الجمهوريه الجزائريه الديمقراطيه الشعبيه

Democratic and Popular Republic of Algeria

وزارة التعليم العالي والبحث العلمي

Ministry of Higher Education and Scientific Research

جامعه غردايه

Ghardaia University

Registration No.

/ / / / /

كليه العلوم والتكنولوجيا

Faculty of Science and Technology

قسم الرياضيات والإعلام الالي

Department of Mathematics and Computer Science

Thesis Submitted in Partial Fulfillment of the Requirements for the

Master's Degree

Domain : Mathematics and Computer Science

Faculty : Mathematics

Specialization : Functional analysis and application

Theme

Nonlinear Eigenvalue Problem Involving The P(x) Laplacian Operator,Existence

Presented by : Ms. Abir Boukraa

Publicly supported the 10/06/2026

In front of the jury composed of :

Mr.Chenini Toufik	MCB	**Ghardaia University**	**President**
Mr. Lalmi Abdelatif	MCB	**Ghardaia University**	**Supervisor**
Mr.Saouli Nabil	MCB	**Ghardaia University**	**Examiner**
Mr.Merabet Brahim	MCB	**Ghardaia University**	**Examiner**

Academic year : 2025 / 2026

Abstract

In this dissertation, we analyze a nonlinear eigenvalue elliptic problem, inspired by models from mathematical physics, particularly in nonlinear quantum mechanics and field theory. More specifically, we examine the $p(x)$-Laplacian equation of the form:

$$\begin{cases} -\Delta_{p(x)}\big(u(x)\big) = \lambda \, \|u(x)\|_{q(x)}^{p(x)-q(x)} \left|u(x)\right|^{q(x)-2} u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

The given equation is defined on an open and regular domain Ω, with $\lambda \in \mathbb{R}$ representing a spectral parameter and u being the corresponding eigenfunction, where:

$$1 < p_- \leq p(x) \leq p_+ < +\infty, \qquad 1 < q_- \leq q(x) \leq q_+ < p^*(x)$$

with

$$p^*(x) = \begin{cases} \dfrac{n\, p(x)}{n - p(x)}, & p_+ < n, \\ \infty, & p_+ \geq n. \end{cases} \tag{2}$$

Such that:

$$r_- = \operatorname*{ess\,inf}_{x \in \Omega} r(x), \qquad r_+ = \operatorname*{ess\,sup}_{x \in \Omega} r(x).$$

The operator $p(x)$-Laplacian is defined by:

$$\Delta_{p(x)}\big(u(x)\big) = \operatorname{div}\Big(|\nabla u(x)|^{p(x)-2}\, \nabla u(x)\Big),$$

and the notation $\|u\|_{q(x)}$ is the norm of u in the space $L^{q(x)}(\Omega)$.

We look into the right variational methods for this case, along with whether solutions exist and how many there are, using concepts from variable exponent Lebesgue and Sobolev space theories, as well as nonlinear spectral theory.

Keywords: variable exponent spaces, $p(x)$-Laplacian, nonlinear eigenvalue problems, Sobolev embeddings, variational methods.

ملخص

في هذه المذكرة، نُحلِّل مسألةً لقيم ذاتية إهليجية لاخطية، مستوحاةً من نماذج الفيزياء الرياضية، ولا سيما ميكانيكا الكم اللاخطية ونظرية الحقل. وبصفة أكثر تحديدًا، ندرس معادلة $p(x)$-لابلاسي بالشكل التالي:

$$\begin{cases} -\Delta_{p(x)}\big(u(x)\big) = \lambda \left\|u(x)\right\|_{q(x)}^{p(x)-q(x)} \left|u(x)\right|^{q(x)-2} u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

المعادلةُ المُعطاة مُعرَّفةٌ على مجالٍ مفتوح ومنتظم Ω، حيث يمثّل $\lambda \in \mathbb{R}$ معاملًا طيفيًا، و u هي الدالة الذاتية المقابلة، مع اشتراط:

$$1 < p_- \le p(x) \le p_+ < +\infty, \qquad 1 < q_- \le q(x) \le q_+ < p^*(x)$$

حيث:

$$p^*(x) = \begin{cases} \dfrac{n\,p(x)}{n-p(x)}, & \text{إذا كان } p_+ < n, \\ \infty, & \text{إذا كان } p_+ \ge n. \end{cases} \tag{2}$$

بحيث:

$$r_- = \operatorname*{ess\,inf}_{x\in\Omega} r(x), \qquad r_+ = \operatorname*{ess\,sup}_{x\in\Omega} r(x).$$

ويُعرَّف مؤثِّر $p(x)$-لابلاسي بالصيغة:

$$\Delta_{p(x)}\big(u(x)\big) = \operatorname{div}\Big(|\nabla u(x)|^{p(x)-2}\,\nabla u(x)\Big),$$

والرمز $\|u\|_{q(x)}$ يدلّ على نظيم u في فضاء $L^{q(x)}(\Omega)$.

نبحث في الأساليب التباينية الملائمة لهذه المسألة، ونتقصّى وجود الحلول وتعدّدها، مستعينين بمفاهيم من نظريات فضاءات ليبيغ وسوبوليف ذات الأس المتغير، إضافةً إلى نظرية الطيف اللاخطية.

الكلمات المفتاحية: فضاءات الأس المتغيّر، مؤثر (p(x))-لابلاس، مسائل القيم الذاتية غير الخطية، تضمينات سوبوليف، الطرق التغايرية.

Résumé

Dans ce mémoire, nous analysons un problème aux valeurs propres elliptique non linéaire, inspiré de modèles issus de la physique mathématique, notamment en mécanique quantique non linéaire et en théorie des champs. Plus précisément, nous étudions l'équation $p(x)$-Laplacien sous la forme suivante :

$$\begin{cases} -\Delta_{p(x)}\big(u(x)\big) = \lambda \, \|u(x)\|_{q(x)}^{p(x)-q(x)} \, \big|u(x)\big|^{q(x)-2} \, u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

L'équation donnée est définie sur un domaine ouvert et régulier Ω, où $\lambda \in \mathbb{R}$ représente un paramètre spectral et u est la fonction propre correspondante, avec les conditions :

$$1 < p_- \leq p(x) \leq p_+ < +\infty, \qquad 1 < q_- \leq q(x) \leq q_+ < p^*(x)$$

où

$$p^*(x) = \begin{cases} \dfrac{n\,p(x)}{n - p(x)}, & \text{si } p_+ < n, \\ \infty, & \text{si } p_+ \geq n. \end{cases} \tag{2}$$

avec :

$$r_- = \operatorname*{ess\,inf}_{x\in\Omega} r(x), \qquad r_+ = \operatorname*{ess\,sup}_{x\in\Omega} r(x).$$

L'opérateur $p(x)$-Laplacien est défini par :

$$\Delta_{p(x)}\big(u(x)\big) = \operatorname{div}\Big(|\nabla u(x)|^{p(x)-2} \, \nabla u(x)\Big),$$

et la notation $\|u\|_{q(x)}$ désigne la norme de u dans l'espace $L^{q(x)}(\Omega)$.

Nous explorons les méthodes variationnelles appropriées à ce problème, ainsi que l'existence et la multiplicité des solutions, en utilisant les concepts des théories des espaces de Lebesgue et de Sobolev à exposant variable, ainsi que la théorie spectrale non linéaire.

Mots clés : espaces à exposant variable, opérateur (p(x))-Laplacien, problèmes aux valeurs propres non linéaires, plongements de Sobolev, méthodes variationnelles.

Dedication

I dedicate this modest work to:

My dear parents for all their sacrifices, unwavering love, support and endless patience, I ask God to provide you with happiness and a long life.

My dear brothers, sisters, and all my family for their encouragement throughout my academic career.

To all my colleagues and friends for the unforgettable moments of pure feelings and happiness. May Allah bless you with success.

To all my professors, I would like to express my sincere gratitude for their guidance, knowledge, and dedication throughout my academic journey.

Sincerely
ABIR

Acknowledgements

First and foremost, I thank Allah Almighty for His guidance and blessings, without which this work would not have been completed.

I would also like to express my sincere gratitude to my family for their continuous support, patience, and constant encouragement throughout my academic journey.

I am equally thankful to my friends for their support, encouragement, and for all the moments of kindness they shared with me during this period.

Finally, I extend my heartfelt appreciation to everyone who stood by my side during this stage, and to all those who offered me kind words, sincere prayers, or moral support that helped me continue.

CONTENTS

NOTATION

- $\Omega \subset \mathbb{R}^N$: bounded open domain with Lipschitz boundary $\partial\Omega$.
- $N \in \mathbb{N}$: space dimension.
- $x \in \Omega$: spatial variable.
- $p : \Omega \to [1, +\infty)$: measurable variable exponent.
- p^-, p^+:
$$p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x).$$
- $L^{p(x)}(\Omega)$: variable exponent Lebesgue space.
- $W^{1,p(x)}(\Omega)$: variable exponent Sobolev space.
- $W_0^{1,p(x)}(\Omega)$: closure of $C_c^\infty(\Omega)$.
- $C_c^\infty(\Omega)$: space of smooth compactly supported functions.
- $C(\overline{\Omega})$: space of continuous functions.
- $\|u\|_{p(x)}$: Luxemburg norm in $L^{p(x)}(\Omega)$.
- $\rho_{p(x)}(u)$:
$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$
- $L^p(\Omega), W^{1,p}(\Omega)$: classical Lebesgue and Sobolev spaces.
- $p'(x)$:
$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

- $L^{p'(x)}(\Omega)$: dual space of $L^{p(x)}(\Omega)$.
- $W^{-1,p'(x)}(\Omega)$: dual space of $W_0^{1,p(x)}(\Omega)$.
- ∇u: weak gradient.
- Δu: Laplacian operator.
- $\Delta_{p(x)} u$: variable exponent $p(x)$-Laplacian.
- $A(u)$: nonlinear operator (typically $p(x)$-Laplacian type).
- $f(x)$: given function / source term.
- φ: test function.
- $\langle \cdot, \cdot \rangle$: duality pairing.
- $|\cdot|$: Euclidean norm in $\mathbb{R}^N$.
- $|\Omega|$: Lebesgue measure of Ω.
- dx: Lebesgue measure element.
- $u_n \rightharpoonup u$: weak convergence.
- $u_n \to u$: strong convergence.
- $u_n \to u$ in measure: convergence in measure.
- ϵ: small positive parameter.
- C, C_1, C_2: generic positive constants.
- $\int_\Omega$: Lebesgue integral over Ω.

INTRODUCTION

The study of nonlinear partial differential equations has witnessed significant development over the past decades, particularly in connection with problems involving nonstandard growth conditions. Among these, equations driven by the $p(x)$-Laplacian operator have attracted considerable attention due to their ability to model heterogeneous and anisotropic phenomena arising in physics, engineering, and applied sciences. Unlike the classical p-Laplacian, the variable exponent framework allows the growth rate to depend on the spatial position, leading to a richer and more flexible mathematical structure.

This dissertation is devoted to the analysis of the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_{p(x)}\big(u(x)\big) = \lambda \left\|u\right\|_{q(x)}^{p(x)-q(x)} \left|u\right|^{q(x)-2}u, & x \in \Omega, \\ \qquad u(x) = 0, & x \in \partial\Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain, $\lambda \in \mathbb{R}$ is a spectral parameter, and $p(x), q(x)$ are variable exponents satisfying appropriate growth conditions. This type of problem is inspired by models from mathematical physics, particularly in nonlinear quantum mechanics and field theory, and involves a rich interplay between elliptic analysis, nonlinear spectral theory, and physical modeling.

The mathematical analysis of such problems requires a careful combination of tools from variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, which generalize the classical function spaces and present significant new challenges due to the lack of homogeneity and the failure of standard techniques.

This dissertation is organized as follows.

The first chapter is devoted to the functional framework. We introduce the main properties of variable exponent spaces, including modular inequalities, embedding theorems, and compactness results. Particular attention is given to the role of log-Hölder continuity, which ensures the validity of key analytical tools such as density results and boundedness of operators. We also present the weak formulation of our main problem.

In the second chapter, we establish several auxiliary results that are essential for the study of our problem. These include monotonicity methods, weak convergence techniques, compact embedding results, and eigenvalue properties of the $p(x)$-Laplacian operator.

The third chapter contains the core contribution of this dissertation. Motivated by the pioneering work of Franzina and Lindqvist [7], we extend and adapt their results to the variable exponent setting involving two exponents $p(x)$ and $q(x)$. More precisely, we establish the following main results:

- Boundedness and regularity of eigenfunctions (Theorem 3.1.1).
- The first eigenfunction has constant sign in Ω (Corollary 3.1.2).
- Simplicity of the first eigenvalue $\lambda_1(p(x), q(x))$ (Theorem 3.2.1 and Theorem 3.2.5).
- Uniqueness of positive solutions (Theorem 3.3.1 and Corollary 3.3.2).
- The spectrum $\sigma(p(x), q(x))$ is a closed set (Theorem 3.4.1).
- The eigenvalues diverge to infinity (Theorem 3.4.2).

