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Sur Des Problèmes Aux Limites Dans Des Espaces De Banach

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Publications

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Abstract

In this thesis, we discussed the existence of solutions for a class of boundary value problem for nonlinear fractional differential equations. Our results have been obtained by the technique of measures of noncompactness and the coincidence degree theory which was first introduced by Mawhin.

Keywords

Fractional differential equations, boundary value problem, Caputo fractional derivative, Riemann-Liouville fractional derivative, measure of noncompactness, measures of weak noncompactness, Pettis integrable, Mönch's fixed point theorem, Darbo fixed point theorem, multi-point, resonance, coincidence degree theory.

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Introduction

Fractional calculus (FC) is an extension of ordinary calculus with more than 300 years of history. Fractional calculus appeared when l'Hopital, one of the founders of Calculus, wrote to Leibnitz the father of Calculus about the meaning of $\frac{d^n}{dx^n}f(x)$ when $n = 1/2$. Leibnitz replied in 1695 saying that it could be $\sqrt{dx:x}$ an apparent paradox from which useful consequences would be drawn one day. The name "fractional calculus" may have originated from the question "what if $n = 1/2$?". Therefore, FC generalizes integrals and derivatives to noninteger orders.

During the last decades, fractional differential equations (FDEs) have become an area of interest to researchers due to its high accuracy and applicability in various fileds of science and technology. As many physical, dynamical, biological and chemical phenomenons are represented in more realistic way by using fractional differential equations instead of integer order differential equation. More realistic approach is the main reason for attracting the attention of researchers. Fractional differential equations are equally suitable not only to the mathematicians but also to engineers and physicists. The fractional order differential equations have a large numbers of applications in many fields of science and technology, we refer the readers to the more recent results, e.g., works of Kilbas et al. [76], Podlubny [99] and Caponetto et al. [46] (control theory), Metzler et al. [93] (relaxation in filled polymer networks), Podlubny et al. [100] (heat propagation), Chern [45] (modeling of the behavior of viscoelastic and viscoplastic materials under external influences), Bai and Feng [21] and Cuesta and Finat Codes [53] (image processing) and Gaul et al. [59] (description of mechanical systems subject to damping).

Many techniques have been developed for studying the existence and uniqueness of solutions of initial (IVPs) and boundary value problem (BVPs) for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorems with the measure of noncompactness (MNCs). The concept of measure of noncompactness was first introduced by Kuratowski [81] in 1930. In 1955, the Italian mathematician Darbo [48] used the Kuratowski measure in order to investigate a class of operators(condensing operators) whose properties can be characterized as being intermediate between those of contraction and compact mappings. Darbo's fixed point theorem is useful in establishing existence results for different classes of operator equations. Other measures of noncompactness have been defined since then. The most important ones are the Hausdorff measure of noncompactness introduced by Goldenstein et al. [60] in 1957 (and later studied by Goldenstein and Markus [61]). The notion of the measure of weak noncompactness was introduced by De Blasi in 1977 [26]. This measure can be regarded as a counterpart of the

Hausdorff measure of noncompactness. This theory plays a significant role in topological fixed point problems (cf. [55]) and many existence results for weak solutions of differential and integral equations in Banach spaces. This technique of measure of weak noncompactness and the fixed point theorem of Mönch type were mainly initiated in the monograph of Banaŝ Goebel [27] and subsequently developed and used in many papers ; see for example, Akhmerov et al.[16], Alvàrez [17], Benchohra et al. [32], Guo et al. [62], and the references therein. Recently, a lot of papers have been devoted to weak solutions of nonlinear fractional differential equations [3, 6, 32, 34, 37, 127].

Moreover, topological degree theory may be one of the most effective tools in solving nonlinear equations. As a measure of the number of solutions of equation $fx = y$ for a fixed y, the degree has fundamental properties such as existence, normalization, additivity, and homotopy invariance. The most powerful one in which the value of the degree is invariant under appropriate perturbations plays a crucial role in the study of nonlinear differential and integral equations. After a pioneering work of Kronecker [80] of 1869, the first definition of degree for maps between Euclidean spaces is due to Brouwer [41] in 1912. In 1951, Nagumo [89] redefines the concept, today commonly known as Brouwer degree. The Brouwer degree $\deg_B(f,\Omega,y)$, defined for any continuous mapping $f\colon \overline{\Omega}\subset\mathbb{R}^n\to\mathbb{R}^n$ such that $y\notin f(\partial\Omega)$, where Ω is open and bounded,

$$\deg_B(f,\Omega,y)=\sum_{x\in f^{-1}(y)}\operatorname{sign} f'(x)$$

where $\deg_B(f,\Omega,y)=0$ if $f^{-1}(y)=\emptyset$, $\operatorname{sign} f'(x)$ is the sign of the determinant of the Jacobian matrix.

In 1934, Leray and Schauder [86] generalized Brouwer degree theory to an infinite Banach space and established the so-called Leray Schauder degree. The latter has been recognized as a very important tool for the study of many problems in ordinary and partial differential equations. Afterwards, many authors defined and developed the topological degree theory for various classes of non-compact nonlinear mappings between Banach spaces ; see,e.g., [51, 108, 109, 118]. Browder [42, 43] introduced a topological degree for nonlinear operators of monotone type in reflexive Banach spaces, where the Galerkin method is used to apply the Brouwer degree. Berkovits [38, 39] gave a new construction of the Browder degree, based on the Leray-Schauder degree. An interesting and accurated description of important applications of the Leray-Schauder degree is due to Mawhin in [91]. Mawhin continuation theorem introduced in [90] and developed in [91] in the frame of a degree theory for mappings of the type $L+N$ between normed vector spaces, with L Fredholm of index zero and N satisfying a suitable compactness property. A fundamental result in proving this continuation theorem is the reduction of the Leray-Schauder degree of some compact perturbation of identity in a normed vector space to the Brouwer degree of the associated mapping in a finite-dimensional vector space (reduction property). Such fractional differential equations can be written as $Lx = Nx$, where L and N are operators from a Banach space X to another Banach space Y (L is a linear and N is a nonlinear). If the kernel of the linear part of the above equation contains only zero, the corresponding BVP is called non-resonant, in this case L is invertible. This means that there exists an integral operator ; then, topological methods can be applied to prove existence theorems. Otherwise, if L is a non-invertible, i.e. $\dim \operatorname{Ker} L \geq 1$, then the problem is said to be at resonance, an important class of resonant problems when L is a Fredholm operator with zero-index, the problem can be solved by using the continuation theorem of coincidence degree theory. More recently, many authors investigated the existence of solutions for fractional differential equations at resonance. For instance see [22, 23, 24, 25, 47, 66, 67, 68, 71, 72, 77, 78, 119] and the references therein.

Let us now briefly describe the organization of this thesis :

Chapter one is devoted to some notations, definitions, and preliminary facts that will be used throughtout this thesis.

Chapter two investigates the existence of weak solutions, for the fractional differential equations that contain both the integral boundary condition and the multi-point boundary condition :

$$\begin{cases} D^{\alpha}_{0^+}x(t)+f(t,x(t))=0, \quad t\in J=[0;1],\\ x^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2,\\ x(1)=\sum_{i=1}^{m-2}\sigma_i\int_0^{\eta_i}x(s)\,\mathrm{d}s+\sum_{i=1}^{m-2}\nu_i x(\eta_i),\end{cases}$$

where D^{α} represents the standard Riemann-Liouville fractional derivative of order α satisfying $n-1<\alpha\leq n$ with $n\geq 3$ and $n\in\mathbb{N}^+$. In addition, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$ and $\sigma_i,\nu_i>0$ with $1\leq i\leq m-2$, where m is an integer satisfying $m\geq 3$. $f\colon [0;1]\times E\to E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$.

The **third Chapter** considers more precisely, in section 3.1, the following boundary value problem of nonlinear hybrid fractional differential equations :

$$\begin{cases} {}^{c}D^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]=g(t,x(\nu(t))),\ t\in J=[0;1],\\ a\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=0}+b\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=1}=c,\end{cases}$$

where $0<\alpha\leq 1, a,b,c$ are real constants such that $a+b\neq 0, {}^{c}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\}), g\in C(J\times\mathbb{R},\mathbb{R}), \mu$ and ν are functions from J into itself.

In section 3.2, we looked for the following boundary value problem for hybrid fractional differential equations with fractional separated integral boundary conditions

$$\begin{cases} {}^{c}D^{\alpha}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)), \quad t\in J=[0;1],\\ a_1\left(\frac{x(t)}{f(t,x(t))}\right)_{t=0}+b_1{}^{c}D^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)_{t=0}=\int_0^1 h(t,x(t))\,\mathrm{d}t,\\ a_2\left(\frac{x(t)}{f(t,x(t))}\right)_{t=1}+b_2{}^{c}D^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)_{t=1}=\int_0^1 k(t,x(t))\,\mathrm{d}t.\end{cases}$$

Where $0<\sigma\leq 1<\alpha\leq 2$, ${}^{c}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, f,g,h,k are a given continuous function and $a_i,b_i, i=1,2$ are real constants such that $a_1\neq 0$.

Our approch will be based on the technique of measures of noncompactness in the Banach algebras and a fixed point theorem for the product of two operators verifying a Darbo type condition.

In Chapter four, we discussed the existence of solutions for the following multi-point boundary value problems by using Mawhin's continuation theorem

$$\begin{cases} \left(\phi(t){}^{c}D^{\alpha}_{0^+}u(t)\right)'=f(t,u(t),u'(t),u''(t),{}^{c}D^{\alpha}_{0^+}u(t)), \quad t\in J=[0;1],\\ u(0)=0,\ {}^{c}D^{\alpha}_{0^+}u(0)=0,\ u''(0)=\sum_{i=1}^{m}a_i u''(\xi_i),\ u'(1)=\sum_{j=1}^{l}b_j u'(\eta_j),\end{cases}$$

where ${}^{c}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $2<\alpha\leq 3$, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_l<1$, $a_i,b_j\in\mathbb{R}$, $(i=1,\dots,m,\ j=1,\dots,l)$, $\phi(t)\in C^1[0;1]$ and $\mu=\min_{t\in J}\phi(t)>0$ while $f\colon [0;1]\times\mathbb{R}^4\to\mathbb{R}$ is a Carathéodory function with a nonlinear growth.

Finally, in **Chapter five**, we established the solvability of multi-point BVP of nonlinear fractional differential equations at resonance with three dimensional kernels :

$$\begin{cases} \left(\phi(t)^c D_{0^+}^{\alpha} u(t)\right)' = f(t,u(t),u'(t),u''(t),u'''(t),{}^c D_{0^+}^{\alpha} u(t)), \quad t \in J = [0;1] \\ u(0) = 0, \quad {}^c D_{0^+}^{\alpha} u(0) = 0, \quad u'''(0) = \sum_{i=1}^{m} a_i u'''(\xi_i), \\ u''(0) = \sum_{j=1}^{l} b_j u''(\eta_j), \quad u'(1) = \sum_{k=1}^{n} c_k u'(\rho_k), \end{cases}$$

where ${}^c D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $3 < \alpha \leq 4$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_l < 1$, $0 < \rho_1 < \cdots < \rho_n < 1$, $a_i, b_j, c_k \in \mathbb{R}$, $(i = 1,\ldots,m,\ j = 1,\ldots,l,\ k = 1,\ldots,n)$, $\phi(t) \in C^1[0;1]$, $\mu = \min_{t\in J}\phi(t) > 0$ and $f\colon [0;1]\times\mathbb{R}^5 \to \mathbb{R}$ is a Carathéodory function.

Chapitre 1

Preliminaries

The aim of this chapter is to introduce some basic concepts, notation and elementary results that are used throughout this thesis.

1.1 Weak Topologies

1.1.1 General Statements

Definition 1.1. Let E be a set and let τ_1 and τ_2 be topologies on E. τ_1 is said to be **weaker** than τ_2 if

$$\tau_1 \subset \tau_2.$$

This means that every τ_1-open set is a τ_2-open set.

Remark 1.2. A topology τ_1 is weaker than a topology τ_2 if for every τ_1-open set A, every point $a \in A$ has a τ_2-neighborhood U_a contained in A, as

$$A = \bigcup_{a \in A} U_a,$$

i.e., A is also τ_2-open.

Remark 1.3. If τ_1 is weaker than τ_2 then

$$Id\colon (E,\tau_1) \to (E,\tau_2)$$

is an open mapping and

$$Id\colon (E,\tau_2) \to (E,\tau_1)$$

is continuous.

Proposition 1.4. *Let τ_1, τ_2 be topologies on a set E with $\tau_1 \subset \tau_2$. If (E,τ_1) is Hausdorff then so is (E,τ_2).*

Proposition 1.5. *Let τ_1, τ_2 be topologies on a set E with $\tau_1 \subset \tau_2$. If (E,τ_1) is Hausdorff and (E,τ_2) is compact then $\tau_1 = \tau_2$.*

This means that one cannot weak en a compact Hausdorff topology without losing the Hausdorff property. This also means that one cannot strengthen a compact Hausdorff space without losing compactness.

Let E be a set and let $\mathfrak{F}$ be a family of mappings from E into topological spaces :

$$\mathfrak{F} = \{f_i \colon E \to G_i \mid i \in I\}.$$

Let τ be the topology generated by the subbase

$$\left\{f_i^{-1}(V) \mid i \in I,\, V \in \tau_{G_i}\right\}.$$

Then τ is the weakest topology on E for which all the fa are continuous maps (it is the intersection of all topologies having this property). It is called the **weak topology induced by** $\mathfrak{F}$, orthe $\mathfrak{F}$-topology of E.

Proposition 1.6. *Let $\mathfrak{F}$ be a family of mappings $E \to G_i$ where E is a set and each G_i is a Hausdorff topological space. If $\mathfrak{F}$ separates points on E then the $\mathfrak{F}$-topology on E is Hausdorff.*

Here separates between point means that $x \neq y$ implies that $f(x-y) \neq f(0)$. In the linear case it reduces to $f(x) \neq f(y)$.

Proposition 1.7. *Let (E,τ) be a compact topological space. If there is a sequence $\{f_n \mid n \in \mathbb{N}\}$ of continuous real-valued functions that separates points in E then E is metrizable.*

Theorem 1.8. *Let E be a vector space (no topology) and let E^* be a separating vector space of linear functionals on E. Denote by τ^* the E^*-topology on E. Then $(E,\, \tau^*)$ is a locally convex topological vector space whose dual space is E^*.*

Local convexity is important because of the Hahn-Banach theorem. An extension of the separation theorem states that if E is a locally convex topological vector space, A is compact, and B closed, then there exists a continuous linear map $f \colon E \to \mathbb{R}$ and $s,t \in \mathbb{R}$ such that

$$f(a) < t < s < f(b)$$

for all $a \in A$ and $b \in B$.

1.1.2 The Weak Topology of a Topological Vector Space

Given a topology, we can determine whether a function is continuous. This argument can be reversed : given a space and functions on that space, we can define a topology with respect these functions are continuous. For example, we can take the discrete topology, which is not interesting (only sequence that are eventually constant converge). In the previous subsection, we have laid the foundations to en dow the space with the weakest topology that makes those function continuous. This is the context of weak topologies over topological vector spaces.

Weak and Original Topologies

Definition 1.9. Let (E,τ) be a topological vector space whose dual E^* (the vector space of continuous linear functionals) separates points. The E^*-topology on E is called the **weak topology** (it is the weakest topology with respect every τ-continuous linear functional is continuous).

We denote the weak topology by τ_w (the space it self is often denoted by E_w).

Corollaire 1.1.1. E_w is locally convex and $E_w^* = E^*$.

An other corollary is :

Corollaire 1.1.2. $(E_w)_w = E_w$.

Note that since every $f \in E^*$ is τ continuous and since τ_w is the weakest topology with this property, it follows that

$$\tau_w \subset \tau,$$

justifying the name of weak topology.

The next propositions shows that weak convergence is consistent with what we know :

Proposition 1.10. *A sequence x_n in a topological vector space (E, τ) weakly converges to zero, $x_n \rightharpoonup 0$, if and only if*

$$\forall f \in E^* \qquad f(x_n) \to 0.$$

Corollaire 1.1.3. Every τ-convergent sequence is τ_w convergent.

Weak and Original Boundedness

Proposition 1.11. *Let (E, τ) be a topological vector space. A set $B \subset E$ is τ_w-bounded (weakly bounded) if and only if*

$$\forall f \in E^* \qquad f \textit{ is a bounded functional on } B.$$

Proposition 1.12. *If (E, τ) is an infinite-dimensional topological vector space then every τ_w-neighborhood of zero contains an infinite-dimensional subspace ; in particular (E, τ_w) is not locally bounded.*

Recall that at opological vector space is metrizable if and only if it has a countable local base. Local boundedness implies the existence of acountable local base. Not being locally bounded does not imply a lack of metrizability.

Weak and Original Closedness

We next come to the concept of closure. If a set is τ_w-closed then it is clearly τ-closed. Let B be a set in a topological vector space (E, τ). Its τ-closure $\overline{B}$ is the intersection of all τ-closed sets that contain it, whereas its τ_w-closure $\overline{B}_w$ is the intersection of all τ_w-closed sets that contain it. Since there are more τ-closed sets than τ_w-closed sets,

$$\overline{B} \subset \overline{B}_w.$$

Theorem 1.13. *Let B be a convex subset of alocally convex topological vector space (E, τ). Then,*

$$\overline{B} = \overline{B}_w.$$

This theorem states that if B is a convex set in a locally convex topological vector space and there is a sequence $x_n \in B$ that weakly converges to x (which is not necessarily in B), then there is also a sequence $y_n \in B$ that originally-converges to x.

Corollaire 1.1.4. For convex subsets of locally convex topological vector spaces :

1. τ-closed equals τ_w-closed.
2. τ-dense equals τ_w-dense.

1.1.3 The Weak-* Topology

Let (E, τ) be a topological vector space. The dual space E^* does not come with a priori topology. In Banach spaces, the dual space has a natural operator norm, and we proved that the dual space end owed with that norm is also a Banach space (in fact, we proved that the dual of a normed space is a Banach space). But for general topological vector spaces, we don't have as for now a topology.

