \\ \quad - \dfrac{\varepsilon^2}{r}\Bigg(\dfrac{11}{5}r\cos\theta\sin\theta - \dfrac{9}{5}r^3\cos^3\theta\sin\theta + r\cos^2 - r^4\cos^3\theta\sin^2\theta \\ \quad - r\cos\theta\sin^2\theta - \dfrac{7}{10}r\cos\theta\sin\theta\Bigg). \end{cases}$$

Next, we derive the first-order differential equation $\frac{dr}{d\theta}$ by dividing $\dot{r}$ by $\dot{\theta}$, and expand the expression keeping only terms of first order in ε. We obtain

$$\frac{dr}{d\theta} = \varepsilon F_1(\theta, r) + \varepsilon^2 F_2(\theta, r) + O(\varepsilon^3),$$

where the functions are given by

$$\begin{aligned} F_1(\theta, r) = &\frac{1}{5}r\sin^2\theta - \frac{4}{5}r^2\cos\theta\sin^2\theta + \frac{4}{5}r^3\cos^2\theta\sin^2\theta + \frac{1}{2}r^4\cos^3\theta\sin^2\theta \\ &- \frac{7}{10}\sin\theta - \frac{6}{10}r\cos\theta\sin\theta + \frac{11}{10}r^2\cos^2\theta\sin\theta - \frac{7}{5}\cos\theta + r\cos^2\theta, \end{aligned}$$

and

$$F_2(\theta, r) = \mathcal{I}(r, \theta) + r\mathcal{II}(r, \theta),$$

where

$$\mathcal{I}(\theta, r) = \frac{11}{5}r\sin^2\theta - \frac{9}{5}r^3\cos^2\theta\sin^2\theta + 2r\cos\theta\sin\theta + r^4\cos^4\theta\sin\theta + \frac{7}{10}r\cos^2\theta,$$

and

$$\begin{aligned} \mathcal{II}(\theta, r) = -&\Bigg(\frac{1}{5}\sin^2\theta - \frac{4}{5}r\cos\theta\sin^2\theta + \frac{4}{5}r^2\cos^2\theta\sin^2\theta + \frac{1}{2}r^3\cos^3\theta\sin\theta \\ &- \frac{7}{10}r^{-1}\sin\theta - \frac{7}{10}\cos\theta\sin\theta - \frac{7}{10}r^{-1}\cos\theta + \cos^2\theta\Bigg) \\ &\Bigg(\frac{1}{5}\cos\theta\sin\theta - \frac{4}{5}r\cos^2\theta\sin\theta + \frac{4}{5}r^2\cos^3\theta\sin\theta + \frac{1}{2}r^3\cos^4\theta\sin\theta \\ &- \frac{7}{10}r^{-1}\cos\theta - \frac{7}{10}\cos^2\theta - \frac{1}{10}\sin^2\theta - \frac{11}{10}r\cos\theta\sin^2\theta + \frac{7}{5}r^{-1}\sin\theta - \cos\theta\sin\theta\Bigg). \end{aligned}$$

Since $F_{10}(r) \equiv 0$, we can proceed to compute $F_{20}(r)$.

Then, the second-order averaged function $F_{20}(r)$ is computed and given by

$$F_{20}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial F_1}{\partial r}(\theta, r) \left(\int_0^{\theta} F_1(\psi, r)\, d\psi \right) + F_2(\theta, r) \right) d\theta$$

As a result

$$F_{20}(r) = \frac{r}{960} \left(384 - \frac{1008}{5} r - 270 r^2 + \frac{176}{5} r^3 + \frac{136955}{2000} r^4 - \frac{144}{25} r^5 - \frac{9}{4} r^6 \right).$$

This equation has exactly three positive roots, $r_1 \approx 1$, $r_2 \approx 2$, $r_3 \approx 4$. According to theorem 3.2, the system (3.21) has three distinct limit cycles, whose existence can be proved using the second-order averaging theory. we have that $\lambda_3 = \left[\frac{n-1}{2}\right] + \left[\frac{l-1}{2}\right] + 1 = 3$ Therefore, the system attains the maximum possible number of limit cycles.