CHAPTER 1

PRELIMINARIES

1.1 Functional framework

We give some results about the Lebesgue and Sobolev spaces with variable exponents, which are well known in [3, 9, 11]. The following notations will be used in the sequel:

$$\|\upsilon\|_{p(\cdot)} = \|\upsilon\|_{L^{p(\cdot)}(\Omega)}, \qquad \|\upsilon\|_{p_-} = \|\upsilon\|_{L^{p_-}(\Omega)}, \qquad \|\upsilon\|_{p_+} = \|\upsilon\|_{L^{p_+}(\Omega)}.$$

Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\cdot) : \Omega \to [1, \infty]$, where Ω is a bounded domain in $\mathbb{R}^n$.

Throughout this work, we study the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_{p(x)}\big(u(x)\big) = \lambda \, \|u(x)\|_{q(x)}^{p(x)-q(x)} \, \big|u(x)\big|^{q(x)-2} u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with Lipschitz boundary, $\lambda \in \mathbb{R}$ is a spectral parameter, and the exponents satisfy:

$$1 < p_- \le p(x) \le p_+ < +\infty, \qquad 1 < q_- \le q(x) \le q_+ < p^*(x) \tag{1.2}$$

with

$$p^*(x) = \begin{cases} \dfrac{n\,p(x)}{n - p(x)}, & p_+ < n, \\ \infty, & p_+ \ge n. \end{cases}$$

A function $p(\cdot)$ is said to satisfy the *log-Hölder continuous condition* in Ω if

$$\forall x, y \in \Omega \text{ with } |x - y| \le \frac{1}{2}, \qquad |p(x) - p(y)| \le \frac{C_0}{-\log(|x - y|)}, \tag{1.3}$$

where $C_0 > 0$ is a constant.We say that $p(\cdot)$ satisfies the log-Hölder decay condition in Ω if

$$\forall x \in \Omega, \qquad |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \tag{1.4}$$

where $p_\infty = \lim_{|x|\to\infty} p(x)$ and $C_\infty > 0$ are constants.By $\mathcal{P}^{\log}(\Omega)$ we denote the class of variable exponents:

$$\mathcal{P}^{\log}(\Omega) = \left\{ p(\cdot) \in \mathcal{P}(\Omega) : \frac{1}{p(\cdot)} \text{ is globally log-Hölder continuous} \right\}.$$

Note that $r(\cdot) \in \mathcal{P}(\Omega)$ is globally log-Hölder continuous in Ω if $r(\cdot)$ satisfies both conditions (1.3) and (1.4).

Definition 1 (Log-Hölder constant)**.** Let $p \in P^{\log}(\Omega)$. The log-Hölder constant of p is defined by

$$C_{\log}(p) = \sup_{\substack{x,y\in\Omega \\ |x-y|\leq\frac{1}{2}}} |p(x) - p(y)| \cdot \log \frac{1}{|x-y|}, \tag{1.5}$$

which is the smallest constant $C_0 > 0$ for which the log-Hölder continuity condition

$$|p(x) - p(y)| \leq \frac{C_0}{-\log|x-y|}, \quad \forall\, x, y \in \Omega, \ |x - y| \leq \frac{1}{2}, \tag{1.6}$$

is satisfied. We say that $p \in P^{\log}(\Omega)$ if and only if $C_{\log}(p) < \infty$.

Proposition 1 (see [2])**.** *Given a domain Ω:*

1. *If $p(\cdot)$ fulfills 1.3, then it is uniformly continuous and fulfills 1.4 on every bounded subset $E \subset \Omega$.*

2. *If $p(\cdot) \in \mathcal{P}(\Omega)$ and $p_+ < +\infty$, then $1/p(\cdot)$ satisfies either conditions1.3, 1.4 or both if and only if $p(\cdot)$ is also.*

Remark 1. From Proposition 1 we deduce that if Ω is bounded, $p(\cdot) \in C(\overline{\Omega})$ and satisfies the conditions 1.2, 1.3, then $p(\cdot), 1/p(\cdot) \in \mathcal{P}^{\log}(\Omega)$.

Now we define the $p(\cdot)$ modular of a measurable function $u : \Omega \to \mathbb{R}$ as follows:

$$\rho_{p(\cdot)}(u) = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}\, dx + \operatorname*{ess\,sup}_{x\in\Omega_\infty} |u(x)|,$$

where

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ is the class of those measurable functions u defined on Ω as follows:

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \in \mathcal{P}(\Omega),\ \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

We define the Luxemburg norm of the space $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{p(x)} = \inf \left\{ \kappa > 0 : \rho_{p(x)}\left(\frac{u}{\kappa}\right) \leq 1 \right\}.$$

The space $L^{p(\cdot)}(\Omega)$ equipped with this norm is a Banach space.Now we present some results that concern variable exponent Lebesgue spaces.

Proposition 2. *Let $u \in L^{p(\cdot)}(\Omega)$, $(u_n)_{n\in\mathbb{N}} \subset L^{p(\cdot)}(\Omega)$, then*

1. $\|u\|_{L^{p(x)}} < 1 \iff \rho_{p(x)}(u) < 1$; $\|u\|_{L^{p(x)}} = 1 \iff \rho_{p(x)}(u) = 1$; $\|u\|_{L^{p(x)}} > 1 \iff \rho_{p(x)}(u) > 1$.
2. *If* $\|u\|_{L^{p(x)}} > 1$, *then* $\|u\|_{L^{p(x)}}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{L^{p(x)}}^{p^+}$.
3. *If* $\|u\|_{L^{p(x)}} < 1$, *then* $\|u\|_{L^{p(x)}}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{L^{p(x)}}^{p^-}$.
4. $\|u_n\|_{L^{p(x)}} \to 0 \iff \rho_{p(x)}(u_n) \to 0$.
5. $\|u_n\|_{L^{p(x)}} \to +\infty \iff \rho_{p(x)}(u_n) \to +\infty$.

Proposition 3. *(see [4]). Let $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$. Then the following statements are equivalent to each other:*

1. $\lim_{n\to\infty} \|u_n - u\|_{L^{p(x)}} = 0$,
2. $\lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$,
3. *u_n converges to u in measure and* $\lim_{n\to\infty} \rho_{p(x)}(u_n) = \rho_{p(x)}(u)$.

Proposition 4. *(see [4]). Assume that Ω has finite measure, $p_1(x), p_2(x) \in \mathcal{P}(\Omega)$. If $p_1(x) \leq p_2(x)$ for almost all $x \in \Omega$ and $1 \leq p_i- \leq p_i+ < +\infty$, then $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ and the embedding is continuous.*

Proposition 5. *(generalized Hölder inequality, see [9, 12]). Let $p(\cdot), p'(\cdot) \in \mathcal{P}(\Omega)$ such that $p^- > 1$ and*

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1, \quad a.e.\ x \in \Omega.$$

Then the inequality

$$\int_\Omega |u(x)v(x)|\,dx \le \left(\frac{1}{p_-}+\frac{1}{p'_-}\right)\|u\|_{L^{p(x)}}\,\|v\|_{L^{p'(x)}}$$

holds for every $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$.
The interpolation inequality given in [14]is also needed.

Lemma 1.1.1. *If* $1 \le p_0 < p_\theta < p_1 \le \infty$, *then*

$$\|u\|_{p_\theta} \le \|u\|_{p_0}^{1-\theta}\|u\|_{p_1}^{\theta}$$

for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ *with* $\theta \in (0,1)$ *defined by*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0}+\frac{\theta}{p_1}.$$

Generally, variable-exponent Lebesgue spaces and classical Lebesgue spaces are similar in many properties. Moreover, several results that concern variable-exponent Lebesgue spaces have been obtained, see for instance [3, 9] for the following statements.

- *The modular* $\rho_{p(\cdot)}$ *and the norm* $\|\cdot\|_{L^{p(\cdot)}}$ *are lower semi-continuous with respect to (sequential) weak convergence and almost everywhere convergence.*
- *The space* $L^{p(\cdot)}(\Omega)$ *is reflexive if and only if* $1 < p_- < p_+ < \infty$.
- *Continuous functions are dense if* $p_+ < \infty$.

The space $W^{1,p(\cdot)}(\Omega)$ *is the variable-exponent Sobolev space defined by*

$$W^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}.$$

This space endowed with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}} + \|\nabla u\|_{L^{p(x)}},$$

is a Banach space.

We define $W_0^{1,p(x)}(\Omega)$ *as the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p(\cdot)}(\Omega)$ *with respect to the norm* $\|u\|_{1,p(\cdot)} = \|\nabla u\|_{L^{p(\cdot)}}$, *since* $p(\cdot)$ *satisfies 1.2 and 1.3. The space* $W^{-1,p'(\cdot)}(\Omega)$ *is the dual space of* $W_0^{1,p(x)}(\Omega)$, *where* $p'(x)$ *is the conjugate exponent function of* $p(x)$ *such that*

$$\frac{1}{p(x)}+\frac{1}{p'(x)} = 1.$$

We denote $X_0 = W_0^{1,p(x)}(\Omega)\setminus\{0\}$, *with the norm*

$$\|u\|_{1,p(x)} = \|\nabla u\|_{L^{p(x)}}.$$

Proposition 6. *(see [3], Theorem 8.1.13). Let $p \in \mathcal{P}(\mathbb{R}^n)$. The space $W_0^{1,p(\cdot)}(\Omega)$ is a Banach space, which is separable if $p(\cdot)$ is bounded,and reflexive and uniformly convex if $1 < p_- \leq p_+ < +\infty$.*

Proposition 7. *Assume that $1 \leq \operatorname{ess\,inf}_{x\in\Omega} p_i(x) \leq p_i(x) \leq \operatorname{ess\,sup}_{x\in\Omega} p_i(x) < +\infty$, $(i = 1, 2)$. If $p_1(x) \leq p_2(x)$, then $W^{1,p_2(\cdot)}(\Omega) \hookrightarrow W^{1,p_1(\cdot)}(\Omega)$.*

Proposition 8. *(see [11]). If Ω is bounded, $p(x) \in C(\overline{\Omega})$ such that $p_+ < n$ and $q(x)$ defined in Ω with $q_- \geq 1$ and*

$$\operatorname*{ess\,inf}_{x\in\Omega}(p^*(x) - q(x)) > 0,$$

then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

Proposition 9. *(see [2], Theorem 6.29, and [3], Theorem 8.3.1).*

1. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < N$, suppose that the maximal operator is bounded on $L^{(p(\cdot)/n)'}(\Omega)$. Then $W_0^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$,and*

$$\|u\|_{p^*(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}.$$

2. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 \leq p_- \leq p_+ < N$. Then for every $u \in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$\|u\|_{L^{p^*(\cdot)}} \leq C\|\nabla u\|_{L^{p(\cdot)}}$$

holds with a constant C depending only on the dimension N, $C_{\log}(p)$, and p_+.

Theorem 1.1.2 (Strong Maximum Principle, see [15])**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, and let $u \in W_0^{1,p(x)}(\Omega)$ be a weak solution of*

$$-\Delta_{p(x)}u = f(x) \quad \text{in } \Omega,$$

where $f \geq 0$ a.e. in Ω. If $u \geq 0$ in Ω and $u \not\equiv 0$, then $u > 0$ in Ω.

Proposition 10 (Weak Maximum Principle)**.** *Let $u \in W_0^{1,p(x)}(\Omega)$ be a weak solution of*

$$-\Delta_{p(x)}u \geq 0 \quad \text{in } \Omega.$$

Then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u = 0.$$

Theorem 1.1.3 (Picone's Identity for the $p(x)$-Laplacian, see [?]). *Let $u, v \in W_0^{1,p(x)}(\Omega)$ with $v > 0$ in Ω. Define the Picone functional:*

$$L(u,v) = |\nabla u|^{p(x)} - |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right).$$

Then the following identity holds:

$$L(u,v) \geq 0 \quad a.e. \ in \ \Omega,$$

with equality if and only if $\nabla\left(\frac{u}{v}\right) = 0$ a.e. in Ω, i.e., $u = cv$ for some constant $c > 0$.

Moreover, the following integral inequality holds:

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right) dx \leq \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Theorem 1.1.4 (Moser Iteration, see [10]). *Let $u \in W_0^{1,p(x)}(\Omega)$ be a weak solution of*

$$-\Delta_{p(x)} u = f(x,u) \quad in \ \Omega,$$

where $|f(x,u)| \leq C|u|^{q(x)-1}$ with $1 < q(x) < p^*(x)$. Then $u \in L^\infty(\Omega)$, and there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W_0^{1,p(x)}(\Omega)}.$$

Definition 2 (Local $C^{1,\alpha}$ Regularity). Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain and $\alpha \in (0,1)$. We say that a function u belongs to the space $C^{1,\alpha}_{\text{loc}}(\Omega)$ if $u \in C^1(\Omega)$ and for every compact subset $K \subset \Omega$, there exists a constant $C = C(K) > 0$ such that

$$|\nabla u(x) - \nabla u(y)| \leq C\,|x-y|^\alpha \quad \forall\, x, y \in K. \tag{1.7}$$

In other words, the first-order partial derivatives of u are locally Hölder continuous with exponent α in Ω.