Recall the natural in clusion from E to linear functionals on E^*,

$$\iota : x \to F_x,$$

where for $f \in E^*$:

$$F_x(f) = f(x),$$

The family of functionals $\{\iota(x) \mid x \in E\}$ (which we can't call continuous because there is no topology on E^*) separates points on E^* as if

$$\iota(x)(f) = \iota(x)(g)$$

for all $x \in E$ then

$$f(x) = g(x)$$

i.e., $f = g$.
If follows from Theorem 1.8 that the $\iota(E)$-topology of E^* turns it in toalocally convex topological vector space whose dual space is $\iota(E)$. The $\iota(E)$-topology of E^* is called the **weak-* topology**. Every linear functional on E^* that is weak-* continuous is of the form $\iota(t)$ for some $t \in E$. The open sets in the weak star topology (E, τ^*) are generated by the subbase :

$$V(x,r) = \{f \in E^* \mid |f(x)| < r\}.$$

Weak-*convergence of a sequence $(f_n) \subset E^*$ to $f \in E^*$ denoted $f_n \overset{*}{\rightharpoonup} f$ means that

$$\forall x \in E \qquad \lim_{n\to\infty} f_n(x) = f(x).$$

The following central theorem states a compactness property of the weak-*topology. It was proved in 1932 by Banach for separable spaces and in 1940 by Alaoglu in the general case. (Leonidas Alaoglu (1914-1981) was a Greek mathematician.)

Theorem 1.14. Banach-Alaoglu. *Let (E, τ) be a topological vector space. Let $V \subset E$ and let*

$$K = \{f \in E^* \mid |f(x)| \le 1 \quad \textit{for all } x \in V\},$$

Then K is weak-compact. (The set of functionals K is call the* **polar** *of the set of vectors V).*

Reflexive Spaces

For any Banach E space, we note its bidual by E^{**}, equipped with the norm

$$\|\xi\| = \sup_{\|f\|_{E^*} \le 1} |\xi(f)|$$

Definition 1.15. The mapping $j\colon E \to E^{**}$ defined by

$$j(x)(x^*) = x^*(x), \quad x^* \in E^*,$$

defines an **isometric embedding** of E into the bidual E^{**} ; this is an immediate consequence of the Hahn-Banach theorem. We shall always identify E with its image in E^{**}.

Proposition 1.16. *E is weak*-dense in E^{**}.*

A Banach space E is called **reflexive** if this embedding is surjective. As a consequence of the Banach-Alaoglu theorem we have the following characterisation of reflexivity :

Proposition 1.17. *A Banach space E is reflexive if and only if $\overline{B}_E$ is weakly compact.*

Proposition 1.18. *A Banach space E is reflexive if and only if its dual E^* is reflexive.*

1.2 Elements From Fractional Calculus Theory

In this section, we have given the definitions of the fractional integrals, Riemann-Liouville and Caputo fractional derivatives on a finite interval $J = [0;T]$ of the real line and present some of their properties in spaces of summable and continuous functions. More detailed see for example [76, 95, 97, 99, 105].

1.2.1 Absolutely Continuous Functions

We consider finite collections of closed intervals $J_k = [a_k; b_k]$, included in a fixed interval $[0;T]$, with $k = 1,2,\ldots,n$, $0 \le b_k \le a_{j+1} < b_{k+1} \le T$ for $1 \le j \le n-1$. We use the notation

$$\mathscr{J}([0;T]) = \{\{J_1,\ldots,J_n\} \mid n \in \mathbb{N}\}$$

to represent the set of all such collections. A partition is an element $P \in \mathscr{J}([0;T])$ such that $[0;T] = \bigcup_{k=1}^{n} J_k$. We denote by $\mathscr{P}([0;T])$ the set of all partitions. We'll write $\mathscr{J}, \mathscr{P}$ when the interval $[0;T]$ is clear from the context.

Definition 1.19. A function $f\colon [0;T] \to \mathbb{R}$ is said to be of bounded variation, $f \in BV[0;T]$ if

$$V_0^T = \sup_{P \in \mathscr{J}([0;T])} \sum_{k=1}^{n} |f(b_k) - f(a_k)|.$$

is finite. $V_0^a(f)$ is called the total variation of f on $[0;T]$.

Clearly, the total variation remains the same if we replace $\mathscr{J}$ by $\mathscr{P}$.

Definition 1.20. We say that the function $f\colon [0;T] \to \mathbb{R}$ is absolutely continuous, $f \in AC[0;T]$ is for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $P \in \mathscr{J}([0;T])$ with $\sum_{k=1}^{n}(b_k - a_k) < \delta$ we have $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$.

Clearly, an absolutely continuous function on $[0;T]$ is uniformly continuous. Moreover, a Lipschitz continuous function on $[0;T]$ is absolutely continuous. Let f and g be two absolutely continuous functions on $[0;T]$. Then $f+g$, $f-g$, and fg are absolutely continuous on $[0;T]$. If, in addition, there exists a constant $C > 0$ such that $|g(t)| \ge C$ for all $t \in [0;T]$, then f/g is absolutely continuous on $[0;T]$.

Proposition 1.21 ([79]). *$AC[0;T] \subset C[0;T] \cap BV[0;T]$*

Proposition 1.22 ([79]). *If $f \in AC[0;T]$ then f' exists almost everywhere, and is integrable.*

Theorem 1.23 ([79]). *Let $f \in AC[0;T]$ and assume $f'(t) = 0$ almost everywhere. Then f is constant.*

Theorem 1.24 ([79]). *Let $h \in L^1[0;T]$. Consider*

$$f(t) = f(0) + \int_0^t h(x)\,\mathrm{d}x$$

with some constant $f(0)$. Then $f \in AC[0;T]$ and $f' = h$ almost everywhere.

Theorem 1.25 ([79]). *Let $f\colon [0;T] \to \mathbb{R}$ the following are equivalent :*

(*i*) *$f \in AC[0;T]$*

(*ii*) *The function f is differentiable almost everywhere, $f' \in L^1[0;T]$ and*

$$f(t) = f(0) + \int_0^t f'(x)\,\mathrm{d}x$$

holds for all $t \in [0;T]$.

Definition 1.26. For $n \in \mathbb{N}^*$ we denote by $AC^n[0;T]$ the space of real-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[0;T]$ such that $f^{(n-1)} \in AC[0;T]$:

$$AC^n[0;T] = \left\{ f\colon [0;T] \to \mathbb{R}, f^{(k)} \in C[0;T], k = 0 \dots n-1,\ f^{(n-1)} \in AC[0;T] \right\}$$

In particular, $AC^1[0;T] = AC[0;T]$.

A characterization of the functions of this space is given by the following lemma :

Lemma 1.27. *[105, Lemma 2.4] $f \in AC^n[0;T], n \in \mathbb{N}^*$, if and only if, it is represented in the form*

$$f(t) = \frac{1}{(n-1)!}\int_0^t (t-x)^{n-1} f^{(n)}(x)\,\mathrm{d}x + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k$$

1.2.2 The Gamma Function

The definition and certain properties of the Gamma function are reviewed in this subsection.

Definition 1.28. The Gamma function is defined as a definite integral over the positive part of the real axis,

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\,\mathrm{d}t. \tag{1.1}$$

For our purposes, we assume that the independent parametric variable, α, is real. Note that singularities occur when α is zero or a negative integer.

Known exact values of the Gamma function are

$$\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t = \sqrt{\pi}.$$
$$\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}, \qquad \Gamma(2) = 1.$$

Other exact values can be deduced using the properties of the Gamma function.

Proposition 1.29. *Integrating by parts on the right-hand side of* (1.1)*, we obtain the important property*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

Working recursively, we obtain

$$\Gamma(\alpha+n-1) = \alpha(\alpha+1)\cdots(\alpha+n-2)\Gamma(\alpha).$$

for any integer, n. Consequently,

$$\Gamma(n+1) = 1\cdot 2\cdots n = n!.$$

for any integer, n, where the exclamation mark denotes the factorial.

A refection property states that

$$\Gamma(1-\alpha) = -\alpha\Gamma(-\alpha) = \frac{1}{\Gamma(\alpha)}\frac{\pi}{\sin(\pi\alpha)}.$$

for $0<\alpha<1$*. Replacing* α *with* 2α *and using the trigonometric identity* $\sin(2\alpha) = 2\sin\alpha\cos\alpha$*, we obtain*

$$\Gamma(1-2\alpha) = -2\alpha\Gamma(-2\alpha) = \frac{1}{2\Gamma(2\alpha)}\frac{\pi}{\sin(\pi\alpha)\cos(\pi\alpha)}.$$

pour $0<\alpha<\frac{1}{2}$.

1.2.3 Riemann-Liouville Integrals

Definition 1.30. The Riemann-Liouville (RL) fractional integral $I^{\alpha}_{0^+}$ of order $\alpha\in\mathbb{R}_+$ is given as

$$I^{\alpha}_{0^+}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}f(x)\,\mathrm{d}x, \quad t>0,$$

where the operator $I^{\alpha}_{0^+}$ is defined on $L^1[0;T]$. Moreover, for $\alpha=1$, we set $I^{1}_{0^+} := Id$, the identity operator.

Remark 1.31. The notation $I^{\alpha}_{0^+}f(t)\mid_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0,\varepsilon)(\varepsilon>0)$ of 0 as follows :

$$I^{\alpha}_{0^+}f(t)\mid_{t=0} = \lim_{t\to 0+} I^{\alpha}_{0^+}f(t).$$

Generally, $I^{\alpha}_{0^+}f(t)\mid_{t=0}$ is not necessarily to be zero. For instance, let $\alpha\in(0;1), f(t)=t^{-\alpha}$. Then

$$I^{\alpha}_{0^+}t^{-\alpha}|_{t=0} = \lim_{t\to 0^+}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\alpha}\,\mathrm{d}s = \lim_{t\to 0^+}\Gamma(1-\alpha) = \Gamma(1-\alpha).$$

Lemma 1.32 ([76])**.** *If* $\alpha>0,\beta>0$*. Then*

$$I^{\alpha}_{0^+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.$$

The semigroup property of the fractional integration operators $I_{0^+}^{\alpha}$ is given by the following result

Lemma 1.33 ([76]).

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} f = I_{0^+}^{\alpha+\beta} f \qquad \alpha > 0,\ \beta > 0. \tag{1.2}$$

Equations (1.2) is satisfied in any point for $f(t) \in C[0;T]$ and in almost every point for $f(t) \in L^1[0;T]$. They are true in any point even for $f(t) \in L^p[0;T]$ $(1 \le p \le \infty)$ if $\alpha + \beta > 1$.

Lemma 1.34. *Let $\alpha > 0, f \in L^1[0;T]$. Then for all $t \in [0;T]$ we have*

$$I_{0^+}^{\alpha+1} f(t) \le \| I_{0^+}^{\alpha} f \|_{L^1}.$$

Proof. Let $f \in L^1[0;T]$, from Lemma 1.33, we have

$$I_{0^+}^{\alpha+1} f(t) = I_{0^+}^{1} I_{0^+}^{\alpha} f(t) = \int_0^t I_{0^+}^{\alpha} f(s)\,\mathrm{d}s \le \int_0^T |I_{0^+}^{\alpha} f(s)|\,\mathrm{d}s = \| I_{0^+}^{\alpha} f \|_{L^1}.$$

□

Lemma 1.35 ([76]). *Let $\alpha > 0$, the fractional integration operators $I_{0^+}^{\alpha}$ are bounded in $L^p[0;T]$ $(1 \le p \le \infty)$;*

$$\| I_{0^+}^{\alpha} f \|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \| f \|_{L^p}.$$

1.2.4 The Riemann-Liouville Fractional Derivative

The Riemann-Liouville fractional derivative of order α of a suitable function, $f(t)$, is defined as

$$(D_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{1}{(t-x)^{1-n+\alpha}} f(x)\,\mathrm{d}x \tag{1.3}$$

where Γ is the Gamma function, $0 \le t$ is a specified lower integration limit,

$$n = [\alpha] + 1 > \alpha. \tag{1.4}$$

is the integral ceiling of the fractional order, α, and the square brackets indicate the integral part. For example, if $0 \le \alpha < 1$, then $n = 1$; if $1 \le \alpha < 2$, then $n = 2$.

Defining $w = t - x$, we obtain

$$(D_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{1}{w^{1-n+\alpha}} f(t-w)\,\mathrm{d}w$$

or

$$(D_{0^+}^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-t}^0 \frac{1}{|v|^{1-n+\alpha}} f(t+v)\,\mathrm{d}v$$

where $v = -w = x - t$, and we recall that $0 \le t$.

Lemma 1.36 ([76]). *If $\alpha > 0, \beta > 0$. Then*

$$D^{\alpha}_{0+} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha-\beta)} t^{\beta-\alpha-1}.$$

In particular, if $\beta = 1$ the Riemann-Liouville fractional derivatives of a constant are, in general, not equal to zero.

The next result characterizes the conditions for the existence of the fractional derivatives D^{α}_{0+} in the space $AC^n[0;T]$.

Lemma 1.37 ([76]). *Let $\alpha > 0$, and $n = [\alpha]+1$. If $f(t) \in AC^n[0;T]$, then the fractional derivatives D^{α}_{0+} exist almost everywhere on $[0;T]$ and can be represented in the forms*

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-x)^{1-n+\alpha}} f^n(x)dx + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)} t^{k-\alpha}$$

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1.38 ([76]). *If $\alpha > 0$ and $f(t) \in L^p[0;T]$ $(1 \leq p \leq \infty)$, then the following equalities*

$$D^{\alpha}_{0+}(I^{\alpha}_{0+} f(t)) = f(t)$$

hold almost everywhere on $[0;T]$.

From Lemmas 1.37-1.38 we derive the following composition relations between fractional differentiation and fractional integration operators.

Lemma 1.39 ([76]). *If $\alpha \leq \beta > 0$ and $f \in L^p[0;T]$ $(1 \leq p \leq \infty)$ then*

$$D^{\beta}_{0+}(I^{\alpha}_{0+} f(t)) = I^{\alpha-\beta}_{0+} f(t)$$

hold almost everywhere on $[0;T]$.

Lemma 1.40 ([76]). *If $\alpha > 0$ and $n = [\alpha]+1$, then*

$$[D^{\alpha}_{0+} f](t) = 0 \iff f(t) = \sum_{j=1}^{n} c_j t^{\alpha-j},$$

where c_j $(j = 1,\ldots,n)$ are arbitrary constants.

The composition of the fractional integration operator I^{α}_{0+} with the fractional differentiation operator D^{α}_{0+} is given by the following result.

Lemma 1.41 ([76]). *Let $\alpha > 0$, $n = [\alpha]+1$, and $f \in C(0;T) \cap L^1(0;T)$; then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} f(t) = f(t) + \sum_{j=1}^{n} c_j t^{\alpha-j},$$

where c_j $(j = 1,\ldots,n)$ are arbitrary constants.

1.2.5 Caputo Fractional Derivative

The Caputo fractional derivative is a variation of the Riemann-Liouville fractional derivative, defined as

$$({}^{c}D^{\alpha}_{0^+}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{1}{(t-x)^{\alpha+1-n}} f^{(n)}(x)\,\mathrm{d}x$$

where

$$n = [\alpha]+1 \text{ si } \alpha \notin \mathbb{N};\ n = \alpha \text{ si } \alpha \in \mathbb{N}^*. \tag{1.5}$$

Defining $v = t-x$, we obtain

$$({}^{c}D^{\alpha}_{0^+}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{1}{v^{\alpha+1-n}} f^{(n)}(t-v)\,\mathrm{d}v$$

Lemma 1.42 ([76]). *Let $\alpha > 0$ and let n be given by (1.5). Also let $\eta > 0$. Then the following relations hold :*

$${}^{c}D^{\alpha}_{0^+}t^{\eta-1} = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha)}t^{\eta-\alpha-1}, \quad (\eta > n),$$

and

$${}^{c}D^{\alpha}_{0^+}t^{k} = 0, \quad (k = 0,\ldots,n-1).$$

In particular, if $\eta = 1$ the Caputo fractional derivatives of a constant are equal to zero.

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1.43 ([76]). *Let $\alpha \geq \beta \geq 0$, and $f \in L^1[0;T]$. Then ${}^{c}D^{\alpha}_{0^+}I^{\beta}_{0^+}f(t) = I^{\alpha-\beta}_{0^+}f(t)$, for all $t \in [0;T]$*

Lemma 1.44 ([76]). *Let $\alpha > 0$ and let n be given by (1.5). If $f(t) \in AC^n[0;T]$ or $f(t) \in C^n[0;T]$, then*

$$I^{\alpha}_{0^+}({}^{c}D^{\alpha}_{0^+}f(t)) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k$$

1.3 Bochner Integral

In this section, we summarize some results about the integration of Banach space valued functions of a single variable. In a rough sense, vector-valued integrals of integrable functions have similar properties, often with similar proofs, to scalar-valued L^1-integrals. Nevertheless, the existence of different topologies (such as the weak and strong topologies) in the range space of integrals that take values in an infinite-dimensional Banach space introduces significant new issues that do not arise in the scalar-valued case.

Let $J = [0;T]$ such that $(T > 0)$ be a finite interval on the real axis $\mathbb{R}$. E will be a Banach space with the norm $\|\cdot\|_E$ and dual space E^*.

1.3.1 Measurability

Definition 1.45. A simple function $f\colon J \to E$ is a function of the form

$$f = \sum_{i=1}^{n} c_i \chi_{A_i} \tag{1.6}$$

where $A_1, \dots, A_n$ are Lebesgue mesurable subsets of J and $c_1, \dots, c_n \in E$.

Here χ_A denotes the indicator function of the set A.

Definition 1.46. A function $f\colon J \to E$ is strongly measurable, or measurable for short, if there exists a sequence of simple functions $f_n\colon J \to E$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ strongly in E (i.e. in norm) for t a.e. in J.

Clearly, if f is simple function then f is measurable.

Measurability is preserved under natural operations on functions.

(1) If $f\colon J \to E$ is measurable then the real function $\|f\|\colon J \to \mathbb{R}$ is measurable.

(2) If $f\colon J \to E$ is measurable and $\varphi\colon J \to \mathbb{R}$ is measurable, then $\varphi f\colon J \to E$ is measurable.

(3) If $\{f_n\colon J \to E\}$ is a sequence of measurable functions and $f_n(t) \to f(t)$ strongly in E for t pointwise a.e. in J, then $f\colon J \to E$ is measurable.

Definition 1.47. A function $f\colon J \to E$ is called weakly measurable if for each $x^* \in E^*$ the real function $x^*(f)\colon J \to \mathbb{R}$ is measurable.

For finite dimensional, or separable, Banach spaces these definitions of measurability and weak measurability are coincide, but for non-separable spaces a weakly measurable function need not be strongly measurable. The relationship between weak and strong measurability is given by the following Pettis theorem (1938).

Theorem 1.48. *[106, Pettis] A function $f\colon J \to E$ is measurable if and only if f is weakly measurable and almost everywhere separable valued, i.e. there is a set $N \subset J$ of measure zero such that the set*

$$\{f(t);\ t \in J \backslash N\} \subset E$$

is separable.

Proposition 1.49 ([106]). *If E is a separable Banach space then $f\colon J \to E$ is measurable if and only if f is weakly measurable.*

Remark 1.50. Since a continuous function is measurable, every almost separably valued, weakly continuous function is strongly measurable.