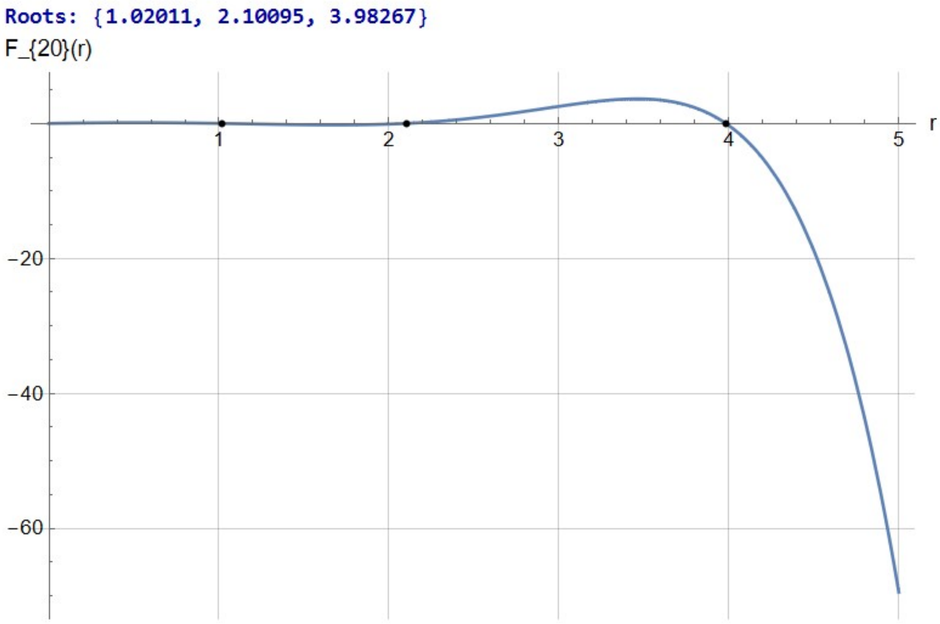

Figure 3.1: Graph of the second-order averaged function $F_{20}(r)$.

3.4 Appendix A

During the computation of the averaged functions F_{10} and F_{20}, several trigonometric integrals involving powers of $\sin\theta$ and $\cos\theta$ arise. These integrals can be evaluated directly or by using classical references such as [46]. Below, we summarize the main integrals with their exact values and validity conditions.

- $\displaystyle\int_0^{2\pi} \cos^{2i}\theta \sin^2\theta\, d\theta = \frac{\pi \alpha_i}{2^i (i+1)!}$,
- $\displaystyle\int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta\, d\theta = 0$
- $\displaystyle\int_0^{2\pi} \cos^{i}\theta \sin\theta\, d\theta = 0$,
- $\displaystyle\int_0^{2\pi} \cos^{i+1}\theta \sin\theta\, d\theta = 0$
- $\displaystyle\int_0^{2\pi} \cos^{2i+1}\theta\, d\theta = 0$,
- $\displaystyle\int_0^{2\pi} \cos^{2i+2}\theta\, d\theta = \frac{2\pi \alpha_{i+1}}{2^{i+1}(i+1)!}$

where $\alpha_i = 1 \cdot 3 \cdot 5 \cdots (2i-1)$ and $\alpha_{i+1} = (2i+1)\alpha_i$ and $i \geq 0$
During the computation of second-order terms in the averaging expansions, expressions containing $\cos^2\theta$ naturally appear. Using the double-angle identity

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2},$$

integrals modulated by $\cos(2\theta)$ arise and play an essential role in simplifying the expansions

- $$\int_0^{2\pi} \cos^{2i}\theta \cos(2\theta)\, d\theta = \frac{\pi i \alpha_i}{2^{i-1}(i+1)!},$$
- $$\int_0^{2\pi} \cos^{2i}\theta \sin^2\theta \cos(2\theta)\, d\theta = \frac{\pi(i-1)\alpha_i}{2^{i+1}(i+2)!}, \quad i \geq 1.$$

For integration with respect to a general angle θ, we use the following useful identities for indefinite integrals.