Theorem 1.1.5 (Hölder Regularity, see [15]). *Let $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of*

$$-\Delta_{p(x)} u = f \quad in \ \Omega,$$

with $f \in L^\infty(\Omega)$. Then there exists $\alpha \in (0,1)$ such that

$$u \in C^{1,\alpha}_{\text{loc}}(\Omega),$$

and the first derivatives of u are locally Hölder continuous in Ω.

Now we define what a weak solution of problem 1.1 means.

Definition 3 (Weak solution)**.** A function

$$u \in W_0^{1,p(x)}(\Omega), \qquad u \neq 0,$$

is called a weak solution of problem (1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega |u|^{q(x)-2}u\varphi \, dx$$

for all

$$\varphi \in W_0^{1,p(x)}(\Omega).$$

Proof.
We derive the weak formulation of problem (1.1). Let $\varphi \in W_0^{1,p(x)}(\Omega)$ be an arbitrary test function. Multiplying both sides of the equation by φ and integrating over Ω, we obtain:

$$-\int_\Omega \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big)\, \varphi \, dx = \lambda \, \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega |u|^{q(x)-2} \, u \, \varphi \, dx.$$

Applying Green's formula to the left-hand side:

$$\int_\Omega \operatorname{div}(\mathbf{F}) \, \varphi \, dx = -\int_\Omega \mathbf{F} \cdot \nabla\varphi \, dx + \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) \, \varphi \, d\sigma,$$

with $\mathbf{F} = |\nabla u|^{p(x)-2}\nabla u$, we get:

$$-\int_\Omega \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big)\, \varphi \, dx = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx - \int_{\partial\Omega} |\nabla u|^{p(x)-2}(\nabla u \cdot \mathbf{n}) \, \varphi \, d\sigma.$$

Since $\varphi \in W_0^{1,p(x)}(\Omega)$, we have $\varphi\big|_{\partial\Omega} = 0$, and therefore the boundary integral vanishes:

$$\int_{\partial\Omega} |\nabla u|^{p(x)-2}(\nabla u \cdot \mathbf{n}) \, \varphi \, d\sigma = 0.$$

Hence, we arrive at the following weak formulation: find $u \in W_0^{1,p(x)}(\Omega)$, $u \neq 0$, such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \, \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega |u|^{q(x)-2} \, u \, \varphi \, dx,$$

holds for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

Definition 4 (Eigenvalue)**.** A real number $\lambda \in \mathbb{R}$ is called an eigenvalue of problem (1.1) if there exists a nontrivial function $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, such that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega |u|^{q(x)-2} u\varphi \, dx \tag{1.8}$$

holds for all $\varphi \in W_0^{1,p(x)}(\Omega)$. The set of all eigenvalues is called the spectrum and is denoted by $\sigma(p(x), q(x))$.

Definition 5 (Eigenfunction)**.** A function $u \in W_0^{1,p(x)}(\Omega)$, $u \not\equiv 0$, is called an eigenfunction associated with an eigenvalue $\lambda \in \sigma(p(x), q(x))$ if it satisfies

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx = \lambda \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega |u|^{q(x)-2} u\varphi \, dx \tag{1.9}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

Definition 6 (Maximal Monotone Operator)**.** Let X be a reflexive Banach space and X^* its dual space. An operator $A : X \to X^*$ is said to be monotone if

$$\langle A(u) - A(v),\ u - v\rangle \geq 0 \quad \forall\, u, v \in X. \tag{1.10}$$

The operator A is called maximal monotone if its graph

$$G(A) = \big\{(u,\ A(u)) : u \in X\big\} \tag{1.11}$$

is not properly contained in the graph of any other monotone operator from X to X^*. Equivalently, A is maximal monotone if and only if

$$R(A + J) = X^*, \tag{1.12}$$

where $J : X \to X^*$ denotes the duality mapping of X.

If X is a normed vector space, X^* will denote the strong dual space, consisting of all the linear functionals on X that are continuous with respect to the topology induced on X by the norm and $\langle \cdot, \cdot \rangle$ will denote the duality pairing between X and X^*. If Y is another normed space, then $\mathcal{L}(X, Y)$ will stand for the space of all continuous linear mappings from X to Y.

1.2 Differentiable functions

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces, $A \subset X$, u an interior point of A and $v \in X$. The *directional derivative* of a function $\mathcal{J} : A \to Y$ at u along the direction v is defined by

$$\partial_v \mathcal{J}(u) = \lim_{t \to 0} \frac{\mathcal{J}(u + tv) - \mathcal{J}(u)}{t},$$

provided the limit exists in Y.

The function $\mathcal{J}$ is said to be *(Fréchet) differentiable* at u if there exists a continuous and linear mapping $L \in \mathcal{L}(X, Y)$ such that the limit

$$\lim_{h \to 0} \frac{\mathcal{J}(u + h) - \mathcal{J}(u) - L(h)}{\|h\|_X} = 0 \tag{1.13}$$

holds in Y. If there exists such a linear function L, then it is uniquely determined, is denoted

$$L = \mathcal{J}'(u),$$

and is said to be the *(Fréchet) differential* of $\mathcal{J}$ at u.

Moreover,

$$\|\mathcal{J}(u + h) - \mathcal{J}(u)\|_Y \leq \|L(h)\|_Y + o(\|h\|_X),$$

as $h \to 0$ in Y and $\mathcal{J}$ is continuous at u.

The function $\mathcal{J}$ is called *Gâteaux differentiable* at u if there exists a linear mapping $T \in \mathcal{L}(X, Y)$ such that the limit

$$\lim_{\varepsilon \to 0} \frac{\mathcal{J}(u + \varepsilon v) - \mathcal{J}(u)}{\varepsilon} = T(v) \tag{1.14}$$

holds in Y. If there exists such a linear mapping T, then it is unique, we denote it by

$$T = D\mathcal{J}(u),$$

and we call it the *Gâteaux differential* of $\mathcal{J}$ at u.

Moreover,

$$\|\mathcal{J}(u + \varepsilon v) - \mathcal{J}(u)\|_Y \leq \varepsilon \|T(v)\|_Y + o(\varepsilon),$$

as $\varepsilon \to 0$ and $\mathcal{J}$ is continuous along all straight lines passing through u.

Proposition 11. *Let X, Y be normed spaces, $A \subset X$, u an interior point of A, and let $\mathcal{J} : A \to Y$ be Gâteaux differentiable in a neighborhood of u. If the Gâteaux differential $D\mathcal{J}$ is continuous at u, then $\mathcal{J}$ is Fréchet differentiable at u.*

CHAPTER 2

AUXILIARY RESULTS

In this chapter, we recall the main mathematical tools and results that will be used throughout this work. These results concern variable exponent Lebesgue and Sobolev spaces, compactness, monotonicity, and eigenvalue problems involving the $L^{p(x)}$-Laplacian operator.

2.1 Variable Exponent Spaces

Proposition 12. *Let $p \in C(\overline{\Omega})$ such that $1 < p^- \leq p(x) \leq p^+ < \infty$. Then the space $W_0^{1,p(x)}(\Omega)$ is separable and reflexive.*

Lemma 2.1.1 (Sobolev inequality, see [8])**.** *There exists a constant $C > 0$, independent of u, such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \qquad \forall u \in W_0^{1,p(x)}(\Omega).$$

In particular, this implies that the first eigenvalue satisfies

$$\Lambda_1 > 0.$$

Proposition 13 (Poincaré inequality)**.** *There exists a constant $C > 0$ such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Proposition 14 (Modular inequalities)**.** *For any $u \in L^{p(x)}(\Omega)$:*

- *If $\|u\|_{p(x)} < 1$, then*

$$\|u\|_{p(x)}^{p^+} \leq \int_\Omega |u|^{p(x)} dx \leq \|u\|_{p(x)}^{p^-}.$$

- *If $\|u\|_{p(x)} > 1$, then*

$$\|u\|_{p(x)}^{p^-} \leq \int_\Omega |u|^{p(x)} dx \leq \|u\|_{p(x)}^{p^+}.$$

Proposition 15. *The space $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(x)}(\Omega)$.*

Theorem 2.1.2 (Lyusternik Schnirelmann Theory, see [13])**.** *Let X be a reflexive Banach space and let $J : X \to \mathbb{R}$ be an even functional of class C^1. Define the constraint manifold*

$$M = \left\{ u \in W_0^{1,p(x)}(\Omega) : \int_\Omega |u|^{q(x)}\, dx = 1 \right\}.$$

For each $k \in \mathbb{N}$, define

$$\lambda_k(p(x), q(x)) = \inf_{A \in \mathcal{F}_k} \sup_{u \in A} \int_\Omega |\nabla u|^{p(x)}\, dx,$$

where

$$\mathcal{F}_k = \{A \subset M : A \text{ is compact, symmetric, and } \gamma(A) \geq k\},$$

and γ denotes the Krasnosel'skiĭ genus. Then $\lambda_k(p(x), q(x))$ is an eigenvalue of problem (1.1) for each $k \in \mathbb{N}$, and

$$\lambda_k(p(x), q(x)) \to +\infty \quad \text{as } k \to \infty.$$

Definition 7 (Krasnosel'skiĭ Genus, see [13])**.** Let $A \subset X \setminus \{0\}$ be a closed symmetric set. The Krasnosel'skiĭ genus of A is defined as

$$\gamma(A) = \inf\{k \in \mathbb{N} : \exists\, \varphi \in C(A, \mathbb{R}^k \setminus \{0\}),\ \varphi \text{ odd}\}.$$

If no such k exists, we set $\gamma(A) = +\infty$, and $\gamma(\emptyset) = 0$.

Proposition 16 (Properties of the Genus)**.** *Let $A, B \subset X \setminus \{0\}$ be closed symmetric sets. Then:*

1. *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
2. *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
3. *If $\gamma(A) \geq 2$, then A contains infinitely many points.*
4. *If A is a $(k-1)$-dimensional sphere, then $\gamma(A) = k$.*

Theorem 2.1.3 (Palais Smale Condition)**.** *Let $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ be the energy functional associated with problem (1.1), defined by*

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)}\, dx.$$

A sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ is called a Palais Smale sequence if:

1. $(J(u_n))$ *is bounded,*
2. $J'(u_n) \to 0$ *strongly in* $W^{-1,p'(x)}(\Omega)$.

The functional J satisfies the Palais Smale condition if every such sequence has a strongly convergent subsequence in $W_0^{1,p(x)}(\Omega)$.

Theorem 2.1.4 (Mountain Pass Theorem, see [16])**.** *Let X be a Banach space and let $J \in C^1(X, \mathbb{R})$ satisfy the Palais–Smale condition. Suppose that:*

1. $J(0) = 0$,
2. *There exist $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $\|u\| = \rho$,*
3. *There exists $e \in X$ with $\|e\| > \rho$ such that $J(e) \leq 0$.*

Then J has a critical value

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \geq \alpha,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = e\}$.

Theorem 2.1.5 (Rellich Kondrachov Compactness Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and let $p \in \mathcal{P}^{\log}(\Omega)$. Assume $q(x) < p^*(x)$ for all $x \in \Omega$. Then:*

$$W_0^{1,p(x)}(\Omega) \hookrightarrow\!\!\!\rightarrow L^{q(x)}(\Omega)$$

is a compact embedding.

Proposition 17 (Existence of minimizer, see [8])**.** *There exists a non-negative minimizer*

$$u \in W_0^{1,p(x)}(\Omega), \qquad u \not\equiv 0,$$

of the Rayleigh quotient

$$\Lambda_1 = \inf_{v \neq 0} \frac{\|\nabla v\|_{p(x)}}{\|v\|_{p(x)}}.$$

Proof. We select a minimizing sequence (v_j) normalized so that

$$\|v_j\|_{p(x)} = 1.$$

Then

$$\Lambda_1 = \lim_{j \to \infty} \|\nabla v_j\|_{p(x)}.$$

By Theorem 2.1.1, we extract a subsequence, still denoted by (v_{j_ν}), such that

$$v_{j_\nu} \to u \quad \text{strongly in } L^{p(x)}(\Omega),$$

and

$$\nabla v_{j_\nu} \rightharpoonup \nabla u \quad \text{weakly in } L^{p(x)}(\Omega).$$

By the weak lower semicontinuity of the norm, we obtain

$$\frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}} \leq \lim_{\nu\to\infty} \frac{\|\nabla v_{j_\nu}\|_{p(x)}}{\|v_{j_\nu}\|_{p(x)}} = \Lambda_1.$$

Hence, u is a minimizer.