1.3.2 Integral

In this subsection, we discuss the vector-valued extension of the Lebesgue integral, the so-called Bochner integral. we will also need its ”weak” companion, the Pettis integral.

The Bochner integral

Definition 1.51. Let

$$f = \sum_{i=1}^{n} c_i \chi_{A_i}$$

be the simple function in (1.6). The integral of f is defined by

$$\int_J f \, \mathrm{d}t = \sum_{i=1}^{n} |A_i| c_i \in E \tag{1.7}$$

where $|A_i|$ denotes the Lebesgue measure of A_i.

It is routine to check that this definition does not depend on the particular representation of f and that

$$\left\| \int_J f \, \mathrm{d}t \right\| \leq \int_J \|f\| \, \mathrm{d}t$$

If f and g are simple functions then

$$\int_J f + g \, \mathrm{d}t = \int_J f \, \mathrm{d}t + \int_J g \, \mathrm{d}t.$$

Moreover, if $A \subset J$ is measurable, it is easy to see that the function $f\chi_A$ is again simple and we set

$$\int_A f \, \mathrm{d}t = \int_J f \chi_A \, \mathrm{d}t.$$

Definition 1.52. A strongly measurable function $f \colon J \to E$ is Bochner integrable, or integrable for short, if there is a sequence of simple functions such that $f_n(t) \to f(t)$ pointwise a.e. in J and

$$\lim_{n \to \infty} \int_J \|f_n - f\| \, \mathrm{d}t = 0.$$

The integral of f is defined by

$$\int_J f \, \mathrm{d}t = \lim_{n \to \infty} \int_J f_n \, \mathrm{d}t.$$

where the limit exists strongly in E.

The value of the Bochner integral of f is independent of the sequence $\{f_n\}$ of approximating simple functions, and

$$\left\| \int_J f \, \mathrm{d}t \right\| \leq \int_J \|f\| \, \mathrm{d}t$$

If f is Bochner integrable and $f = g$ almost everywhere, then g is Bochner integrable and the Bochner integrals of f and g agree.

Moreover, if $L \colon E \to F$ is a bounded linear operator between Banach spaces E, F and $f \colon J \to E$ is integrable, then $Lf \colon J \to F$ is integrable and

$$L\left(\int_J f \, \mathrm{d}t \right) \leq \int_J Lf \, \mathrm{d}t \tag{1.8}$$

More generally, this equality holds whenever $L \colon D(L) \subset E \to F$ is a closed linear operator and $f \colon J \to D(L)$, in which case $\int_J f \, \mathrm{d}t \in D(L)$.

Remark 1.53. For the case $E = \mathbb{R}$, i.e. for $f\colon J \to \mathbb{R}$, the definition of Bochner integral gives an alternative approach to Lebesgue integral. This means that $f\colon J \to \mathbb{R}$ is Bochner integrable in the sense of if and only if f is Lebesgue integrable and the two integrals of f have the same value.

The following result, due to Bochner (1933), characterizes integrable functions as ones with integrable norm.

Theorem 1.54 ([69])**.** *A function $f\colon J \to E$ is Bochner integrable if and only if it is strongly measurable and*

$$\int_J \|f\| \, \mathrm{d}t < +\infty.$$

Thus, in order to verify that a measurable function f is Bochner integrable one only has to check that the real valued function $\|f\|\colon J \to \mathbb{R}$, which is necessarily measurable, is integrable.

The dominated convergence theorem holds for Bochner integrals. The proof is the same as for the scalar-valued case, and we omit it.

Theorem 1.55 ([69])**.** *Suppose that $f_n\colon J \to E$ is Bochner integrable for each $n \in \mathbb{N}$,*

$$f_n(t) \to f(t) \quad \text{as}\, n \to \infty \text{ strongly in}\, E \,\text{for } t \text{ a.e. in } J$$

and there is an integrable function $g\colon J \to \mathbb{R}$ such that

$$\|f_n(t)\| \le g(t) \quad \text{for } t \text{ a.e. in } J \text{ and every } n \in \mathbb{N}.$$

Then $f\colon J \to E$ is Bochner integrable and

$$\int_J f_n \, \mathrm{d}t \to \int_J f \, \mathrm{d}t, \quad \int_J \|f_n - f\| \, \mathrm{d}t \to 0 \quad \text{as } n \to \infty.$$

The definition and properties of L^1-spaces of E-valued functions are analogous to the case of real-valued functions.

Definition 1.56. For $1 \le p < \infty$ the space $L^p(J;E)$ consists of all strongly measurable functions $f\colon J \to E$ such that

$$\int_J \|f\|^p \, \mathrm{d}t < \infty$$

equipped with the norm

$$\|f\|_{L^p(J;E)} = \left[\int_J \|f\|^p \, \mathrm{d}t \right]^{\frac{1}{p}}.$$

The space $L^\infty(J;E)$ consists of all strongly measurable functions $f\colon J \to E$ such that

$$\|f\|_{L^\infty(J;E)} = \sup_{t \in J} \|f(t)\|,$$

where sup denotes the essential supremum.

Note that the elements of $L^1(J;E)$ are precisely the (equivalence classes of) Bochner integrable functions. For $1 \le p \le \infty$ we write

$$L^p(J) = L^p(J;E)$$

Theorem 1.57 ([69])**.** *If E is a Banach space and $1 \le p \le \infty$, then $L^p(J)$ is a Banach space.*

The Pettis integral

Although the theory of Bochner integration is very satisfactory, the conditions for Bochner integrability are sometimes quite restrictive. In this subsection, we briefly sketch a more general integral, the Pettis integral, which can be thought of as the weak analogue of the Bochner integral.

Suppose that $f\colon J \to E$ is a function with the property that $x^*(f)$ belongs to $L^1(J)$ for all $x^* \in E^*$. Such a function induces a linear mapping $L_f\colon E^* \to L^1(J)$ by putting

$$L_f(x^*) = x^*(f), \quad x^* \in E^*.$$

We claim that L_f is a closed operator. Suppose that $\lim_{n\to\infty} x_n^* = x^*$ in E^* and $\lim_{n\to\infty} L_f x_n^* = g$ in $L^1(J)$. By passing to a subsequence, we may assume that $\lim_{n\to\infty} L_f x_n^* = g$ almost everywhere on J. On the other hand, $\lim_{n\to\infty} L_f x_n^* = \lim_{n\to\infty} x_n^*(f) = x^*(f)$ pointwise on J. Therefore $g = x^*(f)$ in $L^1(J)$ and the claim is proved. By the closed graph theorem, L_f is bounded.

Let $L_f^*\colon (L^1(J))^* = L^\infty(J) \to E^{**}$ be the adjoint of the operator L_f defined by

$$L_f^*(g)(x^*) = \int_J g \cdot L_f(x^*)\,\mathrm{d}t = \int_J g \cdot x^*(f)\,\mathrm{d}t \in \mathbb{R}, \quad g \in L^\infty(J).$$

$L_f^*(g)$ is a linear functional on E^* for any $g \in L^\infty(J)$ because

$$\int_J g \cdot (ax_1^* + bx_2^*)(f)\,\mathrm{d}t = a\int_J g \cdot x_1^*(f)\,\mathrm{d}t + b\int_J g \cdot x_2^*(f)\,\mathrm{d}t$$

and it is also bounded because the boundedness of the operator L_f gives

$$|L_f^*(g)(x^*)| = \left|\int_J g \cdot L_f(x^*)\,\mathrm{d}t\right| \le \|g\|_{L^\infty} \cdot \|L_f\| \cdot \|x^*\|_{E^*}.$$

Hence $L_f^*(g) \in E^{**}$ for every $g \in L^\infty(J)$.

Assuming $g = \chi_A$ where $A \subset J$ is measurable, we have

$$L_f^*(\chi_A)(x^*) = \int_J \chi_A \cdot x^*(f)\,\mathrm{d}t = \int_A x^*(f)\,\mathrm{d}t.$$

Then $L_f^*(\chi_A) \in E^{**}$ for every mesurable $A \subset J$.

For each measurable set $A \subset J$ we now define

$$\tau(E,E^*) - \int_A f\,\mathrm{d}t = L_f^*(\chi_A).$$

We call $\tau(E,E^*) - \int_A f dt$ the $\tau(E,E^*)$-integral of f over A. It is the unique element in E^{**} which satisfies

$$L_f^*(\chi_A)(x^*) = \int_A x^*(f)\,\mathrm{d}t, \quad x^* \in E^{**}.$$

Definition 1.58 ([69])**.** A weakly integrable function $f\colon J \to E$ is called Pettis integrable if the adjoint of the operator $L_f\colon x^* \mapsto x^*(f)$ maps $L^\infty(J)$ into E.

We denote by $P(J,E)$ the set of all Pettis integrable functions $f\colon J \to E$.

Note that if f is Pettis integrable, then for every mesurable set $A \subset J$, there exists an element $x_A \in E$ such that for all $x^* \in E^*$ we have

$$x^*(x_A) = \int_A x^*(f) \tag{1.9}$$

The requirement (1.9) defines the element $x_A \in X$ uniquely ; in fact, $x_A = L_f^*(\chi_A)$. We call x_A the Pettis integral of f over A, notation

$$x_A = (P) - \int_A f \, \mathrm{d}t.$$

Clearly, every Bochner integrable function is Pettis integrable and the integrals agree on every measurable set $A \in J$.

The following result gives a sufficient condition for the Pettis integrability of a strongly measurable function.

Theorem 1.59. [69, *Pettis*] *Let $1 < p \leq \infty$ and $1 \leq q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : J \to E$ be a strongly measurable function satisfying $x^*(f) \in L^p(J)$ for all $x^* \in E^*$. Then for all $\varphi \in L^q(J)$ the function*

$$s \mapsto \varphi(s) f(s)$$

is Pettis integrable.

Corollaire 1.3.1 ([69])**.** Let $f : J \to E$ be a strongly measurable function satisfying $x^*(f) \in L^p(J), p > 1$ for all $x^* \in E^*$ then f is Pettis integrable.

For $p = 1$ and $q = \infty$, Theorem 1.59 and Corollary 1.3.1 break down :

Proposition 1.60 ([98])**.** *If $f(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $f(\cdot)h(\cdot)$ is Pettis integrable.*

Let us now recall the definitions of the fractional Pettis integral.

Definition 1.61 ([76])**.** Let $h : J \to E$ be a function. The fractional Pettis integral of the function h of order $\alpha > 0$ is defined by

$$I_{0^+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \, \mathrm{d}s,$$

where the sign $\int$ denotes the Pettis integral.

1.4 Measure of Noncompactness in Banach Spaces

Measures of noncompactness are very useful tools in the theory of operator equations in Banach spaces. They are very often used in the theory of functional equations, including ordinary differential equations, equations with partial derivatives, integral and integro-differential equations, optimal control theory, etc. In particular, the fixed point theorems derived from them have many applications. Most facts provided here come from the books [16, 19, 26, 27, 29].

1.4.1 Kuratowski and Hausdorff MNCs in Banach Spaces

In this subsection, we review the concept of Kuratowski and Hausdorff measure of noncompactness in Banach spaces, also we consider specific properties and examples in some special spaces.

Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element 0. By $B(x,r)$, we denote the closed ball in E centred at x with the radius r. By B_r, we denote the ball $B(0,r)$. Moreover, in place of B_1, we will write B_E (the unit ball in the space E). If Ω is nonempty subset of E, then $\overline{\Omega}$ and $\mathrm{Co}\Omega$ denote the closure and the convex hull of Ω, respectively.

Definition 1.62 ([16])**.** The Kuratowski measure of noncompactness $\alpha(\Omega)$ of the set Ω, is the infimum of the numbers $d > 0$, such that Ω admits a finite covering by sets of diameter smaller than d.

Definition 1.63 ([16])**.** The Hausdorff measure of noncompactness $\chi(\Omega)$ of the set Ω, is the infimum of the numbers $\varepsilon > 0$, such that Ω has a finite ε-net in E.

Recall that a set $S \subset E$ is called an ε-net of Ω if $\Omega \subset S + \varepsilon \overline{B}_E = \{s + \varepsilon b : s \in S, b \in \overline{B}_E\}$.

The next properties are common to α and χ, and so we are going to use ϕ to denote either of them.

Proposition 1.64 ([16])**.** *Let ϕ denote α or χ. Then the following properties are satisfied in any Banach space E :*

(a) *Regularity : $\phi(\Omega) = 0$ if and only if $\overline{\Omega}$ is compact.*

(b) *Semi-additivity : $\phi(\Omega_1 \cup \Omega_2) = \max\{\phi(\Omega_1), \phi(\Omega_2)\}$.*

(c) *Monotonicity : $\Omega_1 \subset \Omega_2$ implies $\phi(\Omega_1) \leq \phi(\Omega_2)$.*

(d) *Non-singularity : If Ω is a finite set, then $\phi(\Omega) = 0$.*

(e) *Semi-homogeneity : $\phi(\lambda\Omega) = |\lambda|\phi(\Omega)$ for any number λ.*

(f) *Algebraic semi-additivity : $\phi(\Omega_1 + \Omega_2) \leq \phi(\Omega_1) + \phi(\Omega_2)$.*

(g) *Invariance under translations : $\phi(x + \Omega) = \phi(\Omega)$, for any $x \in E$.*

(h) *Lipschitzianity : $|\phi(\Omega_1) - \phi(\Omega_2)| \leq L_\phi \rho(\Omega_1, \Omega_2)$, where $L_\chi = 1, L_\alpha = 2$ and ρ denotes the Hausdorff metric. $(\rho(\Omega_1, \Omega_2) = \inf\{\varepsilon > 0 : \Omega_2 \subset \Omega_1 + \varepsilon\overline{B}_E, \Omega_1 \subset \Omega_2 + \varepsilon\overline{B}_E\})$.*

(i) *Continuity : for any $\Omega \subset E$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that $|\phi(\Omega) - \phi(\Omega_1)| < \varepsilon$ for all Ω_1 satisfying $\rho(\Omega, \Omega_1) < \delta$.*

Some less trivial properties of these measures of noncompactness are obtained in the next theorems.

Theorem 1.65 ([16])**.** *The Kuratowski and Hausdorff MNCs are invariant under passage to the closure and to the the convex hull : $\phi(\Omega) = \phi(\overline{\Omega}) = \phi(\mathrm{Co}\Omega)$.*

Theorem 1.66 ([16])**.** *Let B_E be the unit ball in a Banach space E. Then, $\alpha(B_E) = \chi(B_E) = 0$ if E is finite dimensional, and $\alpha(B_E) = 2, \chi(B_E) = 1$ otherwise.*

Since α and χ are invariant under passage to the convex hull, we obtain the following corollary :

Corollaire 1.4.1. Let S_E be the unit sphere in a Banach space E. Then, $\alpha(S_E) = \chi(S_E) = 0$ if E is finite dimensional, and $\alpha(S_E) = 2, \chi(S_E) = 1$ otherwise.

Theorem 1.67 ([16]). *The Kuratowski and Hausdorff MNCs are related by the inequalities*

$$\chi(\Omega) \leq \alpha(\Omega) \leq 2\chi(\Omega).$$

In the class of all infinite dimensional Banach spaces, these inequalities are the best possible.

Remark 1.68. Though, in general, α and χ are different MNCs, in some Banach spaces, we can find a direct relation between them.

Now, we mention the Hausdorff MNC in special spaces ℓ^p, c_0, $C[0;T]$, $L^p[0;T]$. For more details see [16].

The Hausdorff MNC in the Spaces $\ell^p(1 \leq p < \infty)$ and c_0

In the spaces $\ell^p(1 \leq p < \infty)$ and c_0 of sequences summable in the p-th power and respectively sequences converging to zero, the MNC χ can be computed by means of the formula,

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \|(Id - P_n)x\|. \tag{1.10}$$

where P_n is the projection onto the linear span of the first n vectors in the standard basis.

The Hausdorff MNC in the Spaces $C[0;T]$

In the space $C[0;T]$ of continuous real-valued functions on the segment $[0;T]$, the value of the set-function χ on a bounded set Ω can be computed by means of the formula,

$$\chi(\Omega) = \frac{1}{2} \lim_{\delta\to 0} \left\{ \sup_{x\in\Omega} \left[\max_{0\leq r\leq\delta} \|x - x_r\| \right] \right\}, \tag{1.11}$$

where x_r denotes the r-translate of the function x,

$$x_r(t) = \begin{cases} x(t+r) \quad , & 0 \leq t \leq T - r, \\ \ \ x(T) \quad , & T - r \leq t \leq T. \end{cases}$$

The Hausdorff MNC in the Spaces $L^p[0;T]$

In the space $L^p[0;T]$ of equivalence classes x of measurable functions $x\colon [0;T] \to \mathbb{R}$ with integrable p-th power, endowed with the norm $\|x\| = \left(\int_0^T |x(s)|^p \,\mathrm{d}s\right)^{\frac{1}{p}}$. Then

$$\frac{1}{2}\mu(\Omega) \leq \chi(\Omega) \leq \mu(\Omega). \tag{1.12}$$

The function μ appearing above is defined by the formula,

$$\mu(\Omega) = \lim_{\delta\to 0} \sup_{x\in\Omega} \max_{0\leq r\leq\delta} \|x - x_r\|,$$

where x_r denotes the r-translate of the function x, or, alternatively, the Steklov function

$$x_r(t) = \frac{1}{2r} \int_{t-r}^{t+r} x(s) \,\mathrm{d}s.$$

1.4.2 Axiomatic Definition of MNC in Banach Spaces

We will mention here the axiomatic approach for measure of noncompactness, developed by Banaŝ and Goebel [27] in 1980.

Let $\mathfrak{M}_E$ denotes the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ indicates the family of all relatively compact sets.

Definition 1.69. A mapping $\mu\colon \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness if it satisfies the following conditions :

(1) The family $\operatorname{Ker}\mu = \{X \in \mathfrak{M}_E;\ \mu(X) = 0\}$ is non-empty and $\operatorname{Ker}\mu \subset \mathfrak{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\mathrm{Co}X) = \mu(X)$.

(4) $\mu(\overline{X}) = \mu(X)$.

(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0;1]$.

(6) If (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n \geq 1)$ and $\lim_{n\to\infty}\mu(X_n) = 0$ then $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

Let us pay attention to the fact that, one of the most important properties of the measure of noncompactness is a consequence of axiom (6). Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any $n = 1, 2, \ldots,$ we get that $\mu(X_\infty) = 0$. This means that the set X_∞, belongs to the kernel $\operatorname{Ker}\mu$ of the measure μ.

Further on, we indicate a few important classes of measures of noncompactness [27].

Definition 1.70. Let μ be a measure of noncompactness in the Banach space E. We will call the measure μ homogeneous if

(7) $\mu(\lambda X) = |\lambda|\mu(X)$

for $\lambda \in \mathbb{R}$. If the measure μ satisfies the condition

(8) $\mu(X+Y) \leq \mu(X) + \mu(Y)$

it is called subadditive. The measure μ being both homogeneous and subadditive is said to be sublinear.