- $$\int_0^{\theta} \cos^{2i+1}\psi\, d\psi = \sum_{l=0}^{i} \gamma_{i,l} \sin\big((2l+1)\theta\big), \text{ where } \gamma_{i,l} = \frac{1}{2^{2i}}\tbinom{2i+1}{i-l}\tfrac{1}{2l+1}$$
- $$\int_0^{\theta} \cos^{2i}\psi\, d\psi = \frac{1}{2^{2i}}\binom{2i}{i}\theta + \sum_{l=1}^{i} \beta_{i,l}\sin(2l\theta),$$
- $$\int_0^{\theta} \cos^{2i+2}\psi\, d\psi = \frac{1}{2^{2i+2}}\binom{2i+2}{i+1}\theta + \sum_{l=1}^{i} \beta_{i,l}\sin(2l\theta)$$
- $$\int_0^{\theta} \cos^{i}\psi \sin\psi\, d\psi = \frac{1}{i+1}\big(1-\cos^{i+1}\theta\big)$$
- $$\int_0^{\theta} \cos^{i+1}\psi \sin\psi\, d\psi = \frac{1}{i+2}\big(1-\cos^{i+2}\theta\big)$$
- Identity
$$\frac{1}{2^{2i}}\binom{2i}{i}\theta - \frac{2i+2}{2i+1}\cdot\frac{1}{2^{2i+2}}\binom{2i+2}{i+1}\theta = 0$$
- $$\int_0^{2\pi} \cos^{i}\theta \sin\theta \sin((2l+1)\theta)\, d\theta = 0, \quad l \geq 0$$
- $$\int_0^{2\pi} \cos^{i}\theta \sin\theta \sin(2l\theta)\, d\theta = 0, \quad l \geq 0$$
- $$\int_0^{2\pi} \cos^{i}\theta \sin^2\theta \sin((2l+1)\theta)\, d\theta = 0, \quad l \geq 0$$
- $$\int_0^{2\pi} \cos^{i}\theta \sin^2\theta \sin(2l\theta)\, d\theta = 0, \quad l \geq 0$$
- $$\int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin((2l+1)\theta)\, d\theta = \pi C_{i,l}, \quad l \geq 0$$

- $\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin(2l\theta)\, d\theta = \pi K_{i,l}, \quad l \geq 1$

Remark 3.1. All integrals of odd functions over the interval $[0, 2\pi]$, such as those involving $\cos^{2k+1}\theta$ or $\cos^k\theta \sin\theta$, vanish due to symmetry. That is,

$$\int_0^{2\pi} f(\theta)\, d\theta = 0, \quad \text{if } f(\theta) \text{ is an odd function.}$$

Conclusion

In this thesis of master, we conducted a general study of limit cycles in planar polynomial differential systems, recognizing their significance in understanding the periodic behavior of dynamical systems. We began by presenting the fundamental concepts related to limit cycles, including their definitions and properties, and provided a historical overview of the main challenges, particularly the second part of Hilbert's 16^{th} problem.
We then focused on a specific class of polynomial differential systems inspired by generalized Liénard systems, analyzing the bifurcation of limit cycles resulting from small polynomial perturbations applied to a linear center. Our approach relied on the first- and second-order averaging theory to establish accurate upper bounds for the number of limit cycles based on the degrees of the perturbations.
The main contribution of this research lies in the construction of an original applied example that attains the computed theoretical upper bound, thereby emphasizing the effectiveness of the analytical method used. Furthermore, our results align with prior studies, particularly those by Llibre and Valls, thus reinforcing the validity and reliability of the theoretical framework adopted in this study.

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الرتبة: أستاذ محاضر أ

بصفتي المشرف المسؤول عن تصحيح **مذكرة التخرج ماستر** المعنونة بـ :

Upper bouds for the Number of limits Cycles for a class of polynomial Differential Systems

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