2.2 Embedding Results

Theorem 2.2.1 (Sobolev embedding)**.** *Let $q \in C(\overline{\Omega})$ such that $q(x) \leq p^*(x)$. Then*

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

continuously.

Theorem 2.2.2 (Compact embedding)**.** *If $q(x) < p^*(x)$, then the embedding*

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact.

Lemma 2.2.3. *If $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, then*

$$u_n \to u \quad \textit{strongly in } L^{q(x)}(\Omega).$$

2.3 Monotonicity and Nonlinear Tools

Proposition 18 (Monotonicity)**.** *For all $u, v \in W_0^{1,p(x)}(\Omega)$,*

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \cdot (\nabla u - \nabla v)\, dx \geq 0.$$

Proposition 19 (Strict monotonicity)**.** *Equality holds if and only if $\nabla u = \nabla v$ almost everywhere.*

Lemma 2.3.1. *Let $\phi = (u - v)^+ = \max(u - v, 0)$. Then*

$$\nabla \phi = \chi_{\{u>v\}}(\nabla u - \nabla v).$$

Lemma 2.3.2.

$$\int_\Omega (A(u) - A(v)) \cdot \nabla (u - v)^+ \geq 0.$$

2.4 Weak Convergence Tools

Definition 8. A function $u \in W_0^{1,p(x)}(\Omega)$ is called a weak solution if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi dx = \int_\Omega f\varphi dx$$

for all $\varphi \in C_0^\infty(\Omega)$.

Theorem 2.4.1. *Every bounded sequence in $W_0^{1,p(x)}(\Omega)$ admits a weakly convergent subsequence.*

Theorem 2.4.2 (Weak lower semicontinuity)**.**

$$\int_\Omega |\nabla u|^{p(x)}dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)}dx.$$

Lemma 2.4.3. *If $u_n \to u$ in $L^{q(x)}(\Omega)$, then*

$$|u_n|^{q(x)-2}u_n \to |u|^{q(x)-2}u.$$

Proposition 20. *The mapping*

$$u \mapsto |u|^{q(x)-2}u$$

is continuous from $L^{q(x)}(\Omega)$ into $L^{q'(x)}(\Omega)$.

2.5 Energy Functional

Proposition 21 (Scaling property)**.** *For $\alpha > 0$,*

$$-\Delta_{p(x)}(\alpha u) = \alpha^{p(x)-1}(-\Delta_{p(x)}u),$$

and

$$|\alpha u|^{q(x)-2}(\alpha u) = \alpha^{q(x)-1}|u|^{q(x)-2}u.$$

Proposition 22. *Let $u \neq 0$ and define*

$$v = \frac{u}{\|u\|_{q(x)}}.$$

Then $\|v\|_{q(x)} = 1$.

Proposition 23 (Energy identity)**.** *If u is a solution of*

$$-\Delta_{p(x)}u = \lambda|u|^{q(x)-2}u,$$

then

$$\int_\Omega |\nabla u|^{p(x)}dx = \lambda \int_\Omega |u|^{q(x)}dx.$$

2.6 Eigenvalue Properties

Lemma 2.6.1. *If u is a nonnegative weak solution, then either $u = 0$ or $u > 0$ in Ω.*

Lemma 2.6.2. *Any first eigenfunction u satisfies $u > 0$ or $u < 0$ in Ω.*

Theorem 2.6.3 (Simplicity of the first eigenvalue)**.** *If u_1 and u_2 are two eigenfunctions associated with λ_1, then there exists a constant c such that*

$$u_1 = cu_2.$$

Definition 9 (Eigenfunction, see [8])**.** A function

$$u \in W_0^{1,p(x)}(\Omega), \qquad u \not\equiv 0,$$

is called an eigenfunction if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \Lambda S \int_\Omega |u|^{p(x)-2} u \varphi \, dx$$

holds for all

$$\varphi \in C_0^\infty(\Omega),$$

where Λ is the corresponding eigenvalue.

2.7 Auxiliary Inequalities

Lemma 2.7.1. *For $q(x) \leq p(x)$,*

$$a^{p(x)} b^{q(x)-p(x)} \leq C(a^{q(x)} + b^{q(x)}).$$

Lemma 2.7.2. *If $\nabla u = 0$, then u is constant.*

Lemma 2.7.3. *If $0 < \eta < 1$ and $u \leq \eta u$, then $u = 0$.*

2.8 Limit and Spectrum

Theorem 2.8.1. *Let $u_n \rightharpoonup u$ and $\lambda_n \to \lambda$. Then u is a solution of the limit equation.*

Proposition 24. *The spectrum $\sigma(p(x), q(x))$ is closed.*

CHAPTER 3

MAIN RESULTS

Introduction

In the constant exponent setting, nonlinear eigenvalue problems involving the p-Laplacian operator have been extensively studied, and several fundamental results have been established. Among these contributions, the work of Franzina provides important insights into the existence and qualitative properties of eigenvalues associated with such problems.

The aim of this chapter is to extend these results to the more general framework of variable exponent spaces. More precisely, we consider the eigenvalue problem driven by the $p(x)$-Laplacian operator and investigate how the techniques and conclusions obtained in the constant exponent case can be adapted to this nonstandard setting.

The transition from constant to variable exponent spaces is far from trivial. Indeed, the lack of homogeneity and the pointwise variability of the exponent introduce significant analytical difficulties. Classical tools must be carefully re-examined, and several arguments require substantial modifications. In particular, the properties of the Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ play a crucial role in the development of the theory.

In this chapter, we reformulate the eigenvalue problem in the variable exponent framework and establish the corresponding weak formulation. We then adapt the main arguments used by Franzina, including variational techniques and monotonicity methods, to prove analogous results in this more general context.

Special attention is devoted to the choice of appropriate test functions, scaling arguments, and the handling of nonlinear terms involving variable growth. These adaptations allow us to derive key inequalities and ultimately extend the principal results to the $p(x)$-setting.

The results obtained in this chapter form a fundamental step toward the study of eigenvalue problems with nonstandard growth and provide a bridge between classical theory and modern developments in variable exponent analysis.

3.1 The Eigenvalue problem

Theorem 3.1.1. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p(x) < \infty$ and $1 < q(x) < p^*(x)$. Let $\lambda > 0$ be an eigenvalue of equation 3 and $u \in W_0^{1,p(x)}(\Omega)$ be a corresponding eigenfunction. Then u is bounded and its first derivatives are locally Hölder continuous. Moreover, if $u \geq 0$ in Ω then $u > 0$ in Ω.*

Proof.

Step 1. Boundedness: We prove that $u \in L^\infty(\Omega)$
Let $u \in W_0^{1,p(x)}(\Omega)$, $1 < p(x) < \infty$, $1 < q(x) < p^*(x)$ and $\lambda > 0$. The Euler-Lagrange equation is given by

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right) = \lambda \|u\|_{q(x)}^{p(x)-q(x)} |u|^{q(x)-2} u, \tag{3.1}$$

Define

$$f(u) = \mathcal{C} |u|^{q(x)-2} u,$$

with $\mathcal{C} = \lambda \|u\|_{q(x)}^{p(x)-q(x)}$ is a strictly positive constant. Then, equation (3.1) becomes

$$-\Delta_p u = \mathcal{C} |u|^{q(x)-2} u, \tag{3.2}$$

Since u is a weak solution, for every $\varphi \in W_0^{1,p(x)}(\Omega)$, we consider the following weak formulation

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx = \mathcal{C} \int_\Omega |u|^{q(x)-2} u\varphi \, dx. \tag{3.3}$$

Using Moser iteration method, we choose

$$\varphi = |u|^{\beta(x)} u,$$

with

$$1 < \beta_- \leq \beta(x) \leq \beta_+ < +\infty. \tag{3.4}$$

We first compute the gradient, we have

$$\nabla(|u|^{\beta(x)} u) = |u|^{\beta(x)} \nabla u + \beta(x) |u|^{\beta(x)} \nabla u = (\beta(x) + 1) |u|^{\beta(x)} \nabla u.$$

Equation (3.3) becomes

$$\int_\Omega (\beta(x) + 1) |u|^{\beta(x)} |\nabla u|^{p(x)} \, dx = \mathcal{C} \int_\Omega |u|^{\beta(x)+q(x)} dx. \tag{3.5}$$

Now, let $w = |u|^{\alpha(x)}$ with $\alpha(x) = \frac{\beta(x)+p(x)}{p(x)}$, therefore

$$|\nabla w|^{p(x)} = (\alpha(x))^{p(x)} |u|^{(\alpha(x)-1)p(x)} |\nabla u|^{p(x)}.$$

We know that

$$\alpha(x) - 1 = \frac{\beta(x) + p(x)}{p(x)} - 1 = \frac{\beta(x)}{p(x)} \implies \beta(x) = (\alpha(x) - 1)p(x),$$

thus, we obtain

$$|\nabla w|^{p(x)} = (\alpha(x))^{p(x)} |u|^{\beta(x)} |\nabla u|^{p(x)}, \tag{3.6}$$

substituting $|u|^{\beta(x)} |\nabla u|^{p(x)} = \dfrac{1}{(\alpha(x))^{p(x)}} |\nabla w|^{p(x)}$ into (3.5), we get

$$\int_\Omega |\nabla w|^{p(x)} \, dx \leq \mathcal{C}_1 \int_\Omega |u|^{\beta(x)+q(x)} dx.$$

where $\mathcal{C}_1 = \dfrac{\mathcal{C}\,(\alpha_+)^{p_+}}{\beta_- + 1}$
that is to say by Proposition (2)we find

$$\max\left\{\|\nabla w\|_{p(x)}^{p_-}, \|\nabla w\|_{p(x)}^{p_+}\right\} \leq \mathcal{C}_2 \int_\Omega |u|^{\beta(x)+q(x)} dx. \tag{3.7}$$

with

$$\mathcal{C}_2 = \frac{\mathcal{C}_1 \max\left\{\|\nabla w\|_{p(x)}^{p_-}, \|\nabla w\|_{p(x)}^{p_+}\right\}}{\min\left\{\|\nabla w\|_{p(x)}^{p_-}, \|\nabla w\|_{p(x)}^{p_+}\right\}} = \frac{\mathcal{C}\,(\alpha_+)^{p_+} \max\left\{\|\nabla w\|_{p(x)}^{p_-}, \|\nabla w\|_{p(x)}^{p_+}\right\}}{(\beta_- + 1) \min\left\{\|\nabla w\|_{p(x)}^{p_-}, \|\nabla w\|_{p(x)}^{p_+}\right\}}$$

Applying Sobolev inequality (Proposition (2)) since $w \in W_0^{1,p}(\Omega)$, and using (3.7) we arrive at

$$\max\left\{S^{-p_-} \|w\|_{p^*(x)}^{p_-}, S^{-p_+} \|w\|_{p^*(x)}^{p_+}\right\} \leq \mathcal{C}_2 \int_\Omega |u|^{\beta(x)+q(x)} dx.$$

i.e.,

$$\max\left\{\|w\|_{p^*(x)}^{p_-}, \|w\|_{p^*(x)}^{p_+}\right\} \leq \mathcal{C}_3 \int_\Omega |u|^{\beta(x)+q(x)} dx. \tag{3.8}$$

where $\mathcal{C}_3 = \mathcal{C}_2 S^{p_\pm}$. Since $w = |u|^{\frac{\beta(x)+p(x)}{p(x)}}$ then

$$\int_\Omega |w|^{p^*(x)} dx = \int_\Omega |u|^{\frac{(\beta(x)+p(x))p^*(x)}{p(x)}} dx$$

Set $r_2(x) = \frac{(\beta(x)+p(x))p^*(x)}{p(x)}$ and $r_1(x) = \beta(x) + q(x)$, in view of Proposition 2, equation (3.8) becomes

$$\int_\Omega |u|^{r_2(x)}\, dx \leq \mathcal{C}_3 \int_\Omega |u|^{r_1(x)}\, dx. \tag{3.9}$$

Proposition (2) gives

$$\|u\|_{r_2(x)}^{(r_2)_\pm} \leq \mathcal{C}_3 \, \|u\|_{r_1(x)}^{(r_1)_\pm}.$$

Consequently, there is a constant $\mathcal{C}_4$ such that

$$\|u\|_{r_2(x)} \leq \mathcal{C}_4 \, \|u\|_{r_1(x)}\,,$$

By repeating the previous argument, setting $r_{n+1}(x) = \frac{(\beta_n(x)+p(x))p^*(x)}{p(x)}$ and $r_n(x) = \beta_n(x) + q(x)$, we have

$$r_{n+1}(x) = \frac{p^*(x)}{p(x)} \left(r_n(x) + p(x) - q(x)\right),$$

that is to say, $r_{n+1}(x) > r_n(x)$ from $p(x) < p^*(x)$ and $(r_n(x))_n$ diverges to $+\infty$. letting n tends to $+\infty$ we ontain $u \in L^\infty(\Omega)$.