Definition 1.71. We say that a measure of noncompactness μ has the maximum property if

(9) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.

The most important class of measures of noncompactness is described in the below given definition.

Definition 1.72. A sublinear measure of noncompactness μ which has the maximum property and is such that $\operatorname{Ker}\mu = \mathfrak{N}_E$ is called the regular measure.

Measure of Noncompactness in $C[0;T]$

Given $Q \in \mathfrak{M}_{C[0;T]}$ and $\delta > 0$, let

$$\omega(Q,\delta) = \sup\{w(x,\delta) : x \in Q\}$$

where

$$\omega(x,\delta) = \sup\{|x(t) - x(s)| : t,s \in [0;T], |t-s| \leq \delta\}.$$

The quantity $\omega(Q,\delta)$ denotes the so-called modulus of continuity of the set Q.

We have the following result, which is due to Banaš and Goebel [27].

Theorem 1.73. *Let $\omega_0 \colon \mathfrak{M}_{C[0;T]} \to \mathbb{R}_+$ be the mapping defined by*

$$\omega_0(Q) = \lim_{\delta \to 0} \omega(Q,\delta). \tag{1.13}$$

Then ω_0 is a measure of noncompactness in $C[0;T]$ in the sense of Definition 1.69. Moreover, we have

$$\omega_0(Q) = 2\chi(Q).$$

Definition 1.74. Let E be a Banach algebra. A measure of noncompactness μ in E said to satisfy **condition (m)** if it satisfies the following condition :

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X),$$

for any $X, Y \in \mathfrak{M}_E$.

This definition appears in [28].

As is known the family of all real valued and continuous functions defined on interval $[0;T]$ is a Banach space with the standard norm

$$\|x\| = \sup\{|x(t)|, t \in [0;T]\}.$$

Notice that $(C[0;T], \|\cdot\|)$ is a Banach algebra, where the multiplication is defined as the usual product of real functions.

Lemma 1.75. *The measure of noncompactness ω_0 on $C[0;T]$ satisfies condition (m).*

Proof. Let X, Y be a fixed subset of $\mathfrak{M}_{C[0;T]}$, $\varepsilon > 0$ and $t, s \in [0;T]$ with $|t-s| \leq \varepsilon$. Then, for $x \in X$ and $y \in Y$, we have

$$\begin{aligned} |x(t)y(t) - x(s)y(s)| &\leq |x(t)y(t) - x(t)y(s)| + |x(t)y(s) - x(s)y(s)| \\ &\leq |x(t)||y(t) - y(s)| + |y(s)||x(t) - x(s)| \\ &\leq \|x\|\omega(y,\varepsilon) + \|y\|\omega(x,\varepsilon). \end{aligned}$$

Thus,

$$\omega(xy,\varepsilon) \leq \|x\|\omega(y,\varepsilon) + \|y\|\omega(x,\varepsilon),$$

so,

$$\omega(XY,\varepsilon) \leq \|X\|\omega(Y,\varepsilon) + \|Y\|\omega(X,\varepsilon).$$

Therefore, we get

$$\omega_0(XY) = \lim_{\varepsilon \to 0} \omega(XY,\varepsilon) \leq \|X\|\omega_0(Y) + \|Y\|\omega_0(X).$$

This completes the proof.

□

1.4.3 The Axiomatic Measure of Weak Noncompactness

The notion of a measure of weak noncompactness was introduced by De Blasi [26] in 1977 and it is the map $\beta : \mathfrak{M}_E \to \mathbb{R}_+$, defined by

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a weakly compact } \varepsilon-\text{net in } E\},$$

Now, we are going to recall some basic properties of $\beta(\cdot)$.

Properties 1.76 ([26]). Let X,Y be to elements of $\mathfrak{M}_E$. The following properties hold :

(1) $X \subset Y$, then $\beta(X) \leq \beta(Y)$.

(2) $\beta(X) = 0$ if and only if $\overline{X}^w$ is weakly compact.

(3) $\beta(\overline{X}^w) = \beta(X)$, where $\overline{X}^w$ denotes the weak closure of X.

(4) $\beta(X \cup Y) = \max\{\beta(X), \beta(Y)\}$.

(5) $\beta(X+Y) \leq \beta(X) + \beta(Y)$.

(6) $\beta(\lambda X) = |\lambda|\beta(X)$, for $\lambda \in \mathbb{R}$.

(7) $\beta(\mathrm{Co}X) = \beta(X)$.

Theorem 1.77 ([26]). *The measure β is regular.*

Theorem 1.78 ([26]). *Let B_E be the unit ball in a Banach space E. Then $\beta(B_E) = 0$ if E is reflexive and $\beta(B_E) = 1$ otherwise.*

Next, we present a theorem of Ambrosetti type.

Theorem 1.79 ([62]). *Let $J = [0;T]$ and $H \subset C(J,E)$ be a bounded and equicontinuous subset. Then the function $t \to \beta(H(t))$ is continuous on J, and*

$$\beta_C(H) = \max_{t \in J} \beta(H(t)), \quad \beta\left(\int_J u(s)\,\mathrm{d}s\right) \leq \int_J \beta(H(s))\,\mathrm{d}s,$$

where $H(s) = \{u(s) : u \in H\}, s \in J$ and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(J,E)$.

1.4.4 Some Fixed Point Theorems Involving a MNC

The main application of measures of noncompactness in the fixed point theory is contained in the following theorem, which is called the fixed-point theorem of Darbo type, as extension of Schauder's theorem. In this subsection, first we recall Schauder's and Darbo's fixed-point theorem, and we review some important generalizations of Darbo's theorem.

Theorem 1.80. *(Schauder [5]) Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in the set Ω.*

The mapping $T : \Omega \to \Omega$ is said to be a μ-contraction if there exists a positive constant $k < 1$ such that

$$\mu(TW) \leq k\mu(W) \tag{1.14}$$

for any bounded closed subset $W \subset \Omega$.

Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and Banach's fixed point theorem.

Theorem 1.81 ([27, 48]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T : \Omega \to \Omega$ be a continuous operator. If T is a μ-contraction, then T has at least one fixed point.*

A generalization of Theorem 1.81 for the case that μ is a regular measure of noncompactness was proved by Sadovskii in [117] and we present it in the following theorem.

Theorem 1.82 ([117]). *Suppose that Ω is a nonempty, bounded, closed, and convex subset of E and $T : \Omega \to \Omega$ a continuous mapping. If for any nonempty subset W of Ω with $\mu(W) > 0$ we have*

$$\mu(TW) \leq \mu(W)$$

where μ is a regular measure of noncompactness in E, then T has at least one fixed point in Ω.

Recently, Aghajani et al. [9] extended the Darbo's fixed point theorem using control functions and presented the following result.

Theorem 1.83 ([9]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator satisfying*

$$\mu(TW) \leq \varphi(\mu(W)), \tag{1.15}$$

for any non-empty subset W of Ω, where μ is a measure of noncompactness in E and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \in \mathbb{R}_+$, where $\varphi^n(t)$ denotes the n-iteration of φ. Then T has at least one fixed point.

In [9] the authors proved the following Lemma which will be useful in our considerations.

Lemma 1.84 ([9]). *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then the following conditions are equivalent :*

(i) $\lim_{n\to 0} \varphi^n(t) = 0$, *for any* $t \geq 0$,

(ii) $\varphi(t) < t$, *for any* $t > 0$.

By commodity, we will denote by $\mathscr{A}$ the class of functions given by

$$\mathscr{A} = \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+ : \varphi \text{ is nondecreasing and } \lim_{n\to\infty} \varphi^n(t) = 0 \text{ for any } \ t \in \mathbb{R}_+\},$$

where $\varphi^n(t)$ denotes the n-iteration of φ.

Remark 1.85. It is easy to see that if $\varphi \in \mathscr{A}$ then $\varphi(t) < t$, for any $t > 0$. Indeed, in contrary case, we can find $t_0 > 0$ and $t_0 \leq \varphi(t_0)$. By using the nondecreasing character of φ, we have

$$0 < t_0 \leq \varphi(t_0) \leq \varphi^2(t_0) \leq \cdots \leq \varphi^2(t_0) \leq \dots,$$

and, consequently, $0 < t_0 \leq \lim_{n\to\infty} \varphi^n(t_0)$ and this contradicts the fact that $\varphi \in \mathscr{A}$. Moreover, this proves that if $\varphi \in \mathscr{A}$ then φ is continuous at $t_0 = 0$.

Remark 1.86. Taking into account Remark 1.85, the contractive condition appearing in Theorem 1.83, i.e.,$\mu(TW) \leq \varphi(\mu(W))$ can be rewritten as $\mu(TW) < \mu(W)$ for any $W \in \mathfrak{M}_E \setminus \operatorname{Ker}\mu$ and, therefore, Theorem 1.83 is a immediate consequence of Sadovskii theorem 1.82.

Examples of functions belonging to $\mathscr{A}$ are $\varphi(t) = \ln(1+t), \varphi(t) = \arctan t$ and $\varphi(t) = \frac{t}{1+t}$.

The following generalization of Darbo's fixed point theorem appears in [74] and it is the version in the context of measures of noncompactness of a recent result about fixed point theorem which appears in [73].

Let Θ be the class of functions $\theta : (0;\infty) \to (1;\infty)$ satisfying the following condition : For every sequence $(t_n) \subset (0;\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$.

Examples of functions belonging to the class of Θ are $\theta(t) = e^{\sqrt{t}}, \theta(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{t^\alpha}\right)$ with $0 < \alpha < 1, \theta(t) = (1+t^2)^\beta$ with $\beta > 0$ and $\theta(t) = t+1$.

Theorem 1.87 ([74]). *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Suppose that there exist $\theta \in \Theta$ and $k \in [0;1)$ such that, for any nonempty subset W of Ω with $\mu(TW) > 0$,*

$$\theta(\mu(TW)) \leq [\theta(\mu(W))]^k,$$

where μ is a measure of noncompactness in E. Then T has a fixed point in Ω.

The following fixed point theorem is a variant of the Darbo-Mönch fixed point theorem given by O'Regan [101] in 1998, which was motived by some ideas used in [18].

Theorem 1.88 ([101]). *Let D be a closed convex and equicontinuous subset of a Banach space E such that $0 \in D$. Assume that $N : D \to D$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{Co}(\{0\} \cup N(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{1.16}$$

holds for every subset $V \subset D$, then N has a fixed point.

1.5 Coincidence Degree Theory

In the 1970s, Mawhin systematically studied a class of mappings of the form $L+N$, where L is a Fredholm mapping of index zero and N is a nonlinear mapping, which he called a L-compact mapping.

In this section, we introduce Mawhin's degree theory for L-compact mappings and various properties of this degree.

1.5.1 Fredholm and L-Compact Mappings

Definition 1.89. Let E and F be normed spaces. A linear mapping $L\colon \operatorname{dom} L \subset E \to F$ is called a Fredholm mapping if

(1) $\operatorname{Ker} L$ has finite dimension ;

(2) $\operatorname{Im} L$ is closed and has finite codimension.

Proposition 1.90 ([102]). *Let E be a Banach space, $T : E \to E$ be a linear bounded Fredholm operator and $K : E \to E$ be a linear continuous compact mapping. Then $T+K$ is a Fredholm mapping.*

Recall that the codimension of $\operatorname{Im} L$ is the dimension of $\operatorname{Coker} L = F/\operatorname{Im} L$. If L is a Fredholm mapping, then its index is defined by

$$\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L.$$

Now, assume that L is a Fredholm mapping. Then, there exist two linear continuous projections $P\colon E \to E$ and $Q\colon F \to F$ such that such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L.$$

Also, we have

$$E = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad F = \operatorname{Im} L \oplus \operatorname{Im} Q$$

as the topological direct sums.

Obviously, the restriction of L_p of L to $\operatorname{dom} L \cap \operatorname{Ker} P$ is one to one and onto $\operatorname{Im} L$ and so its inverse $K_P\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ is defined. We denote by $K_{P,Q}\colon F \to \operatorname{dom} L \cap \operatorname{Ker} P$ the generalized inverse of L defined by

$$K_{P,Q} = K_P(Id - Q).$$

Let $L\colon E \to F$ be a Fredholm operator of index zero. Then, there exists a bijection $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ (the existence of J is ensured by the fact that $\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = n$). It is easy to prove that $L + JP\colon \operatorname{dom} L \to F$ is a bijection and

$$(L+JP)^{-1} = J^{-1}Q + K_{P,Q}.$$

Notice that $JP\colon E \to F$ is linear continuous operator of finite rank ($\dim \operatorname{Im} JP = \dim \operatorname{Im} Q < \infty$. Denote by $\mathfrak{F}(L)$ the set of all continuous mapping $A\colon E \to F$ such that $\operatorname{Im} A$ has finite dimension and $L + A\colon \operatorname{dom} L \to F$ is an bijection. Since $JP \in \mathfrak{F}(L)$, then $\mathfrak{F}(L) \neq \emptyset$. Assume that $\Lambda \subset E$ and $N\colon \Lambda \to F$ a mapping (generally nonlinear operators). Then, for each $A \in \mathfrak{F}(L)$, we have

$$Lx = Nx, \qquad x \in \operatorname{dom} L \cap \Lambda$$

is equivalent to the equation

$$(L+A)x = (N+A)x,$$

leading to the fixed point problem

$$x = (L+A)^{-1}(N+A)x, \qquad x \in \Lambda,$$

because $(L+A)^{-1}(N+A)x \in \operatorname{dom} L$ for all $x \in \Lambda$.

In particular, for $A = JP$, the equation $Lx = Nx$, $x \in \operatorname{dom} L \cap \Lambda$ is equivalent to the fixed point problem

$$x = (P + J^{-1}QN + K_{P,Q}N)x, \quad x \in \Lambda.$$

Let Λ be a metrique space and $N\colon \Lambda \to N$ be a mapping.

Lemma 1.91 ([91]). *If there exists $A \in \mathfrak{F}(L)$ such that $(L+A)^{-1}N$ is compact on Λ, then for all $B \in \mathfrak{F}(L)$, $(L+B)^{-1}N$ is also compact on Λ.*

Proof. Let $B \in \mathfrak{F}(L)$, then

$$\begin{aligned}(L+B)^{-1}N &= (L+B)^{-1}(L+A)(L+A)^{-1}N \\ &= (L+B)^{-1}(L+B+A-B)(L+A)^{-1}N \\ &= (Id+(L+B)^{-1}(A-B))(L+A)^{-1}N \\ &= (L+A)^{-1}N+(L+B)^{-1}(A-B)(L+A)^{-1}N.\end{aligned}$$

As $A-B$ is continuous with finite rank and $(L+B)^{-1}$ is bijective, therefore $(L+B)^{-1}(A-B)$ is a linear continuous mapping of finite rank, hence compact. On the other hand, we have by hypothesis $(L+A)^{-1}N$ is compact then so is $(L+B)^{-1}(A-B)(L+A)^{-1}N$.

□

Definition 1.92. We say that $N\colon \Lambda \to F$ is L-compact on Λ if there exists $A \in \mathfrak{F}(L)$ such that $(L+A)^{-1}N\colon \Lambda \to E$ is compact on Λ.

This definition is justied by Lemma 1.91.

Definition 1.93. For the operator $N\colon \Lambda \to F$ to be L-compact on Λ, it is necessary and sufficient that the operator

$$M = P+J^{-1}QN+K_{P,Q}N,$$

is compact on Λ, in this case $L+N$ is called L-compact perturbation of the Fredholm operator L.

For $E=F$ and $L=Id$, this concept reduces to the classical one of compact mapping.

It is easy to verify that $N\colon \Lambda \to F$ is L -compact on Λ if and only if $QN\colon \Lambda \to F$ is continuous, $QN(\Lambda)$ is bounded and $K_{P,Q}N = K_p(Id-Q)N\colon \Lambda \to E$ is compact.

If L is invertible, it is sufficient to take $0=A\in\mathfrak{F}(L)$ and consequently the L-compactness of N on is reduced to the compactness of $L^{-1}N$ on Λ.

Proposition 1.94 ([91])**.** (1) *If the operator $N\colon \Lambda \to F$ is L-compact on Λ, then N is L-compact on every subset Ω of Λ.*

(2) *The sum of two L-compact operators on the same set is as well L-compact on Λ.*

Definition 1.95. If $N\colon \Lambda \to F$ is L-compact on each bounded subset B of E, we shall say that N is L-completely continuous on E.

Proposition 1.96 ([91])**.** *If the linear operator $A\colon E \to F$ is L-completely continuous on E with* $\mathrm{Ker}(L+A)=\{0\}$*, then the operator $L+A$ is bijective, and for any operator $N\colon \Lambda \to F$ L-compact on Λ, the operator $(L+A)^{-1}N\colon \Lambda \to E$ is compact on Λ.*

Proof. For $B \in \mathfrak{F}(L)$ we have

$$L+A = (L+B)\left(Id+(L+B)^{-1}(A-B)\right),$$

as $L+B$ is bijective, then

$$\mathrm{Ker}(Id+(L+B)^{-1}(A-B)) = \mathrm{Ker}(L+A) = \{0\},$$

Moreover, $(L+B)^{-1}(A-B)$ is completely continuous on E ; because $A-B$ is L-completely continuous on E. Therefore, $Id+(L+B)^{-1}(A-B)$ is a completely continuous linear perturbation of identity and one-to-one, then $Id+(L+B)^{-1}(A-B)$ is a linear homeomorphism of E onto E. As a consequence, $(L+A)\colon \operatorname{dom} L \to F$ is a bijection. Suppose now that $N\colon \Lambda \to F$ is L-compact on Λ ; then

$$\begin{aligned}(L+B)^{-1}N &= (L+B)^{-1}(L+A)(L+A)^{-1}N\\ &= (L+B)^{-1}(L+B+A-B)(L+A)^{-1}N\\ &= Id+(L+B)^{-1}(A-B)(L+A)^{-1}N.\end{aligned}$$

Since $(L+B)^{-1}(A-B)$ is bijective and $(L+B)^{-1}N$ is compact on Λ , then $(L+A)^{-1}N = Id+(L+B)^{-1}(A-B)^{-1}(L+B)^{-1}N$ is compact on Λ.

□

1.5.2 Degree Theory for L-Compact Mappings

Let E,F be two real normed vector space and $L\colon \operatorname{dom} L \subset E \to F$ a Fredholm operator of index 0 ; denoted by C_L the set of pairs $(L+N,\Omega)$, where Ω is an open bounded subset in E with $N\colon \overline{\Omega} \to F$ is L-compact on which satisfies the condition $(L+N)(x) \neq 0$, for all $x \in \operatorname{dom} L \cap \partial\Omega$.

A mapping $D_L\colon C_L \to \mathbb{Z}$ will be called a degree with respect to L if it is not identically zero and satisfies the following axioms.