Step 2. Local Hölder Continuity of First Derivatives: Since $u \in L^\infty(\Omega)$ (from Step 1), the right hand side of equation (3.2) satisfies:

$$f(x) := \mathcal{C}|u|^{q(x)-2}u \in L^\infty(\Omega),$$

since u is bounded. Therefore, equation (3.2) can be written as:

$$-\Delta_{p(x)} u = f(x), \quad f \in L^\infty(\Omega).$$

By the regularity result of Fan and Zhao [6], any weak solution $u \in W_0^{1,p(x)}(\Omega)$ of

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x), \quad f \in L^\infty(\Omega),$$

belongs to $C^{1,\alpha}_{\text{loc}}(\Omega)$ for some $\alpha \in (0,1)$. More precisely, for every compact subset $K \subset \Omega$, there exist constants $\alpha = \alpha(N, p_-, p_+) \in (0,1)$ and $C = C(K, \|u\|_{L^\infty}, \|f\|_{L^\infty}) > 0$ such that:

$$\|\nabla u\|_{C^{0,\alpha}(K)} \leq C.$$

Hence, the first derivatives of u are locally Hölder continuous in Ω.

Step 3. Strong Maximum Principle:
Suppose $u \geq 0$ in Ω. Since $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ and u satisfies:

$$-\Delta_{p(x)} u = \mathcal{C}|u|^{q(x)-2}u \geq 0 \quad \text{in } \Omega,$$

by the strong maximum principle for the $p(x)$-Laplacian (see [5]), either $u \equiv 0$ in Ω or $u > 0$ in Ω. Since u is an eigenfunction, $u \not\equiv 0$, and therefore $u > 0$ in Ω.

Corollary 3.1.2. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, and let $p, q \in \mathcal{P}^{\log}(\Omega)$ such that $1 < p^- \leq p(x) \leq p^+ < \infty$ and $1 < q(x) < p^*(x)$ for all $x \in \Omega$. Let $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ be a first eigenfunction corresponding to $\lambda_1(p(x), q(x))$. Then either $u > 0$ or $u < 0$ in Ω.*

proof.
Since u is a first eigenfunction corresponding to $\lambda_1(p(x), q(x))$, it follows from the definition of $\lambda_1(p(x), q(x))$ that u is a minimizer of the functional

$$\frac{\displaystyle\int_\Omega |\nabla v|^{p(x)}\, dx}{\displaystyle\int_\Omega |v|^{q(x)}\, dx}$$

among all nontrivial functions in $W_0^{1,p(x)}(\Omega)$.

We observe that replacing u with its absolute value does not change the value of this quotient. Indeed, it holds almost everywhere in Ω that

$$|\nabla |u|| = |\nabla u|,$$

and therefore

$$\int_\Omega |\nabla |u||^{p(x)}\, dx = \int_\Omega |\nabla u|^{p(x)}\, dx.$$

Moreover,

$$\int_\Omega ||u||^{q(x)}\, dx = \int_\Omega |u|^{q(x)}\, dx.$$

Hence, the quotient remains unchanged when u is replaced by $|u|$, which implies that $|u|$ is also a minimizer and therefore a first eigenfunction.

Since $|u| \geq 0$, we can apply the strong maximum principle (see (3.1.1)), which ensures that a nonnegative eigenfunction cannot vanish in Ω. Hence,

$$|u| > 0 \quad \text{in } \Omega.$$

If u changed sign in Ω, then it would vanish at some point in Ω, which contradicts the previous conclusion. Therefore, u does not change sign, and we conclude that either

$$u > 0 \quad \text{or} \quad u < 0 \quad \text{in } \Omega.$$

3.2 On The Simplicity Of $\lambda_1(p(x), q(x))$

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain with finite measure, and let $p, q : \Omega \to (1, \infty)$ be measurable functions such that*

$$1 < q(x) \leq p(x) < \infty \quad \textit{for all } x \in \Omega.$$

Then $\lambda_1(p(x), q(x))$ is simple; i.e., the eigenfunctions corresponding to $\lambda_1(p(x), q(x))$ form a linear space of dimension one.

Proof.

To prove (3.2.1),we use picone's identity for the $p(x)$-Laplacian.(see [1]and [7])
Let $u, v > 0$ be two weak solutions for the following eigenvalue problem (1.1),i.e.,

$$-\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda_1\|u\|_{q(x)}^{p(x)-q(x)}u^{q(x)-1} \tag{3.10}$$

$$-\text{div}\left(|\nabla v|^{p(x)-2}\nabla v\right) = \lambda_1\|v\|_{q(x)}^{p(x)-q(x)}v^{q(x)-1} \tag{3.11}$$

With $u = v = 0$ on $\partial\Omega$.Let $\varphi \in C_0^\infty(\Omega)$ be a test function, such that:

$$\varphi = \frac{u^{q(x)}}{v^{q(x)-1}}$$

Multiplying equation (3.11) by φ and integrating over Ω, we have:

$$\begin{aligned} I_v &= \int_\Omega |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\left(\frac{u^{q(x)}}{v^{q(x)-1}}\right)dx \\ &= \lambda_1\|v\|_{q(x)}^{p(x)-q(x)}\int_\Omega v^{q(x)-1}\frac{u^{q(x)}}{v^{q(x)-1}}dx \\ &= \lambda_1\|v\|_{q(x)}^{p(x)-q(x)}\int_\Omega u^{q(x)}dx. \end{aligned}$$

Moreover, multiplying equation (3.10) by u and integrating over Ω, we get:

$$I_u = \int_\Omega |\nabla u|^{p(x)}dx = \lambda_1\|u\|_{q(x)}^{p(x)-q(x)}\int_\Omega u^{q(x)}dx.$$

Using Picone's identity for the $p(x)$-Laplacian, that is

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v \cdot \nabla\left(\frac{u^{q(x)}}{v^{q(x)-1}}\right)dx \leqslant \int_\Omega |\nabla u|^{p(x)}dx.$$

Yields that:

$$\lambda_1 \|v\|_{q(x)}^{p(x)-q(x)} \int_\Omega u^{q(x)} dx \leqslant \int_\Omega |\nabla u|^{p(x)} dx \ldots\ldots (1)$$

Now, proceeding in the same way with equation (1) with u, using the test function $\psi = \frac{v^{q(x)}}{u^{q(x)-1}}$, we find

$$\lambda_1 \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega v^{q(x)} dx \leqslant \int_\Omega |\nabla v|^{p(x)} dx \ldots\ldots (2)$$

On the other hand, we know that u, v are minimizers that satisfy the Rayleigh quotient, i.e.,

$$\lambda_1 \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega |u|^{q(x)} dx = \int_\Omega |\nabla u|^{p(x)} dx \ldots\ldots (3)$$

$$\lambda_1 \|v\|_{q(x)}^{p(x)-q(x)} \int_\Omega |v|^{q(x)} dx = \int_\Omega |\nabla v|^{p(x)} dx \ldots\ldots (4)$$

So, from (1) and (3) we obtain,

$$\lambda_1 \|v\|_{q(x)}^{p(x)-q(x)} \int_\Omega u^{q(x)} dx \leqslant \lambda_1 \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega u^{q(x)} dx \ldots\ldots (5)$$

and the same from (2) and (4) we get:

$$\lambda_1 \|u\|_{q(x)}^{p(x)-q(x)} \int_\Omega v^{q(x)} dx \leqslant \lambda_1 \|v\|_{q(x)}^{p(x)-q(x)} \int_\Omega |v|^{q(x)} dx \ldots\ldots (6)$$

From inequality (5), since $\int_\Omega u^{q(x)}\, dx > 0$, we can divide both sides to obtain:

$$\|v\|_{q(x)}^{p(x)-q(x)} \leq \|u\|_{q(x)}^{p(x)-q(x)} \quad (7)$$

From inequality (6), since $\int_\Omega v^{q(x)}\, dx > 0$, we similarly obtain:

$$\|u\|_{q(x)}^{p(x)-q(x)} \leq \|v\|_{q(x)}^{p(x)-q(x)} \quad (8)$$

Combining (7) and (8), we conclude:

$$\|u\|_{q(x)} = \|v\|_{q(x)} \quad (9)$$

Now, equality in Picone's identity holds if and only if $\nabla \left(\frac{u}{v}\right) = 0$ a.e. in Ω, which implies that $\frac{u}{v} = c$ for some positive constant $c > 0$, that is:

$$u = cv.$$

Since both u and v are normalized so that $\|u\|_{q(x)} = \|v\|_{q(x)}$, it follows that $c = 1$, and hence $u = v$.

Therefore, the eigenspace associated with $\lambda_1(p(x), q(x))$ is one-dimensional, which proves simplicity.

Theorem 3.2.2. *Let Ω be a ball in $\mathbb{R}^N$ centered at zero. Assume that $p(x), q(x) \in \mathcal{P}(\Omega)$ are radial functions such that $1 < p^- \leq p^+ < \infty$ and $1 < q(x) < p^*(x)$ for all $x \in \Omega$. Then the eigenfunctions corresponding to $\lambda_1(p(x), q(x))$ are radial functions.*

proof.

We prove that if the domain Ω is a ball, then the first eigenfunction u is radially symmetric and decreasing. Let Ω be a ball in $\mathbb{R}^N$. We assume that the exponents $p(x)$ and $q(x)$ are radial functions, denoted by:

$$p(x) = p(|x|) \quad \text{and} \quad q(x) = q(|x|).$$

Set $u(x) = \psi(r)$ where $r = |x|$,so we have

$$\frac{\partial u}{\partial x_i} = \psi'(r)\frac{x_i}{r}$$

Consequently,

$$\nabla u = \psi'(r)\frac{x}{r} \implies |\nabla u| = |\psi'(r)|,$$

and

$$|\nabla u|^{p(x)-2}\nabla u = |\psi'(r)|^{p(x)-2}\psi'(r)\frac{x}{r},$$

Recall the

$$\operatorname{div}\left(f(r)\frac{x}{r}\right) = f'(r) + \frac{N-1}{r}f(r)$$

By setting $f(r) = |\psi'(r)|^{p(x)-2}\psi'(r)$, we get

$$\operatorname{div}\left(f(r)\frac{x}{r}\right) = \left(|\varphi'|^{p(x)-2}\varphi'\right)' + \frac{N-1}{r}|\varphi'|^{p(x)-2}\varphi'$$

By substituting this into our main equation, we obtain:

$$-\left[\left(|\varphi'|^{p(x)-2}\varphi'\right)' + \frac{N-1}{r}|\varphi'|^{p(x)-2}\varphi'\right] = \lambda_1|\varphi|^{q(x)-2}\varphi$$

Multiplying both sides by r^{N-1} leads to the following simplified form:

$$-(r^{N-1}|\varphi'|^{p(x)-2}\varphi')' = \lambda_1 r^{N-1}|\varphi|^{q(x)-2}\varphi$$

suppose that the solution is positive,i.e., $u \geq 0 \implies \varphi \geq 0$, it follows that the right-hand side is positive,that is to say:

$$\lambda_1 r^{N-1}|\varphi|^{q(x)-2}\varphi \geq 0,$$

this implies that the derivative of the radial flux is non positive,i.e.,

$$(r^{N-1}|\varphi'|^{p(x)-2}\varphi')' \leq 0$$

Given the condition $\varphi'(0) = 0$, we conclude that:

$$\varphi'(r) \leq 0 \quad \text{for all } r > 0$$

Hence, the radial solution $\varphi(r)$ is decreasing.

Lemma 3.2.3. *Let $1 < p(x) < \infty$, $1 < q(x) < p^*(x)$ and $\lambda, c > 0$. Then the Cauchy problem*

$$\begin{cases} -(r^{N-1}|\phi'|^{p(r)-2}\phi')' = \lambda r^{N-1}|\phi|^{q(r)-2}\phi, \quad r \in (0, R), \\ \phi(0) = c, \quad \phi'(0) = 0, \end{cases} \tag{3.12}$$

has at most one positive solution

$$\phi \in C^1([0, R]) \cap C^2((0, R)).$$

Proof.

To study the Cauchy problem in the variable exponent case, we write.

$$(r^{N-1}|\phi'|^{p(r)-2}\phi')' = -\lambda r^{N-1}|\phi|^{q(r)-2}\phi.$$

Integrating from 0 to r, we have

$$\int_0^r (t^{N-1}|\phi'|^{p(t)-2}\phi')'dt = -\lambda \int_0^r t^{N-1}|\phi|^{q(t)-2}\phi\, dt.$$

Using the condition $\phi'(0) = 0$, we get

$$r^{N-1}|\phi'|^{p(r)-2}\phi' = -\lambda \int_0^r t^{N-1}|\phi|^{q(t)-2}\phi\, dt.$$

For each r, we define

$$g_r(t) = |t|^{p(r)-2}t,$$

it is easy to show that the function $g_r(t)$ is strictly increasing , then

$$g_r(\phi'(r)) = |\phi'|^{p(r)-2}\phi'.$$

Since $1 < p_- \leq p(r) \leq p_+ < \infty$, the function g_r is strictly increasing and invertible, thus

$$\phi'(r) = g_r^{-1}\left(-\frac{\lambda}{r^{N-1}}\int_0^r t^{N-1}|\phi|^{q(t)-2}\phi\,dt\right).$$

Since also,

$$\phi(r) = \phi(0) + \int_0^r \phi'(t)\,dt,$$

with $\phi(0) = c$, we obtain

$$\phi(r) = c + \int_0^r \phi'(t)\,dt,$$

replacing the expression of ϕ', we find

$$\phi(r) = c - \int_0^r g_t^{-1}\left(\frac{\lambda}{t^{N-1}}\int_0^t s^{N-1}|\phi|^{q(s)-2}\phi\,ds\right)dt.$$

Now, define the operator T:C1[0,R] inter C2(0,R)$\rightarrow$ $\mathbb{R}$,by

$$T(\phi)(r) = c - \int_0^r g_t^{-1}\left(\frac{\lambda}{t^{N-1}}\int_0^t s^{N-1}|\phi|^{q(s)-2}\phi\,ds\right)dt.$$

we prove that T has a fixed point, i.e.,

$$\phi = T(\phi).$$

this solve the cauchy problem (3.12) Assume that there exist two solutions ϕ_1 and ϕ_2. Then

$$T(\phi_1) = \phi_1, \qquad T(\phi_2) = \phi_2.$$

So

$$\phi_1 - \phi_2 = T(\phi_1) - T(\phi_2).$$

and

$$\|\phi_1 - \phi_2\| = \|T(\phi_1) - T(\phi_2)\|.$$

We have,

$$|T(\phi_1)(r) - T(\phi_2)(r)| \leq \int_0^r \left|g_t^{-1}(\phi_1(t)) - g_t^{-1}(\phi_2(t))\right|dt,$$

where

$$\phi_i(t) = \frac{\lambda}{t^{N-1}}\int_0^t s^{N-1}|\phi_i|^{q(s)-2}\phi_i\,ds, \quad i = 1, 2.$$

Since g_t is strictly increasing and $p(t)$ is bounded,then the inverse g_t^{-1} is locally Lipschitz. Hence there exists a constant $C > 0$ such that

$$|g_t^{-1}(a) - g_t^{-1}(b)| \leq C|a - b|.$$

Therefore,

$$|T(\phi_1) - T(\phi_2)| \leq C \int_0^r |\phi_1(t) - \phi_2(t)|dt.$$

Let

$$f(x, u) = |u|^{q(x)-2}u.$$

Since $q(x)$ is bounded, f is locally Lipschitz. hence,

$$|f(u) - f(v)| \leq C|u - v|.$$

So,

$$|\phi_1(t) - \phi_2(t)| \leq C \int_0^t |\phi_1 - \phi_2|ds,$$

we obtain

$$|T(\phi_1) - T(\phi_2)| \leq C \int_0^r \int_0^t |\phi_1 - \phi_2|ds\, dt,$$

that is to say,

$$\|T(\phi_1) - T(\phi_2)\| \leq C_1(\varepsilon)\|\phi_1 - \phi_2\|.$$

Choosing $\varepsilon > 0$ small enough such that

$$C_1(\varepsilon) < 1,$$

we conclude that T is a contraction on $[0, \varepsilon]$.

Therefore ,from the Banach fixed point theorem,we deduce that

$$\phi_1 = \phi_2 \quad \text{on } [0, \varepsilon].$$

By repeating the argument, we extend the equality to the whole interval [0,R],i.e.,

$$\phi_1 = \phi_2 \quad \text{on } [0, R].$$

Finally from the above we conclude that the Cauchy problem admits at most one positive solution.

Lemma 3.2.4. *Let $N > 1$, $1 < p(r) < \infty$, $1 < q(r) < p^*(r)$ and $c_1, c_2 > 0$. Let $\phi_1, \phi_2 \in C^1[0, R] \cap C^2(0, R)$ be two positive solutions to the Cauchy problem (3.12) with $c = c_1, c_2$ respectively.*

If $c_1 \leq c_2$, then

$$\phi_1(r) \leq \phi_2(r) \quad \textit{for all } r \in [0, R].$$

Proof.
Let:

$$
\begin{aligned}
w(r) &= \phi_1(r) - \phi_2(r) \\
w(0) &= c_1 - c_2 \leq 0 \\
w'(0) &= 0
\end{aligned}
$$

Suppose the contrary: There exists r_0 such that:

$$w(r_0) > 0$$

Since $w(0) \leq 0$: There exists a point r_1 such that for $r > r_1$:

$$w(r) > 0 \quad \text{and} \quad w(r_1) = 0$$

We have:

$$\phi_1(r_1) = \phi_2(r_1)$$

and

$$w'(r_1) \geq 0$$

For $i = 1, 2$:

$$-\left(r^{N-1}|\phi_i'|^{p(r)-2}\phi_i'\right)' = \lambda r^{N-1}\phi_i^{q(r)-1}$$

After subtraction, we obtain:

$$-\left(r^{N-1}(A_1 - A_2)\right)' = \lambda r^{N-1}\left(\phi_1^{q(r)-1} - \phi_2^{q(r)-1}\right)$$

where:

$$A_i = |\phi_i'|^{p(r)-2}\phi_i'$$

Since $\phi_1 > \phi_2$ after r_1 and $q(r) > 1$, the function $t \to t^{q(r)-1}$ is increasing. Therefore:

$$\phi_1^{q(r)-1} - \phi_2^{q(r)-1} > 0$$

Since:

$$-\left(r^{N-1}(|\phi_1'|^{p(r)-2}\phi_1' - |\phi_2'|^{p(r)-2}\phi_2')\right)' > 0$$

It follows that:

$$\left(r^{N-1}(|\phi_1'|^{p(r)-2}\phi_1' - |\phi_2'|^{p(r)-2}\phi_2')\right)' < 0$$

That is, the function $r^{N-1}(A_1 - A_2)$ is a decreasing function.

At r_1:

$$\phi_1 = \phi_2$$
$$w'(r_1) \geq 0$$

This implies that the difference between the derivatives is non-negative. However, the function must be decreasing.

Therefore:

$$w(r) \leq 0 \implies \phi_1(r) \leq \phi_2(r)$$

Theorem 3.2.5. *Let Ω be a ball in $\mathbb{R}^N$. Let $p, q \in C(\overline{\Omega})$ such that*

$$1 < p^- \leq p(x) \leq p^+ < \infty, \qquad 1 < q(x) < p^*(x) \quad \text{for all } x \in \Omega.$$

Then the first eigenvalue $\lambda_1(p(x), q(x))$ is simple.

Proof.

Case 1: $N = 1$

In this case the main equation is written as:

$$-\frac{d}{dx}\left(|u'|^{p(x)-2}u'\right) = \lambda|u|^{q(x)-2}u,$$

with boundary conditions

$$u(0) = u(1) = 0$$

Assume the existence of two positive solutions $u_1 > 0$ and $u_2 > 0$.

Consider the test function $\varphi \in C_0^\infty(\Omega)$

$$\varphi = \frac{u_1^{p(x)}}{u_2^{p(x)-1}}.$$

Multiplying the equation with u_2 by φ and integrating over $[0, 1]$, we obtain:

$$\int_0^1 |u_2'|^{p(x)-2}u_2'\left(\frac{u_1^{p(x)}}{u_2^{p(x)-1}}\right)' dx = \lambda\int_0^1 u_2^{q(x)-1}\frac{u_1^{p(x)}}{u_2^{p(x)-1}}dx.$$

Picone's inequality, we have:

$$\int_0^1 |u_1'|^{p(x)}dx \geq \int_0^1 |u_2'|^{p(x)-2}u_2'\left(\frac{u_1^{p(x)}}{u_2^{p(x)-1}}\right)' dx.$$

It follows that

$$\int_0^1 |u_1'|^{p(x)}dx \geq \lambda \int_0^1 u_1^{p(x)} u_2^{q(x)-p(x)} dx.$$

From the main equation with u_1, we also have:

$$\int_0^1 |u_1'|^{p(x)}dx = \lambda \int_0^1 u_1^{q(x)} dx.$$

Similarly we arrive at

$$\int_0^1 u_1^{q(x)} dx \geq \int_0^1 u_1^{p(x)} u_2^{q(x)-p(x)} dx.$$

Note that

$$u_1^{p(x)} u_2^{q(x)-p(x)} = u_1^{q(x)} \left(\frac{u_1}{u_2}\right)^{p(x)-q(x)}.$$

Using the fact that $p(x) \geq q(x)$, it follows that $\left(\frac{u_1}{u_2}\right)^{p(x)-q(x)} \geq 0$. Since by the normalization condition

$$\|u_1\|_{L^{q(x)}(\Omega)} = \|u_2\|_{L^{q(x)}(\Omega)} = 1,$$

we can deduce that:

$$u_1 = ku_2 \quad \text{for some constant } k > 0,$$

that is to say,

$$\|u_1\|_{L^{q(x)}(\Omega)} = |k| \cdot \|u_2\|_{L^{q(x)}(\Omega)} = 1.$$

Which implies $k = 1$. Consequently, we conclude that

$$u_1 = u_2.$$

Case 2: $N > 1$

By Theorem 2.9.3, it follows that $u_i(x) = \phi_i(r)$. In this case, the equation is rewritten in radial coordinates as

$$-\left(r^{N-1}|\phi'|^{p(r)-2}\phi'\right)' = \lambda r^{N-1}|\phi|^{q(r)-2}\phi.$$

Suppose that

$$\phi_1(0) < \phi_2(0),$$

by Lemma 3.4, we obtain:

$$\phi_1(r) < \phi_2(r) \implies u_1(x) < u_2(x).$$

Given the normalization condition

$$\|u_i\|_{L^{q(x)}(\Omega)} = 1, \quad \text{for } i = 1, 2.$$

The strict inequality $u_1 < u_2$ implies

$$\|u_1\|_{L^{q(x)}(\Omega)} < \|u_2\|_{L^{q(x)}(\Omega)}.$$

which is a contradiction.

Consequently, we must have

$$u_1 = u_2.$$

This proves that the first eigenvalue $\lambda_1(p(x), q(x))$ is simple.

3.3 Further Uniqueness Results

Theorem 3.3.1. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, and let $p, q \in C(\overline{\Omega})$ such that*

$$1 < q(x) \leq p(x) \quad \textit{for all } x \in \Omega.$$

$$\eta^{p(x)-1} \leq \eta^{p^- -1}$$

Proof.

Assume, that $\lambda_1 < \lambda$. Let u_1 and u be positive eigenfunctions associated with λ_1 and λ, respectively. They satisfy

$$\begin{cases} -\operatorname{div}\left(|\nabla u_1|^{p(x)-2}\nabla u_1\right) = \lambda_1 |u_1|^{q(x)-2} u_1 \\ -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda |u|^{q(x)-2} u \end{cases}$$

Set $\tilde{u} = \eta u$. We have

$$\nabla(\eta u) = \eta \nabla u,$$

Consequently, we obtain

$$|\nabla(\eta u)|^{p(x)-2}\nabla(\eta u) = \eta^{p(x)-1}|\nabla u|^{p(x)-2}\nabla u,$$

and

$$-\operatorname{div}\left(|\nabla(\eta u)|^{p(x)-2}\nabla(\eta u)\right) = \eta^{p(x)-1}\lambda |u|^{q(x)-2} u,$$

choose η as follows

$$\eta = \left(\frac{\lambda_1}{\lambda}\right)^{\frac{1}{p^- - 1}}.$$

Since $p(x) \geq p^-$, it follows that

$$\eta^{p(x)-1} \leq \frac{\lambda_1}{\lambda}.$$

Thus, the equation with ηu satisfies

$$-\text{div}\left(|\nabla(\eta u)|^{p(x)-2}\nabla(\eta u)\right) \leq \lambda_1 |u|^{q(x)-2}u.$$

Now, let $\varphi \in W_0^{1,p(x)}(\Omega)$ be a non-negative test function ($\varphi \geq 0$), for u_1, the weak form is:

$$\int_\Omega |\nabla u_1|^{p(x)-2}\nabla u_1 \cdot \nabla\varphi \, dx = \lambda_1 \int_\Omega u_1^{q(x)-1}\varphi \, dx,$$

for ηu, we have:

$$\int_\Omega |\nabla(\eta u)|^{p(x)-2}\nabla(\eta u) \cdot \nabla\varphi \, dx \leq \lambda_1 \int_\Omega (\eta u)^{q(x)-1}\varphi \, dx.$$

By choosing the test function $\varphi = (u_1 - \eta u)^+$, we may write

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla(\eta u)|^{p(x)-2}\nabla(\eta u)\right) \cdot \nabla(u_1 - \eta u)^+ dx \leq 0.$$

We utilize the monotonicity of the operator $A(x,\xi) = |\xi|^{p(x)-2}\xi$, that is

$$(A(x,\xi) - A(x,\zeta)) \cdot (\xi - \zeta) \geq 0.$$

By substituting $\xi = \nabla u_1$, $\zeta = \nabla(\eta u)$, and using the test function $\varphi = (u_1 - \eta u)^+$, we get:

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla(\eta u)|^{p(x)-2}\nabla(\eta u)\right) \cdot \nabla(u_1 - \eta u)^+ dx \geq 0.$$

From the weak inequality derived previously, we also have

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla(\eta u)|^{p(x)-2}\nabla(\eta u)\right) \cdot \nabla\varphi \, dx \leq 0.$$

Combining these two results, it follows that:

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2}\nabla u_1 - |\nabla(\eta u)|^{p(x)-2}\nabla(\eta u)\right) \cdot \nabla\varphi \, dx = 0.$$

This implies

$$\nabla(u_1 - \eta u)^+ = 0 \implies (u_1 - \eta u)^+ = 0,$$

which leads to

$$u_1 \leq \eta u$$

By repeating this argument , we obtain:

$$u_1 \leq \eta^k u \quad \forall k \in \mathbb{N}.$$

Since $\eta < 1$, taking the limit as $k \to \infty$ then $u \to 0$, which contradicts the assumption that u_1 is a positive eigenfunction. Thus, λ_1 must equal λ.