(a) Additivity-excision property : If $(L+N,\Omega) \in C_L$; and Ω_1,Ω_2 are two disjoint open subsets in Ω such that

$$(L+N)x \neq 0, \quad \text{for all } x \in \operatorname{dom} L \cap (\overline{\Omega}\backslash\Omega_1 \cup \Omega_2),$$

then $(L+N;\Omega_1) \in C_L$, $(L+N,\Omega_2) \in C_L$ and

$$D_L(L+N,\Omega) = D_L(L+N,\Omega_1) + D_L(L+N,\Omega_2).$$

(b) Homotopy invariance axiom : Let $H\colon (\operatorname{dom} L \times [0;1]) \cap \overline{\Gamma} \to F$ be the operator defined by

$$H(x,\lambda) = Lx + N(x,\lambda),$$

where Γ is an opan bunded in $E \times [0;1]$, and $N\colon \overline{\Gamma} \to F$ is L-Compact on $\overline{\Gamma}$. Assume that

$$H(x,\lambda) \neq 0, \quad \text{for all } (x,\lambda) \in \operatorname{dom} L \cap (\partial\Gamma)_\lambda \times [0;1],$$

where $(\partial\Gamma)_\lambda = \{x \in E : (x,\lambda) \in \partial\Gamma\}$, then for each $\lambda \in [0;1]$ we have $(H(\cdot,\lambda),\Gamma_\lambda) \in C_L$ and $D_L(H(\cdot,\lambda),\Gamma_\lambda)$ is constant on $[0;1]$, where Γ_λ denotes the set

$$\Gamma_\lambda = \{x \in E : (x,\lambda) \in \Gamma\}.$$

From these axioms, we can immediately deduce the following properties :

(1) Excision property : Suppose that $(L+N,\Omega) \in C_L$ and $\Omega_1 \subset \Omega$ an open subset such that $(L+N)(x) \neq 0$; for all $x \in \operatorname{dom} L \cap (\overline{\Omega}\backslash\Omega_1)$, then

$$(L+N,\Omega_1) \in C_L \text{ and } D_L(L+N,\Omega) = (L+N,\Omega_1).$$

(2) Existence property : If $D_L(L+N,\Omega)\neq 0$; then the equation $Lx+Nx=0$ has at least one solution in Ω.

(3) Invariance on the boundary : If $(L+N_1,\Omega)$ and $(L+N_2,\Omega)$ belong to C_L ; such that $N_1x=N_2x$ for all $x\in D(L)\cap\partial\Omega$, then $D_L(L+N_1,\Omega)=D_L(L+N_2;\Omega)$.

(4) Normalization property : If $(L+N,\Omega)\in C_L$, where $L+N=A|_{\overline{\Omega}}$ with A is a linear one-to-one mapping from $\operatorname{dom}A$ into F ; then

$$|D_L(L+N-b,\Omega)|=\begin{cases}1 & \text{if } b\in(L+N)(\operatorname{dom}L\cap\Omega),\\ 0 & \text{if } b\notin(L+N)(\operatorname{dom}L\cap\Omega).\end{cases}$$

For further details, we refer to [92] and [91].

Brouwer degree

Assume that $\dim E=\dim F<\infty$, $L=0$ and N is continuous on $\overline{\Omega}$ such that $Nx\neq 0$ for all $x\in\partial\Omega$. As N is a continuous mapping of finite rank, then $N=0+N\in C_0$. In this case D_0 is reduced to the Brouwer degree and is usually denoted by

$$D_0(N,\Omega)=\deg_B(N,\Omega,0).$$

Proposition 1.97. *Let E,F,G be three real normed vector space such that* $\dim E=\dim F=\dim G<\infty$, $N\colon\overline{\Omega}\subset E\to F$, $(N,\Omega)\in C_0$ *and* $A\colon F\to G$ *is a bijection, then*

$$D_0(AN,\Omega)=\operatorname{sign}(\det A)D_0(N,\Omega).$$

Leray-Schauder degree

If $E=F$ is a real Banach space, $L=Id$ and N is compact (hence Id-compact) such that for all $x\in\partial\Omega$, $x+Nx\neq 0$. Then $Id+N\in C_{Id}$ and D_{Id} is the same so-called Leray-Schauder degree denoted by

$$D_{Id}(Id+N,\Omega)=\deg_{LS}(Id+N,\Omega,0).$$

Let $\delta=\inf_{x\in\partial\Omega}\|(Id+N)x\|>0$, N is continuous compact mapping. Then, for any $\varepsilon>0$, there exist a finite dimensional space F^ε and a continuous mapping $N^\varepsilon\colon\overline{\Omega}\to F^\varepsilon$ such that $\|N^\varepsilon x-Nx\|<\varepsilon$ for all $x\in\overline{\Omega}$. If $0<\varepsilon<\frac{\delta}{2}$ then

$$\begin{aligned}\inf_{x\in\partial\Omega}\|(Id+N^\varepsilon)x\| &= \inf_{x\in\partial\Omega}\|(Id+N^\varepsilon)x-(Id+N)x-(Id+N)x\|\\ &\geq \inf_{x\in\partial\Omega}|\|(Id+N)x\|-\|(Id+N^\varepsilon)x-(Id+N)x\||\\ &> \frac{\delta}{2}>0.\end{aligned}$$

which proves that $0\notin(Id+N^\varepsilon)(\partial\Omega)$ and in particular $0\notin(Id+N^\varepsilon)(F^\varepsilon\cap\partial\Omega)$. Consequently, the degree $\deg_B(Id+N^\varepsilon|_{F^\varepsilon},\Omega\cap F^\varepsilon,0)$ is well defined, and so we define

$$\deg_{LS}(Id+N,\Omega,0)=\deg_B(Id+N^\varepsilon|_{F^\varepsilon},\Omega\cap F^\varepsilon,0)$$

this definition is independent of the chosen subspace F^ε and approximation N^ε (see [113]).

Proposition 1.98. *For $(Id+N,\Omega)\in C_{Id}$, Let E_0 be a closed subspace of E and $N(\overline{\Omega})\subset E_0$, then*

$$\deg_{LS}(Id+N,\Omega,0)=\deg_{LS}(Id+N|_{E_0},\Omega\cap E_0,0)$$

Proof. Since $N(\overline{\Omega})\subset E_0$, we may choose a finite dimensional space $E_0^{\varepsilon}\subset E_0$ and a continuous mapping $N^{\varepsilon}\colon \overline{\Omega}\to E_0^{\varepsilon}$ such that $\|N^{\varepsilon}x-Nx\|<\varepsilon$ for small $\varepsilon>0$. Then we have

$$\deg_{LS}(Id+N,\Omega,0)=\deg_{B}(Id+N^{\varepsilon}|_{E_0^{\varepsilon}},\Omega\cap E_0^{\varepsilon},0)=\deg_{LS}(Id+N|_{E_0},\Omega\cap E_0,0).$$

□

Remark 1.99. (1) If $dim\,E<\infty$, then $\deg_{LS}(Id+N,\Omega,0)=\deg_{B}(Id+N,\Omega,0)$.

(2) Brouwer degree and Leray-Schauder degree satisfy all the axioms above and properties which will be used subsequently.

1.5.3 Coincidence Degree for L-Compact Mappings

In this subsection, we define coincidence degree for L-compact mappings and give some properties of coincidence degree

Denote by $C(L)$ the set of all linear mappings $A\colon E\to F$ L-completely continuous on E such that $\mathrm{Ker}(L+A)=\{0\}$. Notice that $\mathfrak{F}(L)\subset C(L)$.

Lemma 1.100 ([91])**.** *For all $A,B\in C(L)$, the following assertions are true*

(1) *$\Delta_{A,B}=(L+A)^{-1}(B-A)$ is completely continuous on E,*

(2) *$Id+(L+B)^{-1}(N-B)=(Id+\Delta_{A,B})(Id+(L+A)^{-1}(N-A))$,*

(3) *for all $r>0$, $|D_{Id}(Id+\Delta_{A,B},B_r)|=1$.*

Proof. It is clear that $\Delta_{A,B}$ is completely continuous on E because if $B\in C(L)$, then in view of proposition 1.96 $(A-B)$ is L-completely continuous on E. On the other hand, we have

$$\begin{aligned}Id+(L+B)^{-1}(N-B)&=Id+(L+B)^{-1}(L+B+A-B)(L+A)^{-1}(N-A+A-B)\\&=Id+\left(Id+(L+B)^{-1}(A-B)\right)(L+A)^{-1}(N-A)+(L+B)^{-1}(A-B)\\&=(Id+\Delta_{A,B})+(Id+\Delta_{A,B})(L+A)^{-1}(N-A)\\&=(Id+\Delta_{A,B})\left(Id+(L+A)^{-1}(N-A)\right)\end{aligned}$$

$Id+\Delta_{A,B}\colon E\to E$ is a completely continuous perturbation of identity with $Id+\Delta_{A,B}=(L+B)^{-1}(L+A)$; then $Id+\Delta_{A,B}$ is an isomorphism, which completes the proof.

□

Lemma 1.101 ([91])**.** *For $(L+N,\Omega)\in C_L$, where N is fixed, $|D_{Id}(Id+(L+A)^{-1}(N-A),\Omega)|$, does not depend on the choice of the operator A in $C(L)$.*

Proof. From Lemma 1.100 and the product formula of Leray-Schauder degree (see [**?**, 102], we find

$$D_{Id}(Id+(L+A)^{-1}(N-A),\Omega)=D_{Id}(Id+\Delta_{A,B},B_r)D_{Id}(Id+(L+B)^{-1}(N-B),\Omega),$$

then
$$|D_{Id}(Id+(L+A)^{-1}(N-A),\Omega)| = |D_{Id}(Id+(L+B)^{-1}(N-B),\Omega)|.$$

□

Let $P\colon E \to E$, $Q\colon F \to F$, $K_{P,Q}\colon F \to E$ and $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ be the operators introduced in subsection 1.5.1, where the isomorphism J is chosen such that $D_{Id}(Id+\Delta_{B,JP},B_r)=1$, then we may note

$$Id+(L+JP)^{-1}(N-JP) = Id+(J^{-1}Q+K_{P,Q})(N-JP) = Id-(P-J^{-1}QN-K_{P,Q}N).$$

As $(L+N,\Omega)\in C_L$, then $P-J^{-1}QN-K_{P,Q}N$ is compact and satisfies $x\neq Px-J^{-1}QNx-K_{P,Q}Nx$, for all $x\in \operatorname{dom} L\cap\partial\Omega$, which justifies the following definition.

Definition 1.102. If $(L+N,\Omega)\in C_L$, the degree of $L+N$ in Ω with respect to L is defined by

$$D_L(L+N,\Omega) = \deg_{LS}(Id-P+J^{-1}QN+K_{P,Q}N,\Omega,0).$$

The degree defined is called the coincidence degree of L and $-N$ on $\operatorname{dom} L\cap\Omega$. This degree was introduced by J. Mawhin in 1972 (see [91]).

Using the properties of Leray-Schauder degree, one can show that D_L satisfies the properties of excision-additivity, invariance by homotopy, and the non-nullity of the degree.

The computation of $D_L(L+N,\Omega)$ is reduced to that of Brouwer degree in the following interesting particular case.

Proposition 1.103 ([91])**.** *If $(L+N,\Omega)\in C_L$ with $N(\overline{\Omega})\subset \operatorname{Im} Q$, then $(N|_{\operatorname{Ker} L},\Omega\cap\operatorname{Ker} L)\in C_0$ and $D_L(L+N,\Omega)=\operatorname{sign}(\det J^{-1})\deg_B(N|_{\operatorname{Ker} L},\Omega\cap\operatorname{Ker} L,0)$.*

Proof. Using the definition of D_L with the same notations, we get $QN=N$, $(Id-P)_{|\operatorname{Ker} L}=0$, $K_{P,Q}N=0$ and $(P-J^{-1}N)(\overline{\Omega})\subset \operatorname{Ker} L$, thus by the definition of the Leray-Schauder degree and propositions 1.97-1.98 we have

$$\begin{aligned}
D_L(L+N,\Omega) &= \deg_{LS}(Id-P+J^{-1}N,\Omega,0)\\
&= \deg_B(Id-P+J^{-1}N|_{\operatorname{Ker} L},\Omega\cap\operatorname{Ker} L,0)\\
&= \deg_B(J^{-1}N|_{\operatorname{Ker} L},\Omega\cap\operatorname{Ker} L,0)\\
&= \operatorname{sign}(\det J^{-1})\deg_B(N|_{\operatorname{Ker} L},\Omega\cap\operatorname{Ker} L,0)
\end{aligned}$$

□

1.5.4 Existence Theorems for Operator Equations

Let E,F be real normed spaces, $L\colon \operatorname{dom} L\subset E\to F$ be a linear Fredholm mapping of index zero and $\Omega\subset E$ be an open bounded subset with $\operatorname{dom} L\bigcap\overline{\Omega}\neq\emptyset$.

Theorem 1.104 ([91, 102])**.** *Let $N,T:\overline{\Omega}\to F$ be two L-compact. If the following conditions are satisfied :*

(1) $Lx+\lambda Nx+(1-\lambda)Tx\neq 0$ *for all* $(x,\lambda)\in \operatorname{dom} L\cap\partial\Omega\times[0;1)$;

(2) $D_L(L+T,\Omega)\neq 0$;

then, $Lx+Nx=0$ *has a solution in* $\operatorname{dom} L \bigcap \overline{\Omega}$.

Proof. If the equation $Lx+Nx=0$ has a solution in $\operatorname{dom} L \cap \partial\Omega$, the theorem is proved. Otherwise, let $H\colon \overline{\Omega}\times[0;1] \to F$ be defined by

$$H(x,\lambda) = \lambda Nx + (1-\lambda)Tx \quad \text{for all } (x,\lambda) \in \overline{\Omega}\times[0;1],$$

H is L-compact on $\overline{\Omega}\times[0;1]$, by assumption we have $Lx+H(x,\lambda) \neq 0$ for all $x \in \operatorname{dom} L \cap \partial\Omega \times [0;1]$. By the invariance homotopy property, we obtain

$$D_L(L+N,\Omega) = D_L(L+T,\Omega) \neq 0,$$

then it is sufficient to use the existence property to complete the proof.

□

The following result is a special case of theorem 1.104 where $\operatorname{Ker} L \neq \{0\}$.

Theorem 1.105 ([91, 102])**.** *Let* $N,T: \overline{\Omega} \to F$ *be L-compact. Let* $F_0 \subset F$ *be a subspace with* $F = \operatorname{Im} L \oplus F_0$ *algebraically and* $T(\overline{\Omega}) \subset F_0$*. Suppose that the following conditions hold :*

(1) $Lx + \lambda Nx + (1-\lambda)Tx \neq 0$ *for all* $(x,\lambda) \in \operatorname{dom} L \cap \partial\Omega \times (0;1)$

(2) $Tx \neq 0$ *for all* $x \in \operatorname{Ker} L \cap \partial\Omega$

(3) $\deg_B(T|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

Then, the equation $Lx+Nx=0$ *has a solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Proof. Let $Q\colon F \to F$ be the projection such that $\operatorname{Im} Q = F_0$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Then $QT = T$ and $(L+T)x = 0$ if and only if

$$Q(L+T)x = 0, \quad (Id-Q)(L+T)x = 0,$$

i.e., $Tx = 0$ and $Lx = 0$. Therefore, by the assumption (2) we deduce $(L+T,\Omega) \in C_L$ and in view of Proposition1.103, we have

$$|D_L(L+T,\Omega,0)| = |\deg_B(T|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0)| \neq 0.$$

Thus, it follows from Theorem 1.104 that $Lx = Nx$ has a solution in $\operatorname{dom} L \cap \Omega$. This completes the proof.

□

A useful consequence of Theorem 1.105 is the following one

Theorem 1.106. *[91, 102, Mawhin's continuation theorem] Let* $N\colon E \to F$ *L-compact. Suppose that the following conditions hold :*

(1) $Lx + \lambda Nx \neq 0$ *for every* $(x,\lambda) \in (\operatorname{dom} L \setminus \operatorname{Ker} L) \bigcap \partial\Omega \times (0;1)$.

(2) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \bigcap \partial\Omega$.

(3) $\deg_B(QN|_{\operatorname{Ker} L}, \Omega \bigcap \operatorname{Ker} L, 0) \neq 0$, *where* $Q\colon F \to F$ *is a projection such that* $\operatorname{Im} L = \operatorname{Ker} Q$.

Then, the abstract equation $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \bigcap \overline{\Omega}$.

Proof. Let us apply Theorem 1.105 with $F_0 = \operatorname{Im} Q$ and $T = QN$ it is clear that T is L-compact. By the assumption (2), we know that

$$QNx \neq 0 \quad \text{for all } x \in \operatorname{Ker} L \cap \partial\Omega.$$

Now, if $Lx + \lambda Nx + (1-\lambda)QNx = 0$ for some $(x, \lambda) \in \operatorname{dom} L \cap \partial\Omega \times (0;1)$, then we have

$$QNx = 0, \quad Lx + \lambda Nx = 0.$$

But $QNx = 0$ implies that $Nx \in \operatorname{Im} L$ hence, $x \in (\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega$ and $Lx + \lambda Nx = 0$, which contradicts the assumption (1), this means that

$$Lx + \lambda Nx + (1-\lambda)QNx \neq 0 \quad \text{for all } (x, \lambda) \in \operatorname{dom} L \cap \partial\Omega \times (0;1)$$

Thus, the conditions of Theorem 1.105 are satisfied, and consequently, $Lx = Nx$ has a solution in $\operatorname{dom} L \bigcap \overline{\Omega}$.

□

Remark 1.107. As $\deg_B\left(J^{-1}QN\,|_{\operatorname{Ker} L}, \Omega \bigcap \operatorname{Ker} L, 0\right) = \pm \deg_B\left(QN\,|_{\operatorname{Ker} L}, \Omega \bigcap \operatorname{Ker} L, 0\right)$ we can replace the condition (3) in Theorem 1.106 by the more general condition

$$\deg_B\left(J^{-1}QN\,|_{\operatorname{Ker} L}, \Omega \bigcap \operatorname{Ker} L, 0\right) \neq 0.$$

Chapitre 2

Weak Solutions for Nonlinear Fractional Differential Equations with Integral and Multi-Point Boundary Conditions [1]

2.1 Introduction

In this chapter we investigate the existence of weak solutions, for the fractional differential equations that contain both the integral boundary condition and the multi-point boundary condition :

$$\begin{cases} D_{0^+}^{\alpha}x(t)+f(t,x(t))=0, \quad t\in J=[0;1], \\ x^{(i)}(0)=0, \quad i=0,1,2,\ldots,n-2, \\ x(1)=\sum_{i=1}^{m-2}\sigma_i\int_0^{\eta_i}x(s)\,\mathrm{d}s+\sum_{i=1}^{m-2}\nu_i x(\eta_i), \end{cases} \tag{2.1}$$

where D^{α} represents the standard Riemann-Liouville fractional derivative of order α satisfying $n-1<\alpha\leq n$ with $n\geq 3$ and $n\in\mathbb{N}^+$. In addition, $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1$ and $\sigma_i,\nu_i>0$ with $1\leq i\leq m-2$, where m is an integer satisfying $m\geq 3$. $f\colon [0;1]\times E\to E$ is a given function satisfying some assumptions that will be specified later, E is a Banach space with norm $\|\cdot\|$.