Since $0 < \eta < 1$, it follows that:

$$\eta^n \to 0 \quad \text{as } n \to \infty$$

By taking the limit in the previous inequality $u_1 \leq \eta^n u$, we obtain:

$$u_1 \leq 0.$$

contradiction with the assumption that u_1 is a positive eigenfunction ($u_1 > 0$).

Hence

$$\lambda = \lambda_1(p(x), q(x))$$

Corollary 3.3.2. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < q(x) < p(x)$. The equation*

$$-\Delta_{p(x)} v = |v|^{q(x)-2} v$$

has a unique positive solution in $W_0^{1,p(x)}(\Omega) \backslash \{0\}$.

Proof.

We aim to prove the existence of a unique positive solution for the following equation:

$$-\Delta_{p(x)} v = |v|^{q(x)-2} v$$

Let $u > 0$ be a solution to the equation:

$$-\Delta_{p(x)} u = \lambda_1 \|u\|_{q(x)}^{p(x)-q(x)} |u|^{q(x)-2} u$$

We define v as follows:

$$v = \frac{u}{\|u\|_{q(x)}}$$

By direct calculation, we have:

$$\nabla v = \frac{\nabla u}{\|u\|_{q(x)}}$$

Using the properties of the $p(x)$-Laplacian operator, we obtain:

$$|\nabla v|^{p(x)-2}\nabla v = \frac{1}{\|u\|_{q(x)}^{p(x)-1}}|\nabla u|^{p(x)-2}\nabla u$$

Therefore:

$$-\Delta_{p(x)}v = \frac{1}{\|u\|_{q(x)}^{p(x)-1}}\left(-\Delta_{p(x)}u\right)$$

Substituting the equation for u into the expression for v, we get:

$$-\Delta_{p(x)}v = \frac{\lambda_1}{\|u\|_{q(x)}^{p(x)-1}}\|u\|_{q(x)}^{p(x)-q(x)}|u|^{q(x)-2}u$$

By simplifying the terms, we arrive at:

$$-\Delta_{p(x)}v = \lambda_1\frac{1}{\|u\|_{q(x)}^{q(x)-1}}|u|^{q(x)-2}u$$

Expressing $|v|^{q(x)-2}v$ in terms of u: Since we have:

$$v = \frac{u}{\|u\|_{q(x)}}$$

It follows that:

$$|v|^{q(x)-2}v = \frac{1}{\|u\|_{q(x)}^{q(x)-1}}|u|^{q(x)-2}u$$

Thus, we obtain

$$-\Delta_{p(x)}v = \lambda_1|v|^{q(x)-2}v$$

This confirms that v is indeed a solution to the problem.

Assume the existence of two positive solutions $v_1, v_2 > 0$ such that

$$-\Delta_{p(x)}v = |v|^{q(x)-2}v$$

We can rewrite the equation as follows

$$-\Delta_{p(x)}v = \lambda\|v\|_{q(x)}^{p(x)-q(x)}|v|^{q(x)-2}v \tag{3.13}$$

where the parameter λ is defined by:

$$\lambda = \frac{1}{\|v\|_{q(x)}^{p(x)-q(x)}}$$

Consequently, every solution is an eigenfunction of the operator. According to Theorem 2.9.7, any positive eigenfunction associated with $\lambda_1(p(x), q(x))$ (given that $\lambda = \lambda_1$) must satisfy:

$$v_1 = cv_2$$

for some constant c. Substituting $v_1 = cv_2$ into the original equation, we evaluate both sides:
Left Hand Side
By the homogeneity property of the $p(x)$-Laplacian, we have:

$$-\Delta_{p(x)}(cv_2) = c^{p(x)-1}\left(-\Delta_{p(x)}v_2\right)$$

Right Hand Side

$$|cv_2|^{q(x)-2}(cv_2) = c^{q(x)-1}|v_2|^{q(x)-2}v_2$$

By comparing both sides of the equation, we obtain:

$$c^{p(x)-1} = c^{q(x)-1}$$

This simplifies to:

$$c^{p(x)-q(x)} = 1$$

Given the assumption that $p(x) \neq q(x)$, it follows that:

$$c = 1$$

And consequently:

$$v_1 = v_2$$

This completes the proof of uniqueness.

Lemma 3.3.3. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p(x) < \infty$ and $1 < q(x) < p^*(x)$ with $q(x) \neq p(x)$. Let $S_{p(x)q(x)}$ be the set of all nontrivial solutions to (3.3.2). If $w \in S_{p(x)q(x)}$ is a point of minimum for the restriction of J to $S_{p(x)q(x)}$ then w is an eigenfunction corresponding to $\lambda_1(p(x), q(x))$. In particular, $\lambda_1(p(x), q(x))$ is simple if and only if the restriction of J to $S_{p(x)q(x)}$ has a unique (up to the sign) point of minimum.*

Proof.
Let the manifold $S_{p(x),q(x)}$ be defined as:

$$S_{p(x),q(x)} = \left\{v \in W_0^{1,p(x)}(\Omega), v \neq 0, \|v\|_{q(x)} = 1\right\}$$

Since $\|v\|_{q(x)} = 1$, it follows from the properties of Luxemburg norms that:

$$\int_\Omega |v|^{q(x)}dx = 1$$

The energy functional J is given by:

$$J(v) = \int_\Omega \frac{1}{p(x)} |\nabla v|^{p(x)} dx - \int_\Omega \frac{1}{q(x)} dx$$

We define the first eigenvalue $\lambda_1(p(x), q(x))$ as:

$$\lambda_1(p(x), q(x)) = \inf_{v \neq 0} \frac{\int_\Omega |\nabla v|^{p(x)} dx}{\|v\|_{q(x)}^{p^+}}$$

Under the constraint $\|v\|_{q(x)} = 1$, we obtain:

$$\lambda_1 = \inf_{v \in S} \int_\Omega |\nabla v|^{p(x)} dx$$

If there exists a minimizer w such that $J(w) = \min J$, then:

$$\int_\Omega |\nabla w|^{p(x)} dx = \lambda_1$$

By applying the Variational Principle, since w is a minimum point, the directional derivative (Euler-Lagrange equation) must vanish. This leads to:

$$\int_\Omega |\nabla w|^{p(x)-2} \nabla w \cdot \nabla \varphi \, dx = \lambda_1 \int_\Omega |w|^{q(x)-2} w \varphi \, dx$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. This is precisely the weak formulation of the eigenvalue problem:

$$-\Delta_{p(x)} w = \lambda_1 |w|^{q(x)-2} w$$

which implies that w is the first eigenfunction. Suppose that $v_1, v_2 \in S$ such that:

$$J(v_1) = J(v_2) = \min J$$

Thus, v_1 and v_2 are both first eigenfunctions.

According to the Simplicity Lemma, there exists a constant c such that:

$$v_2 = c v_1$$

By substituting into the norm conditions: Since $\|v_1\|_{q(x)} = 1$ and $\|v_2\|_{q(x)} = 1$, we have:

$$\|c v_1\|_{q(x)} = |c| \|v_1\|_{q(x)} = |c|$$

It follows that:

$$|c| = 1 \implies c = \pm 1$$

And hence:

$$v_2 = \pm v_1$$

The minimizer is unique up to the sign. Therefore, we conclude that J has a unique minimizer (up to the sign). Let us consider two such minimizers u_1 and u_2.

We define the normalized functions:

$$v_1 = \frac{u_1}{\|u_1\|_{q(x)}}, \quad v_2 = \frac{u_2}{\|u_2\|_{q(x)}}$$

Clearly, $\|v_1\|_{q(x)} = \|v_2\|_{q(x)} = 1$. From our initial hypothesis, we have:

$$J(v_1) = J(v_2) = \min J$$

By substituting the previously established uniqueness up to a sign:

$$v_1 = \pm v_2$$

This leads to:

$$\frac{u_1}{\|u_1\|_{q(x)}} = \pm \frac{u_2}{\|u_2\|_{q(x)}}$$

Which implies:

$$u_1 = cu_2$$

where c is a constant.

The results show that the eigenspace associated with λ_1 is one-dimensional. Consequently, λ_1 is simple.

Theorem 3.3.4. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure and $1 < q(x) < p(x)$.(3.3.2) has a unique (up to the sign) nontrivial solution $w \in W_0^{1,p(x)}(\Omega)$ with the following property: $J(w) \leq J(v)$ if $v \in W_0^{1,p(x)}(\Omega)$ is a nontrivial solution to (3.3.2). Moreover, w is an eigenfunction corresponding to $\lambda_1(p(x), q(x))$.*

Proof.
We define the following set:

$$S = \left\{ v \in W_0^{1,p(x)}(\Omega) : \|v\|_{q(x)} = 1 \right\}$$

Let m be the infimum of the functional J over S:

$$m = \inf_{v \in S} J(v)$$

Thus, there exists a minimizing sequence $(v_n) \subset S$ such that $J(v_n) \to m$.

The functional is given by:

$$J(v_n) = \int_\Omega \frac{1}{p(x)} |\nabla v_n|^{p(x)} dx - \int_\Omega \frac{1}{q(x)} |v_n|^{q(x)} dx$$

Since $v_n \in S$, we have $\|v_n\|_{q(x)} = 1$. This implies that the term $\int_\Omega |\nabla v_n|^{p(x)} dx$ remains bounded. Consequently, the sequence (v_n) is bounded in the Sobolev space $W_0^{1,p(x)}(\Omega)$.

Since the space $W_0^{1,p(x)}(\Omega)$ is reflexive, there exists a subsequence, still denoted by (v_n), that converges weakly to some w in $W_0^{1,p(x)}(\Omega)$:

$$v_n \rightharpoonup w \quad \text{in } W_0^{1,p(x)}(\Omega)$$

Moreover, due to the compact embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

it follows that v_n converges strongly to w in $L^{q(x)}(\Omega)$.

Since $v_n \to w$ strongly in $L^{q(x)}(\Omega)$, we have:

$$\|w\|_{q(x)} = 1 \implies w \in S$$

By the weak lower semi-continuity of the modular functional, we obtain:

$$\int_\Omega |\nabla w|^{p(x)} dx \leq \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^{p(x)} dx$$

It follows that:

$$J(w) \leq \liminf_{n\to\infty} J(v_n) = m$$

Given that m is the infimum, we conclude $J(w) = m$, which proves that w is a minimizer.

Taking the Gateaux derivative of the functional at w:

$$\left. \frac{d}{dt} J(w + t\varphi) \right|_{t=0} = 0$$

leads to the following integral identity:

$$\int_\Omega |\nabla w|^{p(x)-2} \nabla w \cdot \nabla \varphi \, dx = \int_\Omega |w|^{q(x)-2} w \varphi \, dx$$

This is the weak formulation of the equation:

$$-\mathrm{div}\left(|\nabla w|^{p(x)-2} \nabla w \right) = |w|^{q(x)-2} w$$

Hence, w is a weak solution to the problem.
We finally define the first eigenvalue λ_1 as:

$$\lambda_1 = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{q(x)}\, dx}$$

Since $\|w\|_{q(x)} = 1$ and $\int_\Omega |\nabla w|^{p(x)} dx = \lambda_1$, it follows that w is the first eigenfunction. Let v_1 and v_2 be two solutions. We define the normalized functions:

$$u_i = \frac{v_i}{\|v_i\|_{q(x)}}$$

Thus, u_1 and u_2 are both first eigenfunctions. By the simplicity of the first eigenvalue λ_1, there exists a constant c such that:

$$u_1 = cu_2 \quad \Longrightarrow \quad v_1 = C_1 v_2$$

Substituting this relation into the $p(x)$-Laplace equation:

$$-\Delta_{p(x)}(C_1 v_2) = |C_1 v_2|^{q(x)-2}(C_1 v_2)$$

Taking into account the homogeneity of the $p(x)$-Laplacian operator, we obtain:

$$C_1^{p(x)-1} = C_1^{q(x)-1}$$

Which can be rewritten as:

$$C_1^{p(x)-q(x)} = 1$$

Since $p(x) \neq q(x)$ almost everywhere in Ω, we must have:

$$C_1 = \pm 1$$

3.4 On the spectrum $\sigma(p(x), q(x))$

Theorem 3.4.1. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, $1 < p(x) < \infty$ and $1 < q(x) < p^*(x)$. Then $\sigma(p(x), q(x))$ is a closed set.*

Proof.
Our objective is to prove that the spectral set $G(p(x), q(x))$ is closed.
Suppose a sequence of eigenvalues $(\lambda_n) \subset G(p(x), q(x))$ such that $\lambda_n \to \lambda$ as $n \to \infty$. For each $n \in \mathbb{N}$, there exists a corresponding eigenfunction $u_n \neq 0$ satisfying the following equation:

$$-\text{div}(|\nabla u_n|^{p(x)-2}\nabla u_n) = \lambda_n |u_n|^{q(x)-2} u_n$$

Without loss of generality, we can normalize the eigenfunctions such that $\|u_n\|_{q(x)} = 1$.