This problem was studied recently in [114] in the scalar case using Krasnoselkii's fixed point theorem, Schauder type fixed point theorem, Banach's contraction mapping principle and nonlinear alternative for single-valued maps.

Here we extend the results of [114] to cover the abstract case. We establish the existence of weak solutions of the problem (2.1) using method associated with the technique of measures of weak noncompactness and a fixed point theorem of Mönch's type, which is an important method for seeking solutions of differential and integral equations.

1. **Z. Baitiche**, K. Guerbati and M. Benchohra, Weak solutions for some nonlinear fractional differential equations with fractional integral boundary conditions in Banach spaces, *PanAmerican Mathematical Journal* Volume 29 (2019), Number 1, 86 - 100.

2.2 Existence Results

In this section we discuss the existence theorem of weak solutions for the problem (2.1). E denotes the real Banach space with norm $\|\cdot\|$ and dual E^* also $E_w = (E,w) = (E,\sigma(E,E^*))$ denotes the space E with its weak topology. $C(J,E)$ is the Banach space of continuous functions $x\colon J \to E$, with the usual supremum norm.

$$\|x\|_\infty = \sup_{t\in J} \|x(t)\|_E,$$

Definition 2.1. A function $h\colon E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E (i.e. for any $(x_n)_n$ in E with $x_n \to x$ in (E,w) then $h(x_n) \to h(x)$ in (E,w) for each $t \in J$).

Lemma 2.2. *Let $n-1 < \alpha \le n$ with $n \ge 3$ and let $h \in C(J,E)$ be a given function, the unique solution of the fractional differential equation*

$$D^{\alpha}_{0^+}x(t) + h(t) = 0, \quad t \in J$$

with multi-point and integral boundary conditions

$$\begin{cases} x^{(i)}(0) = 0, \quad i = 0,1,2,\dots,n-2, \\ x(1) = \sum_{i=1}^{m-2} \sigma_i \int_0^{\eta_i} x(s)\,\mathrm{d}s + \sum_{i=1}^{m-2} \nu_i x(\eta_i), \end{cases}$$

is given by

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)\,\mathrm{d}s \\ &- \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\frac{\sigma_i}{\Gamma(\alpha+1)}\int_0^{\eta_i}(\eta_i - s)^{\alpha}h(s)\,\mathrm{d}s - \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\frac{\nu_i}{\Gamma(\alpha)}\int_0^{\eta_i}(\eta_i - s)^{\alpha-1}h(s)\,\mathrm{d}s \\ &= \int_0^1 G(t,s)h(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)h(s)\,\mathrm{d}s \\ &+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)h(s)\,\mathrm{d}s \end{aligned}$$

where $\xi = 1 - \frac{1}{\alpha}\sum_{i=1}^{m-2}\sigma_i\eta_i^{\alpha} - \sum_{i=1}^{m-2}\nu_i\eta_i^{\alpha} > 0$ and

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} & , \quad 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1} & , \quad 0 \le t \le s \le 1. \end{cases}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha+1)}\begin{cases} t^{\alpha}(1-s)^{\alpha-1} - (t-s)^{\alpha} & , \quad 0 \le s \le t \le 1, \\ t^{\alpha}(1-s)^{\alpha-1} & , \quad 0 \le t \le s \le 1. \end{cases}$$

The proof is similar to the one given in [114].

Remark 2.3. From the expression of $G(t,s)$ and $H(t,s)$, it is obvious that $G(t,s)$ and $H(t,s)$ are continuous and nonnegative on $J \times J$. It is easy to figure out that $0 \le G(t,s) \le \frac{1}{\Gamma(\alpha)}$ and $0 \le H(t,s) \le \frac{1}{\Gamma(\alpha+1)}$ hold for all $t,s \in J$.

To prove the main results, we need the following assumptions :

(H_1) For each $t \in J$, the function $f(t,\cdot)$ is weakly sequentially continuous ;

(H_2) For each $x \in C(J,E)$, the function $f(\cdot,x(\cdot))$ is Pettis integrable on J ;

(H_3) There exist $p \in L^\infty(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that
$$\|f(t,x(t))\| \leq p(t)\psi(\|x\|_\infty);$$

(H_4) There exists a constant $R > 0$ such that
$$\frac{R}{\|p\|_{L^\infty}\psi(R)K} > 1, \tag{2.2}$$
where
$$K = \frac{1}{\Gamma(\alpha)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\sigma_i\frac{(\alpha+1)\eta_i^\alpha - \alpha\eta_i^{\alpha+1}}{\alpha\Gamma(\alpha+2)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\nu_i\frac{\eta_i^{\alpha-1}-\eta_i^\alpha}{\Gamma(\alpha+1)}$$

(H_5) For each bounded and measurable set $D \subset E$, and for each $t \in J$, we have
$$\beta(f(t,D)) \leq p(t)\beta(D).$$

Theorem 2.4. *Assume that the hypotheses $(H_1)-(H_5)$ hold. If*
$$\|p\|_{L^\infty} < \frac{1}{K}, \tag{2.3}$$
then the boundary value problem (2.1) has at least one solution,

Proof. Transform the integral equation (2.1) into a fixed point equation. Consider the operator $\mathcal{N} : C(J,E) \to C(J,E)$ defined by :
$$\mathcal{N}x(t) = \int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)f(s,x(s))\,\mathrm{d}s$$
$$+\frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)f(s,x(s))\,\mathrm{d}s.$$

Since, $s \mapsto G(t,s), s \mapsto H(t,s)$ are $\in L^\infty(J)$ then $G(t,\cdot)f(\cdot,x(\cdot)), H(t,\cdot)f(\cdot,x(\cdot))$ for all $t \in J$ are Pettis integrable (Proposition 1.60) and thus, the operator $\mathcal{N}$ is well defined.
Let $R > 0$ and consider the set
$$D = \left\{ \begin{array}{c} x \in C(J,E) : \|x\|_\infty \leq R, \quad \|x(t_2)-x(t_1)\|_E \leq \\ \frac{\|p\|_{L^\infty}\psi(R)}{\Gamma(\alpha+1)}\left[(1+\Gamma(\alpha+1)K)(t_2^{\alpha-1}-t_1^{\alpha-1}) + (t_2^\alpha - t_1^\alpha)\right], \quad t_1,t_2 \in J, t_1 < t_2 \end{array}\right\}$$

It is clear that the convex closed and equicontinuous subset $D \subset C(J,E)$.

We will show that the operator $\mathcal{N}$ satisfies all the assumptions of Theorem 1.88 ; the proof will be given in three steps.

Step 1 : We shall show that the operator $\mathcal{N}$ maps into itself. First of all, we begin to show that $\mathcal{N}\colon D \to D$. To see this, let $x \in D, t \in J$. Without loss of generality, assume that $\mathcal{N}x(t) \neq$

0. By the Hahn-Banach theorem, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|\mathscr{N}x(t)\|_E = \varphi(\mathscr{N}x(t))$. Thus

$$\begin{aligned}\|\mathscr{N}x(t)\|_E &= \varphi\Big(\int_0^1 G(t,s)f(s,x(s))\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)f(s,x(s))\,\mathrm{d}s \\ &\quad + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)f(s,x(s))\,\mathrm{d}s\Big) \\ &\leq \int_0^1 G(t,s)\varphi(f(s,x(s)))\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)\varphi(f(s,x(s)))\,\mathrm{d}s \\ &\quad + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)\varphi(f(s,x(s)))\,\mathrm{d}s.\end{aligned}$$

Using hypotheses (H_3) we get

$$\begin{aligned}&\|\mathscr{N}x(t)\|_E \\ &\leq \|p\|_{L^\infty}\psi(R)\Big(\int_0^1 G(t,s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)\,\mathrm{d}s\Big) \\ &\leq \|p\|_{L^\infty}\psi(R)\Big(\frac{1}{\Gamma(\alpha)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\sigma_i\frac{(\alpha+1)\eta_i^{\alpha} - \alpha\eta_i^{\alpha+1}}{\alpha\Gamma(\alpha+2)} + \frac{1}{\xi}\sum_{i=1}^{m-2}\nu_i\frac{\eta_i^{\alpha-1} - \eta_i^{\alpha}}{\Gamma(\alpha+1)}\Big) \\ &\leq \|p\|_{L^\infty}\psi(R)K \leq R.\end{aligned}$$

then $\|\mathscr{N}x\|_\infty = \sup_{t\in J}\|\mathscr{N}x(t)\|_E \leq R$.

Next, let $t_1, t_2 \in J, t_1 < t_2, x \in D$, without loss of generality, assume that $\mathscr{N}x(t_2) - \mathscr{N}x(t_1) \neq 0$. By the Hahn-Banach theorem, there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and

$$\|\mathscr{N}(x)(t_2) - \mathscr{N}(x)(t_1)\|_E = \varphi\big(\mathscr{N}(x)(t_2) - \mathscr{N}(x)(t_1)\big).$$

So,

$$\begin{aligned}
\|\mathcal{N}(x)(t_2)-\mathcal{N}(x)(t_1)\|_E &= \varphi\Big(\int_0^1 G(t_2,s)f(s,x(s))\,\mathrm{d}s - \int_0^1 G(t_1,s)f(s,x(s))\,\mathrm{d}s \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)f(s,x(s))\,\mathrm{d}s \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)f(s,x(s))\,\mathrm{d}s\Big) \\
&\leq \int_0^1 |G(t_2,s)-G(t_1,s)|\|f(s,x(s))\|\,\mathrm{d}s \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)\|f(s,x(s))\|\,\mathrm{d}s \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)\|f(s,x(s))\|\,\mathrm{d}s \\
&\leq \|p\|_{L^\infty}\psi(R)\Big(\int_0^1 |G(t_2,s)-G(t_1,s)|\,\mathrm{d}s \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)\,\mathrm{d}s \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)\,\mathrm{d}s\Big) \\
&\leq \frac{\|p\|_{L^\infty}\psi(R)}{\Gamma(\alpha+1)}\big[(1+K\Gamma(\alpha+1))(t_2^{\alpha-1}-t_1^{\alpha-1})+(t_2^{\alpha}-t_1^{\alpha})\big]
\end{aligned}$$

This estimation shows that $\mathcal{N}$ maps D into itself.

Step 2 : We will show that the operator $\mathcal{N}$ has a weakly sequentially continuous. To see this, by Lemma 9 of [96], a sequence $x_n(\cdot)$ weakly convergent to $x(\cdot)\in D$ if and only if $x_n(\cdot)$ tends weakly to $x(\cdot)$ for each $t\in J$. Let (x_n) be a sequence in D and let $x_n(t)\to x(t)$ in (E,w) for each $t\in J$. Fix $t\in J$. Since f satisfies assumption (H_1), we have $f(t,x_n(t))$ converges weakly uniformly to $f(t,x(t))$. Hence the Lebesgue Dominated Convergence theorem for Pettis integral implies $\mathcal{N}x_n(t)$ converges weakly uniformly to $\mathcal{N}x(t)$ in (E,w). We do it for each $t\in J$ so $\mathcal{N}x_n\to\mathcal{N}x$. Then $\mathcal{N}: D\to D$ is weakly sequentially continuous.

Step 3 : The implication (1.16) holds. Now let V be a subset of D such that $\overline{V}=\overline{\mathrm{Co}}(\mathcal{N}(V)\cup\{0\})$. Clearly, $V(t)\subset\overline{\mathrm{Co}}(\mathcal{N}(V(t))\cup\{0\}), t\in J$. Further, as V is bounded and equicontinuous, by theorem 1.79 the function $t\to v(t)=\beta(V(t))$ is continuous on J. By assumption

(H_5), and the properties of the measure β, for any $t \in J$, we have

$$
\begin{aligned}
v(t) &\le \beta\left(\overline{\mathrm{Co}}(\mathcal{N}(V)(t)\cup\{0\})\right) \le \beta(\mathcal{N}(V)(t)) \\
&\le \beta\Big(\int_0^1 G(t,s)f(s,V(s))\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)f(s,V(s))\,\mathrm{d}s \\
&+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)f(s,V(s))\,\mathrm{d}s\Big) \\
&\le \int_0^1 G(t,s)\beta(f(s,V(s)))\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)\beta(f(s,V(s)))\,\mathrm{d}s \\
&+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)\beta\big(f(s,V(s))\big)\,\mathrm{d}s \\
&\le \int_0^1 G(t,s)p(s)\beta(V(s))\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)p(s)\beta(V(s))\,\mathrm{d}s \\
&+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)p(s)\beta(V(s))\,\mathrm{d}s \\
&\le \int_0^1 G(t,s)p(s)v(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)p(s)v(s)\,\mathrm{d}s \\
&+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)p(s)v(s)\,\mathrm{d}s \\
&\le \|p\|_{L^\infty}\|v\|_\infty\Big(\int_0^1 G(t,s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\sigma_i\int_0^1 H(\eta_i,s)\,\mathrm{d}s \\
&+ \frac{t^{\alpha-1}}{\xi}\sum_{i=1}^{m-2}\nu_i\int_0^1 G(\eta_i,s)\,\mathrm{d}s\Big)
\end{aligned}
$$

which gives

$$\|v\|_\infty \le \|p\|_{L^\infty}\|v\|_\infty K.$$

This means that

$$(1-\|p\|_{L^\infty}K)\|v\|_\infty \le 0.$$

By (2.3) it follows that $\|v\|_\infty = 0$, that is $v(t) = \beta\big(V(t)\big) = 0$ for each $t \in J$, and then by Theorem 2 of [96], V is weakly relatively compact in E. Applying Theorem 1.88 we conclude that $\mathcal{N}$ has a fixed point which is a solution of the problem (2.1).

□

2.2.1 Example

Let

$$E = \ell^1 = \left\{x = (x_1,x_2,\ldots,x_n,\ldots),\ \sum_{n=1}^{\infty}|x_n| < \infty\right\},$$

be the Banach space with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

We consider the following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\frac{9}{2}} x_n(t) + e^{-t}\sin x_n(t) = 0, \quad \forall t \in J = [0;1], \\ x_n'(0) = x_n''(0) = x_n'''(0) = 0, \\ x_n(1) = 3\int_0^{\frac{1}{3}} x_n(s)\,\mathrm{d}s + 2\int_0^{\frac{1}{2}} x_n(s)\,\mathrm{d}s + 4\int_0^{\frac{2}{3}} x_n(s)\,\mathrm{d}s \\ \qquad + x_n(\frac{1}{3}) + \frac{5}{2}x_n(\frac{1}{2}) + \frac{1}{2}x_n\left(\frac{2}{3}\right). \end{cases} \tag{2.4}$$

Set

$$x = (x_1, x_2, \ldots, x_n, \ldots), \quad f = (f_1, f_2, \ldots, f_n, \ldots),$$
$$f(t, x(t)) = \{e^{-t}\sin x_n(t)\}_{n\geq 1}, \quad t \in J.$$

From the equation above, it is clear that $\alpha = \frac{9}{2}, m = n = 5, \sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 4, \nu_1 = 1, \nu_2 = \frac{5}{2}, \nu_3 = \frac{1}{2}, \eta_1 = \frac{1}{3}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{2}{3}$. Using the Matlab program, we can find $\xi = 0.4689, K = 0.0949$.

For each $x \in \ell^1$, $t \in J$ we have

$$|f_n(t,x)| \leq e^{-t}|x_n|.$$

Hence conditions $(H_1), (H_2)$ and (H_3) hold with

$$p(t) = e^{-t},\ t \in J \quad \text{and} \quad \psi(x) = |x|,\ x \in [0,\infty).$$

For any bounded set $D \subset E$, we have

$$\beta(f(t,D)) \leq e^{-t}\beta(D), \text{ for each } t \in J.$$

Hence (H_5) is satisfied. On the other hand we have

$$\frac{R}{\|p\|_{L^\infty}\psi(R)K} > 1.$$

is equivalent to

$$R > \frac{10K}{1-K},$$

Hence the condition (H_4) holds for $R > 1.0485$. Consequently, Theorem 1.88 implies that problem (2.4) has a solution defined on J.

Chapitre 3

Boundary Value Problems for Hybrid Fractional Differential Equations

Perturbation techniques are useful in the nonlinear analysis for studying the dynamical systems represented by nonlinear differential and integral equations. Evidently, some differential equations representing a certain dynamical system have no analytical solution, so the perturbation of such problems can be helpful. The perturbed differential equations are categorized in to various types. An important type of these such perturbations is called a hybrid differential equation (i.e.quadratic perturbation of a nonlinear differential equation). This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. The study of hybrid differential equations is implicit in the works of Krasnose'lskii, Dhage and Lakshmikantham and extensively studied by many researchers, we refer [13, 49, 52, 64, 65, 75, 88, 107, 110, 112, 121, 122].

Dhage and Lakshmikantham [52] discussed the existence and uniqueness theorems of the solution to the ordinary first-order hybrid differential equation with perturbation of first type

$$\begin{cases} \frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \text{ a.e. } t \in J, \\ x(t_0) = x_0 \in \mathbb{R}. \end{cases}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$, where $J = [t_0; t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some t_0 and $a \in \mathbb{R}$ with $a > 0$.

Zhao et al.[121] studied existence and uniqueness results for the following hybrid differential equations involving Riemann-Liouville differential operators :

$$\begin{cases} D^q_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \text{ a.e. } t \in J = [0;T], \\ x(0) = 0, \end{cases}$$

where $0 < q < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. A fixed point theorem in Banach algebras was the main tool used in this work.

Hilal and Kajouni [65] extended the results to the following boundary value problem for fractional hybrid differential equations involving Caputo's derivative

$$\begin{cases} {}^cD^\alpha_{0^+}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \text{ a.e. } t \in J = [0;T], \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{3.1}$$

where $0<\alpha<1,\ f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\}),\ g\in C(J\times\mathbb{R},\mathbb{R})$ and a,b,c are real constants with $a+b\neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Caratheodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions.

Darwish and Sadarngani [49], where using a measure of noncompactness argument combined with the generalized version of Darbo's theorem, authors provide sufficient conditions for the following fractional hybrid initial value problem with supremum

$$\begin{cases} {}^cD^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t),\max_{0\leq\tau\leq t}x(\tau))}\right]=g(t,x(t)), \quad 0<t<1,\\ x(0)=0,\end{cases}$$

where $\alpha\in(0;1), f\in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\}),\ g\in C(J\times\mathbb{R},\mathbb{R})$.

Benchohra et al. [35] studied the existence and uniqueness of solutions of the following nonlinear fractional differential equations :

$$\begin{cases} {}^cD^{\alpha}_{0^+}y(t)=f(t,y(t)), \text{ for each } t\in J=[0;T],\ 0<\alpha<1,\\ ay(0)+by(T)=c,\end{cases} \tag{3.2}$$

where where ${}^cD^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f\colon[0;T]\times\mathbb{R}\to\mathbb{R}$ is continuous and a,b,c are real constants with $a+b\neq 0$.

Ahmad and Ntouyas [13] discussed the following fractional boundary value problem with fractional separated boundary conditions

$$\begin{cases} {}^cD^{q}_{0^+}x(t)=f(t,x(t)), \quad t\in[0;1], \quad 1<q\leq 2\\ \alpha_1x(0)+\beta_1{}^cD^{p}_{0^+}x(0)=\gamma_1, \quad \alpha_2x(1)+\beta_2{}^cD^{p}_{0^+}x(1)=\gamma_2, \quad 0<p\leq 1,\end{cases} \tag{3.3}$$

where ${}^cD^{q}_{0^+}$ is the Caputo fractional derivative, f is a given continuous function, and $\alpha_i,\beta_i,\gamma_i$ $(i=1,2)$ are real constants such that $\alpha_1\neq 0$. The results is obtained by using appropriate standard fixed point theorems.

Motivated by some recent studies on hybrid fractional differential equations (HFDEs), in this chapter, we shall establish sufficient conditions for the existence of solutions for two boundary value problems for hybrid Caputo fractional differential equations.

In section 3.1, we give our main result for the following boundary value problem of nonlinear fractional hybrid differential equations :

$$\begin{cases} {}^cD^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]=g(t,x(\nu(t))), \quad t\in J=[0;1],\\ a\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=0}+b\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=1}=c,\end{cases} \tag{3.4}$$

where $0<\alpha\leq 1, a,b,c$ are real constants such that $a+b\neq 0, {}^cD^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\}), g\in C(J\times\mathbb{R},\mathbb{R}), \mu$ and ν are functions from J into itself. Note that if $\mu(t)=\nu(t)=t$, then the first problem of (3.4) is reduces to the problem (3.1). Also if $\mu(t)=\nu(t)=t,\ f(t,x(t))=1$, then the problem (3.4) is reduces to the problem (3.2).

In section 3.2, we consider the following boundary value problem for hybrid fractional differential equations with fractional separated integral boundary conditions

$$\begin{cases} {}^cD^{\alpha}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x(t)), \quad t\in J=[0;T],\\ a_1\left(\frac{x(t)}{f(t,x(t))}\right)_{t=0}+b_1{}^cD^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)_{t=0}=\int_0^1 h(t,x(t))\,\mathrm{d}t,\\ a_2\left(\frac{x(t)}{f(t,x(t))}\right)_{t=1}+b_2{}^cD^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)_{t=1}=\int_0^1 k(t,x(t))\,\mathrm{d}t.\end{cases} \tag{3.5}$$

Where $0 < \sigma \leq 1 < \alpha \leq 2$, ${}^cD^{\alpha}_{0^+}$ is the Caputo fractional derivative, f,g,h,k are a given continuous functions and $a_i, b_i, i = 1,2$ are real constants such that $a_1 \neq 0$. Note that if $f(t,x(t)) = 1, h(t,x(t)) = c_1$ and $k(t,x(t)) = c_2$, c_1, c_2 are real constants, then the first problem of (3.5) is reduces to the problem (3.3).

The main tools in our analysis are Darbo fixed point theorem and the measure of non-compactness related to monotonicity which was introduced by Banaŝ and Olszowy [30].

3.1 Boundary Value Problems for Hybrid Fractional Differential Equations [1]

3.1.1 Existence of Solutions

In this subsection, we discuss the existence of solutions of the the problem (3.4).

Lemma 3.1. *For any $y \in C(J)$, the unique solution of the hybrid fractional differential equation,*

$$ {}^cD^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(\mu(t)))}\right] = y(t), \quad 0 < t < 1, \tag{3.6}$$

with boundary conditions

$$ a\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=0} + b\left[\frac{x(t)}{f(t,x(\mu(t)))}\right]_{t=1} = c, \tag{3.7}$$

is given by

$$ x(t) = f(t,x(\mu(t)))\left\{\int_0^1 G(t,s)y(s)\,\mathrm{d}s + \frac{c}{a+b}\right\}, \tag{3.8}$$

where

$$ G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (t-s)^{\alpha-1} - \dfrac{b}{a+b}(1-s)^{\alpha-1} & , \quad 0 \leq s \leq t \leq 1, \\ -\dfrac{b}{a+b}(1-s)^{\alpha-1} & , \quad 0 \leq t \leq s \leq 1. \end{cases} \tag{3.9}$$

Here $G(t,s)$ is called the Green function of the boundary value problem (3.6) and (3.7).

Proof. We may apply Lemma 1.44 to reduce (3.6) to an equivalent integral equation

$$ \frac{x(t)}{f(t,x(\mu(t)))} = I^{\alpha}_{0^+}y(t) + c_0, \quad c_0 \in \mathbb{R}. \tag{3.10}$$

Consequently, the general solution of (3.6) is

$$ x(t) = f(t,x(\mu(t)))(I^{\alpha}_{0^+}y(t) + c_0). \tag{3.11}$$

Applying the boundary conditions (3.7) in (3.2.1) we find that

$$ ac_0 + b(I^{\alpha}_{0^+}y(1) + c_0) = c. $$

1. **Z. Baitiche**, K. Guerbati, M. Benchohra and Yong Zhou, Boundary Value Problems for Hybrid Fractional Differential Equations, *Mathematics.* **2019** 7(3), 282.

Therefore, we have

$$c_0 = \frac{1}{a+b}(c - bI_{0^+}^{\alpha} y(1)).$$

Substituting the value of c_0 in (3.2.1) we get (3.17).

$$x(t) = f(t, x(\mu(t)))\left\{ I_{0^+}^{\alpha} y(t) - \frac{b}{a+b} I_{0^+}^{\alpha} y(1) + \frac{c}{a+b} \right\}. \tag{3.12}$$

that can be written as

$$x(t) = f(t, x(\mu(t)))\left\{ \int_0^1 G(t,s) y(s)\, \mathrm{d}s + \frac{c}{a+b} \right\},$$

where G is defined by (3.9). The proof is complete.

□

Remark 3.2. From the expression of $G(t,s)$, it is obvious that $G(t,s)$ is continuous on $J \times J$.

Thanks to Lemma 3.1, the proposed problem is equivalent to the following integral equation

$$x(t) = f(t, x(\mu(t)))\left\{ \int_0^1 G(t,s) g(s, x(\nu(s)))\, \mathrm{d}s + \frac{c}{a+b} \right\}.$$

Firstly, we list some assumptions :

(A_1) The functions $\mu, \nu\colon J \to J$ are continuous.

(A_2) $f \in C(J \times \mathbb{R},\ \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.

(A_3) There exists a constant $k \in (0;1)$ such that

$$|f(t,x_1) - f(t,x_2)| \leq (|x_1 - x_2| + 1)^k - 1,\ t \in J,\ x_1, x_2 \in \mathbb{R}.$$

(A_4) There exists a continuous nondecreasing function $\psi\colon \mathbb{R}_+ \to (0, +\infty)$ such that

$$|g(t,x)| \leq \psi(|x|), \quad t \in J,\ x \in \mathbb{R}_+.$$

(A_5) There exists $r > 0$ such that

$$[(r+1)^k - 1 + M]\left\{ \frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} \psi(r) + \frac{|c|}{|a+b|} \right\} \leq r,$$

and

$$\frac{|a| + 2|b|}{|a+b|\Gamma(\alpha+1)} \psi(r) + \frac{|c|}{|a+b|} \leq 1,$$

where

$$M = \sup\{|f(t,0)| : t \in J\}.$$

Now, we are in a position to state and prove our main result in this paper.

Theorem 3.3. *Assume that assumptions $(A_1) - (A_5)$ hold. Then the Problem (3.4) has at least one solution in the Banach algebra $C(J)$.*

Proof. To prove this result using Theorem 1.87, we consider the operator $\mathscr{T}$ on the Banach algebra $C(J)$ as follows

$$\mathscr{T}x(t) = f(t,x(\mu(t)))\left\{\int_0^1 G(t,s)g(s,x(\nu(s)))\,\mathrm{d}s + \frac{c}{a+b}\right\}$$

for $t \in J$. By virtue of Lemma 3.1, a fixed point of $\mathscr{T}$ gives us the desired result.

We define operators $\mathscr{F}$ and $\mathscr{G}$ on the Banach algebra $C(J)$ in the following way :

$$\mathscr{F}x(t) = f(t,x(\mu(t))),$$

and

$$\mathscr{G}x(t) = \int_0^1 G(t,s)g(s,x(\nu(s)))\,\mathrm{d}s + \frac{c}{a+b}$$

for $t \in J$. Then $\mathscr{T}x = (\mathscr{F}x)\cdot(\mathscr{G}x)$ for any $x \in C(J)$.
We divide the proof into five steps.

Step 1 : $\mathscr{T}$ transforms $C(J)$ into itself.
In fact, since the product of continuous functions is a continuous function, it is sufficient to prove that $\mathscr{F}x, \mathscr{G}x \in C(J)$ for any $x \in C(J)$. Now, from the assumptions (A_1) and (A_2), it follows that if $x \in C(J)$ then $\mathscr{F}x \in C(J)$. Next, we will prove that if $x \in C(J)$ then $\mathscr{G}x \in C(J)$. To do this, let $\varepsilon > 0$ be fixed, take $x \in C(J)$ and $t_1, t_2 \in J$ with $t_2 - t_1 \leq \varepsilon$ and we can assume that $t_1 \leq t_2$. Then, in view of assumption (A_4), we get

$$\begin{aligned}
|\mathscr{G}x(t_2) - \mathscr{G}x(t_1)| &\leq \int_0^{t_1} |G(t_1,s) - G(t_2,s)||g(s,x(\nu(s)))|\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)||g(s,x(\nu(s)))|\,\mathrm{d}s \\
&\quad + \int_{t_2}^{1} |G(t_1,s) - G(t_2,s)||g(s,x(\nu(s)))|\,\mathrm{d}s \\
&\leq \psi(\|x\|)\Big(\textstyle\int_0^{t_1} |G(t_1,s) - G(t_2,s)|\,\mathrm{d}s \\
&\quad + \displaystyle\int_{t_1}^{t_2} |G(t_1,s) - G(t_2,s)|\,\mathrm{d}s + \int_{t_2}^{1} |G(t_1,s) - G(t_2,s)|\,\mathrm{d}s\Big) \\
&\leq \frac{\psi(\|x\|)}{\Gamma(\alpha+1)}\Big(2(t_2-t_1)^\alpha + (t_1^\alpha - t_2^\alpha)\Big) \\
&\leq \frac{2\psi(\|x\|)}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha \\
&\leq \frac{2\psi(\|x\|)}{\Gamma(\alpha+1)}\varepsilon^\alpha.
\end{aligned}$$

From the above inequality, we conclude that $|\mathscr{G}x(t_2) - \mathscr{G}x(t_1)| \to 0$ when $\varepsilon \to 0$. Therefore, $\mathscr{G}x \in C(J)$. This proves that if $x \in C(J)$ then $\mathscr{T}x \in C(J)$.

Step 2 : An estimate of $\|\mathscr{T}x\|$ for $x \in C(J)$.

Now, let us fix $x \in C(J)$, then using our assumptions for $t \in J$, we obtain

$$\begin{aligned}
|(\mathscr{T}x)(t)| &= \left| f(t,x(\mu(t))) \left\{ I_{0^+}^{\alpha} g(t,x(\nu(t))) - \frac{b}{a+b} I_{0^+}^{\alpha} g(1,x(\nu(1))) + \frac{c}{a+b} \right\} \right| \\
&\leq (|f(t,x(\mu(t))) - f(t,0)| + |f(t,0)|) \\
&\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|g(s,x(\nu(s)))|}{(t-s)^{1-\alpha}} \mathrm{d}s + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^1 \frac{|g(s,x(\nu(s)))|}{(1-s)^{1-\alpha}} \mathrm{d}s + \frac{|c|}{|a+b|} \right\} \\
&\leq [(|x(\mu(t))|+1)^k - 1 + M] \\
&\quad \times \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi(|x(\nu(s)|)}{(t-s)^{1-\alpha}} \mathrm{d}s + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^1 \frac{\psi(|x(\nu(s)|)}{(1-s)^{1-\alpha}} \mathrm{d}s + \frac{|c|}{|a+b|} \right\} \\
&\leq \left[(\|x\|+1)^k - 1 + M \right] \left\{ \frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)} \psi(\|x\|) + \frac{|c|}{|a+b|} \right\}.
\end{aligned}$$

Therefore,

$$\|\mathscr{T}x\| \leq \left[(\|x\|+1)^k - 1 + M \right] \left\{ \frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)} \psi(\|x\|) + \frac{|c|}{|a+b|} \right\}.$$

By assumption (A_5), we deduce that the operator $\mathscr{T}$ maps the ball $B_r \subset C(J)$ into itself. Moreover, let us observe that from the last estimates, we obtain

$$\begin{cases} \|\mathscr{F}B_r\| & \leq (r+1)^k - 1 + M, \\ \|\mathscr{G}B_r\| & \leq \frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)} \psi(r) + \frac{|c|}{|a+b|}. \end{cases} \tag{3.13}$$

Step 3 : The operators $\mathscr{F}$ and $\mathscr{G}$ are continuous on the ball B_r.
In fact, firstly we prove that the operator $\mathscr{F}$ is continuous on the ball B_r. To do this, fix $\varepsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x-y\| \leq \varepsilon$. Then for $t \in J$, we have

$$\begin{aligned}
|(\mathscr{F}x)(t) - (\mathscr{F}y)(t)| &= |f(t,x(\mu(t))) - f(t,y(\mu(t)))| \\
&\leq (|x(\mu(t)) - y(\mu(t))| + 1)^k - 1 \\
&\leq (\|x-y\| + 1)^k - 1 \\
&\leq (\varepsilon + 1)^k - 1,
\end{aligned}$$

and, since $(\varepsilon+1)^k - 1 \to 0$ when $\varepsilon \to 0$. Thus, from the above inequality the operator $\mathscr{F}$ is continuous on the ball B_r.

Next, we prove that the operator $\mathscr{G}$ is continuous on the ball B_r. To do this, we take a sequence $\{x_n\} \subset B_r$ and $x \in B_r$ such that $\|x_n - x\| \to 0$ as $n \to \infty$, and we have to prove that $\|\mathscr{G}x_n - \mathscr{G}x\| \to 0$ as $n \to \infty$. Since $G(t,s)$ and $g(t,x)$ are uniformly continuous on the compact $J \times J$ and $J \times [-r;r]$, respectively, we may denote

$$\begin{cases} K & = \sup\{|G(t,s)| : t,s \in J\}, \\ H & = \sup\{|g(t,x)| : t \in J, x \in [-r;r]\}. \end{cases}$$

Since $\mu\colon J \to J$ is continuous, then for any n and $t \in J$, we have $|x_n(\nu(t))| \leq r$. Thus, for any n and $t \in J$, we get

$$|G(t,s)||g(t,x_n(\nu(t)))| \leq KH, \quad s \in J.$$

By applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\lim_{n\to\infty}(\mathscr{G}x_n)(t) &= \lim_{n\to\infty}\int_0^1 G(t,s)g(s,x_n(\nu(t)))\,\mathrm{d}s+\frac{c}{a+b}\\
&= \int_0^1 G(t,s)g(s,x(\nu(s)))\,\mathrm{d}s+\frac{c}{a+b}\\
&= (\mathscr{G}x)(t).
\end{aligned}$$

Thus, the above inequality shows that the operator $\mathscr{G}$ is continuous in B_r. Hence we conclude that $\mathscr{T}$ is continuous operator on B_r.

Step 4 : Estimates of $\omega_0(\mathscr{F}X)$ and $\omega_0(\mathscr{G}X)$ for $\emptyset \neq X \subset B_r$.
Firstly, we estimate $\omega_0(\mathscr{F}X)$. Let $\varepsilon > 0$ be fixed, since $\mu : J \to J$ is uniformly continuous, we can find $\delta > 0$ (which can be taken with $\delta < \varepsilon$) such that, for $|t_1 - t_2| < \delta$ we have $|\mu(t_1)-\mu(t_2)| < \varepsilon$. Let $x \in X$ and $t_1,t_2 \in J$ with $|t_1-t_2| \leq \delta < \varepsilon$. Then, in view of assumption (A_3), we have

$$\begin{aligned}
|(\mathscr{F}x)(t_1)-(\mathscr{F}x)(t_1)| &= |f(t_1,x(\mu(t_1)))-f(t_2,x(\mu(t_2)))|\\
&\leq |f(t_1,x(\mu(t_1)))-f(t_1,x(\mu(t_2)))|+|f(t_1,x(\mu(t_2)))-f(t_2,x(\mu(t_2)))|\\
&\leq \left[(|x(\mu(t_1))-x(\mu(t_2))|+1)^k-1\right]+\omega(f,\varepsilon)\\
&\leq \left[(\omega(X,\varepsilon)+1)^k-1\right]+\omega(f,\varepsilon),
\end{aligned}$$

where

$$\omega(f,\varepsilon)=\sup\left\{|f(t_1,x)-f(t_2,x)| : t_1,t_2\in J, |t_1-t_2|\leq\varepsilon,\, x\in[-r;r]\right\}.$$

So,

$$\omega(\mathscr{F}X,\varepsilon)\leq\left[(\omega(X,\varepsilon)+1)^k-1\right]+\omega(f,\varepsilon).$$

Observe that the function $f(t,x)$ is uniformly continuous on the set $J\times[-r;r]$. Hence, we deduce that $\omega(f,\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus, from the above inequality, we conclude

$$\omega_0(\mathscr{F}X)\leq(\omega_0(X)+1)^k-1. \tag{3.14}$$

Next, we estimate $\omega_0(\mathscr{G}X)$. Fix $\varepsilon > 0$, since $G(t,s)$ is uniformly continuous on $J\times J$, there exists $\delta > 0$ (which can be taken with $\delta < \varepsilon$) such that, for any $t_1,t_2 \in J$ with $|t_2-t_1| \leq \delta < \varepsilon$,

$$|G(t_1,s)-G(t_2,s)|\leq\frac{\varepsilon}{H},\quad s\in J.$$

Thus,

$$\begin{aligned}
|\mathscr{G}x(t_2)-\mathscr{G}x(t_1)| &= \int_0^1|G(t_1,s)-G(t_2,s)||g(s,x(\nu(s)))|\,\mathrm{d}s\\
&\leq H\int_0^1|G(t_1,s)-G(t_2,s)|\,\mathrm{d}s<\varepsilon.
\end{aligned}$$

So,

$$\omega(\mathscr{G}x,\varepsilon)\leq\varepsilon.$$

Taking $\varepsilon\to 0$, we get

$$\omega_0(\mathscr{G}X)=0. \tag{3.15}$$

Step 5 : An estimate of $\omega_0(\mathscr{T}X)$ for $\emptyset \neq X \subset B_r$.
From Lemma 1.75 and the estimates (3.13), (3.14) and (3.15), we have

$$\begin{aligned}
\omega_0(\mathscr{T}X) &= \omega_0(\mathscr{F}X.\mathscr{G}X) \leq \|\mathscr{F}X\|\omega_0(\mathscr{G}X) + \|\mathscr{G}X\|\omega_0(\mathscr{F}X) \\
&\leq \|\mathscr{F}B_{r_0}\|\omega_0(\mathscr{G}X) + \|\mathscr{G}B_{r_0}\|\omega_0(\mathscr{F}X) \\
&\leq \left[(\omega_0(X)+1)^k - 1\right]\left[\frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)}\psi(r_0) + \frac{|c|}{|a+b|}\right].
\end{aligned}$$

Under the assumption (A_5), we know that

$$\frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)}\psi(r_0) + \frac{|c|}{|a+b|} \leq 1.$$

Hence,

$$\omega_0(\mathscr{T}X) + 1 \leq (\omega_0(X)+1)^k.$$

Thus, the contractive condition appearing in Theorem 1.81 is satisfied with $\theta(t) = t+1$, where $\theta \in \Theta$. By applying Theorem 1.87 we get that the operator $\mathscr{T}$ has at least one fixed point in the ball B_r. Consequently, the problem (3.4) has at least one solution in B_r. This completes the proof.