The weak formulation of the problem is:

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\varphi\, dx = \lambda_n \int_\Omega |u_n|^{q(x)-2} u_n \varphi\, dx$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$. By choosing $\varphi = u_n$ as a test function, we get:

$$\int_\Omega |\nabla u_n|^{p(x)} dx = \lambda_n \int_\Omega |u_n|^{q(x)} dx$$

Given the normalization $\|u_n\|_{q(x)} = 1$, the integral on the right hand side is constant (related to the modular), leading to:

$$\int_\Omega |\nabla u_n|^{p(x)} dx = \lambda_n$$

Since the sequence (λ_n) converges to λ, it is bounded. Thus, the sequence of integrals $\int_\Omega |\nabla u_n|^{p(x)} dx$ is also bounded. Consequently, we conclude that the sequence (u_n) is bounded in the Sobolev space $W_0^{1,p(x)}(\Omega)$. By the reflexivity of the space, there exists a subsequence such that:

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega)$$

Thanks to the compact embedding, we obtain:

$$u_n \to u \quad \text{strongly in } L^{q(x)}(\Omega)$$

Since $\|u_n\|_{q(x)} = 1$, the strong convergence implies $\|u\|_{q(x)} = 1$, hence $u \neq 0$.

On the left-hand side of the weak formulation, by utilizing the monotonicity of the $p(x)$-Laplacian and Minty's trick, we have:

$$|\nabla u_n|^{p(x)-2}\nabla u_n \rightharpoonup |\nabla u|^{p(x)-2}\nabla u \quad \text{weakly}$$

On the right hand side, the strong convergence in $L^{q(x)}(\Omega)$ ensures:

$$|u_n|^{q(x)-2} u_n \to |u|^{q(x)-2} u \quad \text{strongly}$$

Passing to the limit as $n \to \infty$, we obtain the following integral identity:

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx = \lambda \int_\Omega |u|^{q(x)-2} u\varphi\, dx$$

which is the weak form of:

$$-\text{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda |u|^{q(x)-2} u$$

Since u is a non-trivial solution ($u \neq 0$), it follows that $\lambda \in \sigma(p(x), q(x))$. Consequently, the spectral set $\sigma(p(x), q(x))$ is closed.

Theorem 3.4.2. *Let Ω be a domain in $\mathbb{R}^N$ with finite measure, and let $p, q \in \mathcal{P}^{\log}(\Omega)$ such that $1 < p^- \le p(x) \le p^+ < \infty$ and $1 < q(x) < p^*(x)$ in Ω.*

Then $\lambda_n(p(x), q(x)) \in \sigma(p(x), q(x))$ for all $n \in \mathbb{N}$, and

$$\lambda_n(p(x), q(x)) \longrightarrow +\infty \quad \text{as } n \to \infty.$$

Proof.

We argue by contradiction. Assume that there exists $L > 0$ such that

$$\lambda_n(p(x), q(x)) \le L \quad \text{for all } n \in \mathbb{N}.$$

Let

$$M = \left\{ u \in W_0^{1,p(x)}(\Omega) : \int_\Omega |u|^{q(x)} dx = 1 \right\}$$

and define the energy functional

$$E(u) = \int_\Omega |\nabla u|^{p(x)} dx.$$

Consider the sublevel set

$$B_L = \{u \in M : E(u) \le L\}.$$

Since B_L is bounded in $W_0^{1,p(x)}(\Omega)$, and this space is reflexive, there exists a sequence $(u_n) \subset B_L$ and $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(x)}(\Omega).$$

By the compact embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

we obtain

$$u_n \to u \quad \text{strongly in } L^{q(x)}(\Omega).$$

Since $u_n \in M$, we have

$$\int_\Omega |u_n|^{q(x)} dx = 1,$$

and passing to the limit yields

$$\int_\Omega |u|^{q(x)} dx = 1,$$

so that $u \neq 0$.

Now, let $P_n : W_0^{1,p(x)}(\Omega) \to X_n$ be a sequence of finite-dimensional projections, where X_n is a subspace of dimension n.

We claim that there exists n such that

$$P_n u \neq 0 \quad \text{for all } u \in B_L.$$

Otherwise, for every n there exists $u_n \in B_L$ such that

$$P_n u_n = 0.$$

Passing to the limit, and using the properties of the projections, we would obtain $u = 0$, which contradicts the fact that $u \neq 0$.

Therefore, there exists n such that

$$\inf_{u \in B_L} \|P_n u\| > 0.$$

Let $A \subset B_L$ be a compact symmetric set such that its Krasnoselskii genus satisfies

$$\gamma(A) \geq k$$

for some large integer k.

Since P_n is odd and continuous and $0 \notin P_n(A)$, we have

$$\gamma(P_n(A)) \geq \gamma(A) \geq k.$$

On the other hand, since $P_n(A) \subset X_n$ and $\dim(X_n) = n$, it follows that

$$\gamma(P_n(A)) \leq n.$$

Choosing $k > n$, we obtain a contradiction:

$$k \leq \gamma(P_n(A)) \leq n.$$

Hence, our assumption is false, and therefore

$$\lambda_n(p(x), q(x)) \to +\infty \quad \text{as } n \to \infty.$$

CHAPTER 4

CONCLUSION

In this dissertation, we have studied a nonlinear eigenvalue problem involving the $p(x)$-Laplacian operator in the framework of variable exponent Lebesgue and Sobolev spaces. More precisely, we have investigated the following problem:

$$\begin{cases} -\Delta_{p(x)}\big(u(x)\big) = \lambda \left\|u\right\|_{q(x)}^{p(x)-q(x)} \left|u\right|^{q(x)-2} u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $p(x)$ and $q(x)$ are variable exponents satisfying appropriate growth conditions.

The analysis of this problem required a careful combination of tools from variable exponent spaces, variational methods, and nonlinear spectral theory. The main results established in this work can be summarized as follows.

In the first chapter, we introduced the functional framework of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, including their main properties such as modular inequalities, embedding theorems, log-Hölder continuity, and the weak formulation of the problem.

In the second chapter, we collected the main auxiliary results needed for our analysis, including monotonicity methods, weak convergence tools, compact embedding results, and eigenvalue properties of the $p(x)$-Laplacian.

The core contribution of this dissertation was presented in the third chapter, where, motivated by the work of Franzina and Lindqvist [8], we extended their results to the variable exponent setting involving two exponents $p(x)$ and $q(x)$. The main results obtained are the following:

- We proved that any eigenfunction is bounded and its first derivatives are locally Hölder continuous (Theorem3.1.1).
- We showed that the first eigenfunction has constant sign in Ω (Corollary3.1.2).

- We established the simplicity of the first eigenvalue $\lambda_1(p(x), q(x))$ using Picone's identity (Theorem3.2.1 and Theorem 3.2.5).
- We proved the uniqueness of positive solutions (Theorem 3.3.1 and Corollary 3.3.2).
- We showed that the spectrum $\sigma(p(x), q(x))$ is a closed subset of $\mathbb{R}$ (Theorem 3.4.1).
- We proved that the eigenvalues diverge to infinity (Theorem 3.4.2).

These results contribute to the ongoing development of nonlinear spectral theory in variable exponent spaces and provide a rigorous mathematical foundation for the study of such problems.

Perspectives.

This work opens the door to several possible extensions and future research directions:

- The study of the second eigenvalue and higher eigenvalues of the $p(x)$-Laplacian.
- The investigation of more general nonlinear problems involving two variable exponents.
- The extension of these results to unbounded domains or to domains with more complex geometry.
- The numerical approximation of eigenvalues in the variable exponent setting.

BIBLIOGRAPHY

[1] Walter Allegretto and Yanxiong Huang. A picone's identity for the p-laplacian and applications. *Nonlinear Analysis: Theory, Methods & Applications*, 32(7):819–830, 1998.

[2] D. V. Cruz-Uribe and A. Fiorenza. *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*. Springer, 2013.

[3] Lars Diening, Petteri Harjulehto, Peter Hästö, and Michael Růžička. *Lebesgue and Sobolev Spaces with Variable Exponents*, volume 2017 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.

[4] X. Fan, Y. Zhao, and D. Zhao. Compact imbedding theorems with symmetry of strauss-lions type for the space $w^{1,p(x)}(\omega)$. *Journal of Mathematical Analysis and Applications*, 255:333–348, 2001.

[5] X. L. Fan and Q. H. Zhang. Existence of solutions for $p(x)$-laplacian dirichlet problem. *Nonlinear Analysis*, 52:1843–1852, 2003.

[6] X. L. Fan and D. Zhao. A class of de giorgi type and hölder continuity. *Nonlinear Analysis*, 36:295–318, 1999.

[7] Giovanni Franzina and Pier Domenico Lamberti. Existence and uniqueness for a p-laplacian nonlinear eigenvalue problem. *Electronic Journal of Differential Equations*, 2010(26):1–10, 2010.

[8] Giovanni Franzina and Peter Lindqvist. An eigenvalue problem with variable exponents. *Nonlinear Analysis: Theory, Methods & Applications*, 85:1–16, 2013.

[9] O. Kováčik and J. Rákosník. On spaces $l^{p(x)}$ and $w^{k,p(x)}$. *Czechoslovak Mathematical Journal*, 41:592–618, 1991.

[10] Olga A. Ladyženskaja, Vsevolod A. Solonnikov, and Nina N. Ural'ceva. *Linear and Quasi-linear Equations of Parabolic Type*, volume 23 of

Translations of Mathematical Monographs. American Mathematical Society, 1988.

[11] Y. Mizuta, T. Ohno, T. Shimomura, and N. Shioji. Compact embeddings for sobolev spaces of variable exponents and existence of solutions for nonlinear elliptic problems involving the $p(x)$-laplacian and its critical exponent. *Annales Academiae Scientiarum Fennicae Mathematica*, 35: 115–130, 2010.

[12] Julian Musielak. *Orlicz Spaces and Modular Spaces*, volume 1034 of *Lecture Notes in Mathematics*. Springer, 2006.

[13] Andrzej Szulkin. Ljusternik-schnirelmann theory on c^1-manifolds. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 5, pages 119–139. Elsevier, 1988.

[14] Luc Tartar. *An Introduction to Sobolev Spaces and Interpolation Spaces*, volume 3 of *Lecture Notes of the Unione Matematica Italiana*. Springer Science & Business Media, Berlin, Heidelberg, 2007.

[15] Peter Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *Journal of Differential Equations*, 51(1):126–150, 1984.

[16] Eberhard Zeidler. *Nonlinear Functional Analysis and Its Applications*. Springer-Verlag, 1986.

الجمهورية الجزائرية الديمقراطية الشعبية
وزارة التعليم العالي والبحث العلمي
جامعة غرداية

Faculté des Sciences et de la Technologie

Département des Mathématiques & d'Informatiques

كلية العلوم والتكنولوجيا

قسم الرياضيات والإعلام الآلي

Rapport de Correction du Mémoire

Enseignant(e) Chargé(e) de la correction, membre du jury est encadreur(se) :

M/Mme/Mlle : **Lalmi Abdellatif**

Thème :

Nonlinear eigenvalue problem involving the p(x)-Laplacian operator: existence and uniqueness results

Après les corrections apportées au mémoire suivant les remarques des jurées,

L'étudiant(e) : **Boukraa Abir**

Est autorisé(e) à déposer le manuscrit au niveau du département.

Ghardaïa le 01/07/2026

Signature
L'encadreur(se)
Lalmi Abdellatif

Signature
Président des Jurés
Chenini Toufik

Signature
Chef du département

جامعة غرداية
قسم الرياضيات و الإعلام الآلي
كلية العلوم والتكنولوجيا

رئيس قسم الرياضيات و الإعلام الآلي
الحاج موسى