□

3.1.2 Example

Consider the following fractional hybrid problem

$$\begin{cases}
{}^cD_{0^+}^{\frac{1}{2}}\left[\frac{x(t)}{\sqrt{1+|x(e^{t-1})|}}\right] = \frac{1}{3}\sin x(\sqrt{t}), \quad t \in J = [0;1] \\
\left[\frac{x(t)}{\sqrt{1+|x(e^{t-1})|}}\right]_{t=0} + \left[\frac{x(t)}{\sqrt{1+|x(e^{t-1})|}}\right]_{t=1} = 0.
\end{cases} \tag{3.16}$$

Corresponding to the problem (3.4), we have that $f(t,x) = \sqrt{1+|x|}$, $|g(t,x)| = \frac{1}{3}\sin x$, $\mu(t) = e^{t-1}, \nu(t) = \sqrt{t}, \alpha = \frac{1}{2}, a = b = 1, c = 0, M = \sup_{t\in J}|f(t,0)| = 1$. It is clear that the assumption $(A_1)-(A_2)$ hold. On the other hand, since the function $\beta(x) = \sqrt{1+|x|} - 1$ is concave (because $\beta''(x) \leq 0$) and $\beta(0) = 0$, we infer that β is subadditive and, therefore, for any $t \in J$ and $x_1, x_2 \in \mathbb{R}$, we have

$$\begin{aligned}
|f(t,x_1) - f(t,x_2)| &= |\beta(x_1) - \beta(x_2)| \leq \beta(x_1 - x_2) \\
&= \sqrt{1+|x_1-x_2|} - 1.
\end{aligned}$$

So, the assumption (A_3) holds, with $k = 1/2$. Moreover, for any $t \in J$ and $x \in \mathbb{R}$, we have

$$|g(t,x)| = \frac{1}{3}|\sin x| \leq \frac{1}{3}|x|.$$

Hence the assumption (A_4) holds, where $\psi(x) = \frac{1}{3}x$.
Observe that the assumption (A_5) is equivalent to

$$\frac{\sqrt{r+1}}{\sqrt{\pi}} \leq 1 \quad \text{and} \quad \frac{r}{\sqrt{\pi}} \leq 1,$$

Thus, assumption (A_5) is satisfied for all $0 < r \leq \sqrt{\pi}$.
So, all the assumption of Theorem 3.3 are satisfied, and consequently problem

3.2 Existence of Solutions for Hybrid Fractional Differential Equations with Fractional Separated Integral Boundary Conditions [2]

3.2.1 Existence of Solutions

In this subsection, we are concerned with the existence of solutions of the problem (3.5).

Lemma 3.4. *Let $y,\rho,\theta \in C(J)$. The unique solution of the hybrid fractional differential equation,*

$$ {}^{c}D^{\alpha}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = y(t), \quad t \in J =: [0;1], \tag{3.17}$$

with separated integral boundary conditions

$$\begin{aligned} a_1\left(\frac{x(t)}{f(t,x(t))}\right)_{t=0} + b_1\,{}^{c}D^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)_{t=0} &= \int_0^1 \rho(t)\,\mathrm{d}t,\\ a_2\left(\frac{x(t)}{f(t,x(t))}\right)_{t=1} + b_2\,{}^{c}D^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right)_{t=1} &= \int_0^1 \theta(t)\,\mathrm{d}t, \end{aligned} \tag{3.18}$$

is given by :

$$\begin{aligned} x(t) = f(t,x(t))\Big\{ & I^{\alpha}_{0^+}y(t) - \frac{a_2 t}{v} I^{\alpha}_{0^+}y(1) + \frac{b_2 t}{v} I^{\alpha-\sigma}_{0^+}y(1)\\ & + \frac{v - a_2 t}{a_1 v}\int_0^1 \rho(t)\,\mathrm{d}t + \frac{t}{v}\int_0^1 \theta(t)\,\mathrm{d}t \Big\}, \end{aligned} \tag{3.19}$$

where

$$ v = \frac{a_2\Gamma(2-\sigma) + b_2}{\Gamma(2-\sigma)}.$$

Proof. By Lemma 1.44, we reduce Eq.(3.17) to an equivalent integral equation

$$\frac{x(t)}{f(t,x(t))} = I^{\alpha}_{0^+}y(t) + c_0 + c_1 t, \qquad c_0, c_1 \in \mathbb{R}.$$

Consequently, the general solution of (3.17) is

$$ x(t) = f(t,x(t))\big(I^{\alpha}_{0^+}y(t) + c_0 + c_1 t\big). \tag{3.20}$$

By Lemma 1.42 and 1.43 we get

$$ {}^{c}D^{\sigma}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = I^{\alpha-\sigma}_{0^+}y(t) + c_1\frac{t^{1-\sigma}}{\Gamma(2-\sigma)}.$$

From the boundary condition (3.18), we have

$$ a_1 c_0 = \int_0^1 \rho(t)\,\mathrm{d}t,$$

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$$a_2\Big(I_{0^+}^{\alpha}y(1)+c_0+c_1\Big)+b_2\Big(I_{0^+}^{\alpha-\sigma}y(1)+\frac{d_1}{\Gamma(2-\sigma)}\Big)=\int_0^1\theta(t)\,\mathrm{d}t.$$

Therfore, we get

$$c_0=\frac{1}{a_1}\int_0^1\rho(t)\,\mathrm{d}t,$$

and

$$c_1=\frac{a_2}{v}I_{0^+}^{\alpha}y(1)+\frac{b_2}{v}I_{0^+}^{\alpha-\sigma}y(1)+\frac{a_2}{a_1v}\int_0^1\rho(t)\,\mathrm{d}t+\frac{1}{v}\int_0^1\theta(t)\,\mathrm{d}t.$$

Substituting the value of c_0,c_1 in (3.20) we get (3.19).

□

By Lemma 3.4, the BVP (3.5) is equivalent to the equation

$$\begin{aligned}x(t)=f(t,x(t))\Big\{&I_{0^+}^{\alpha}g(t,x(t))-\frac{a_2t}{v}I_{0^+}^{\alpha}g(1,x(1))+\frac{b_2t}{v}I_{0^+}^{\alpha-\sigma}g(1,x(1))\\&+\frac{v-a_2t}{a_1v}\int_0^1h(t,x(t))\,\mathrm{d}t+\frac{t}{v}\int_0^1k(t,x(t))\,\mathrm{d}t\Big\}.\end{aligned}$$

In [9] the authors proved the following Lemma which will be useful in our considerations.

Lemma 3.5 ([50]). *Let $f\colon\mathbb{R}_+\to\mathbb{R}_+$ be the function defined by $f(t)=t^{\alpha}$.*

(i) If $\alpha\geq 1$ and $t_1,t_2\in J$ with $t_2>t_1$, then $t_2^{\alpha}-t_1^{\alpha}\leq\alpha(t_2-t_1)$,

(ii) If $0<\alpha<1$ and $t_1,t_2\in J$ with $t_2>t_1$, then $t_2^{\alpha}-t_1^{\alpha}\leq(t_2-t_1)^{\alpha}$.

For the next theorem we use the assumptions :

(C_1) $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g,h,k\in C(J\times\mathbb{R},\mathbb{R})$.

(C_2) There exists an upper semi-continuous function $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\varphi(t)<t$, for any $t>0$, φ is nondecreasing, and

$$|f(t,x_1)-f(t,x_2)|\leq\varphi(|x_1-x_2|),\ t\in J,\ x_1,x_2\in\mathbb{R}.$$

(C_3) There exist functions $\phi_1,\phi_2,\phi_3\in L^1(J,\mathbb{R}_+)$ and $\psi_1,\psi_2,\psi_3\colon\mathbb{R}_+\to\mathbb{R}_+$ continuous, non-decreasing such that

$$|g(t,x)|\leq\phi_1(t)\psi_1(|x|),\quad |h(t,x)|\leq\phi_2(t)\psi_2(|x|),\quad |k(t,x)|\leq\phi_3(t)\psi_3(|x|).$$

(C_4) There exists $r>0$ such that

$$[\varphi(r)+M]\,(A_g\psi_1(r)+A_h\psi_2(r)+A_k\psi_3(r))\leq r,$$

and

$$A_g\psi_1(r)+A_h\psi_2(r)+A_k\psi_3(r)\leq 1,$$

where

$$M=\sup_{t\in J}|f(t,0)|,\quad A_g=\left(\frac{v+|a_2|}{|v|\Gamma(\alpha)}+\frac{|b_2|}{|v|\Gamma(\alpha-\sigma)}\right)\|\phi_1\|_{L^1},$$

$$A_k=\frac{\|\phi_3\|_{L^1}}{|v|},\quad A_h=\frac{|v|+|a_2|\|\phi_2\|_{L^1}}{|a_1v|}.$$

Theorem 3.6. *Assume that assumptions $(C_1)-(C_4)$ hold. Then the Problem (3.5) has at least one solution in the Banach algebra $C(J)$.*

Proof. To prove this result using Theorem 1.83, we consider the operator $\mathscr{T}$ on the Banach algebra $C(J)$ as follows

$$\begin{aligned}\mathscr{T}x(t) = f(t,x(t))\Big\{ & I_{0^+}^{\alpha}g(t,x(t)) - \frac{a_2 t}{v} I_{0^+}^{\alpha}g(1,x(1)) + \frac{b_2 t}{v} I_{0^+}^{\alpha-\sigma}g(1,x(1)) \\ & + \frac{v-a_2 t}{a_1 v}\int_0^1 h(t,x(t))\,\mathrm{d}t + \frac{t}{v}\int_0^1 k(t,x(t))\,\mathrm{d}t \Big\}\end{aligned}$$

for $t \in J$. By virtue of Lemma 3.4, a fixed point of $\mathscr{T}$ gives us the desired result. We define operators $\mathscr{F}$ and $\mathscr{P}$ on the Banach algebra $C(J)$ in the following way :

$$\mathscr{F}x(t) = f(t,x(t)),$$

and

$$\begin{aligned}\mathscr{P}x(t) = {} & I_{0^+}^{\alpha}g(t,x(t)) - \frac{a_2 t}{v} I_{0^+}^{\alpha}g(1,x(1)) + \frac{b_2 t}{v} I_{0^+}^{\alpha-\sigma}g(1,x(1)) \\ & + \frac{v-a_2 t}{a_1 v}\int_0^1 h(t,x(t))\,\mathrm{d}t + \frac{t}{v}\int_0^1 k(t,x(t))\,\mathrm{d}t\end{aligned}$$

for $t \in J$. Then $\mathscr{T}x = (\mathscr{F}x)\cdot(\mathscr{P}x)$ for any $x \in C(J)$.
We divide the proof into several steps.

Step 1 : $\mathscr{T}$ transforms $C(J)$ into itself.
In fact, since the product of continuous functions is a continuous function, it is sufficient to prove that $\mathscr{F}x, \mathscr{P}x \in C(J)$ for any $x \in C(J)$. Now, from the assumptions (C_1), it follows that if $x \in C(J)$ then $\mathscr{F}x \in C(J)$. Next, we will prove that if $x \in C(J)$ then $\mathscr{P}x \in C(J)$. To do this, let $t \in J$ be fixed, take $x \in C(J)$ and let (t_n) be a sequence in J such that $t_n \to t$ as $n \to \infty$. Without restriction of the generality, we may assume that $t_n \geq t$ for n large enough. For every n, we have

$$\begin{aligned} & |\mathscr{P}x(t_n) - \mathscr{P}x(t)| \\ & \leq \int_0^t \frac{(t_n-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|\,\mathrm{d}s + \int_t^{t_n}\frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|\,\mathrm{d}s \\ & + \frac{t_n-t}{v}\Big\{|a_2|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|\,\mathrm{d}s + |b_2|\int_0^1\frac{(1-s)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}|g(s,x(s))|\,\mathrm{d}s \\ & \quad + |a_2|\int_0^1 |h(s,x(s))|\,\mathrm{d}s + \int_0^1 |k(s,x(s))|\,\mathrm{d}s\Big\} \\ & \leq \int_0^t \frac{(t_n-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|\,\mathrm{d}s + \int_t^{t_n}\frac{(t_n-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|\,\mathrm{d}s \\ & + \frac{t_n-t}{v}\Big\{|a_2|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|g(s,x(s))|\,\mathrm{d}s + |b_2|\int_0^1\frac{(1-s)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}|g(s,x(s))|\,\mathrm{d}s \\ & \quad + |a_2|\|\phi_2\|_{L^1}\psi_2(\|x\|) + \|\phi_3\|_{L^1}\psi_3(\|x\|)\Big\}. \end{aligned}$$

Since $g \in C(J\times\mathbb{R},\mathbb{R}), g$ will be bounded on the compact $J\times[-\|x\|;\|x\|]$ and we put $L=\sup\{|g(s,x)|: s\in J, x\in[-\|x\|;\|x\|]\}$. From the last estimate and Lemma 3.5, we get

$$
\begin{aligned}
&|\mathscr{P}x(t_n)-\mathscr{P}x(t)| \\
&\leq \frac{L(t_n^\alpha-t^\alpha)}{\Gamma(\alpha+1)}+\frac{t_n-t}{v}\left\{\frac{|a_2|L}{\Gamma(\alpha+1)}+\frac{|b_2|L}{\Gamma(\alpha-\sigma+1)}+|a_2|\|\phi_2\|_{L^1}\psi_2(\|x\|)+\|\phi_3\|_{L^1}\psi_3(\|x\|)\right\} \\
&\leq \frac{L(t_n-t)}{\Gamma(\alpha)}+\frac{t_n-t}{v}\left\{\frac{|a_2|L}{\Gamma(\alpha+1)}+\frac{|b_2|L}{\Gamma(\alpha-\sigma+1)}+|a_2|\|\phi_2\|_{L^1}\psi_2(\|x\|)+\|\phi_3\|_{L^1}\psi_3(\|x\|)\right\}.
\end{aligned}
$$

From the above inequality, we conclude that $(\mathscr{P}x)(t_n)\to(\mathscr{P}x)(t)$ when $n\to\infty$. Therefore, $\mathscr{P}x\in C(J)$. This proves that if $x\in C(J)$ then $\mathscr{T}x\in C(J)$.

Step 2 : An estimate of $\|\mathscr{T}x\|$ for $x\in C(J)$.
Now, let us fix $x\in C(J)$, then using the assumptions (C_2) and (C_3), for $t\in J$ we obtain

$$
\begin{aligned}
|(\mathscr{T}x)(t)| &= \Bigg| f(t,x(t))\Big\{I_{0^+}^\alpha g(t,x(t))-\frac{a_2t}{v}I_{0^+}^\alpha g(1,x(1))+\frac{b_2t}{v}I_{0^+}^{\alpha-\sigma}g(1,x(1)) \\
&\quad +\frac{v-a_2t}{a_1v}\int_0^1 h(s,x(s))\,\mathrm{d}s+\frac{t}{v}\int_0^1 k(s,x(s))\,\mathrm{d}s\Big\}\Bigg| \\
&\leq \big(|f(t,x(t))-f(t,0)|+|f(t,0)|\big)\Big\{I_{0^+}^\alpha|g(t,x(t))|+\frac{|a_2|}{|v|}I_{0^+}^\alpha|g(1,x(1))| \\
&\quad +\frac{|b_2|}{|v|}I_{0^+}^{\alpha-\sigma}|g(1,x(1))|+\frac{|v|+|a_2|}{|a_1v|}\int_0^1|h(s,x(s))|\,\mathrm{d}s+\frac{1}{|v|}\int_0^1|k(s,x(s))|\,\mathrm{d}s\Big\} \\
&\leq \big[\varphi(|x(t)|)+M\big]\Big\{\psi_1(\|x\|)I_{0^+}^\alpha\phi_1(t)+\frac{|a_2|\psi_1(\|x\|)}{|v|}I_{0^+}^\alpha\phi_1(1) \\
&\quad +\frac{|b_2|\psi_1(\|x\|)}{|v|}I_{0^+}^{\alpha-\sigma}\phi_1(1)+\frac{(|v|+|a_2|)\|\phi_2\|_{L^1}}{|a_1v|}\psi_2(\|x\|)+\frac{\|\phi_3\|_{L^1}}{|v|}\psi_3(\|x\|)\Big\}.
\end{aligned}
$$

It follows from, Lemma 1.34 and 1.35 that

$$
\begin{aligned}
|(\mathscr{T}x)(t)| &\leq \big[\varphi(\|x\|)+M\big]\Big\{\frac{\|\phi_1\|_{L^1}}{\Gamma(\alpha)}\psi_1(\|x\|)+\frac{|a_2|\|\phi_1\|_{L^1}}{|v|\Gamma(\alpha)}\psi_1(\|x\|) \\
&\quad +\frac{|b_2|\|\phi_1\|_{L^1}}{|v|\Gamma(\alpha-\sigma)}\psi_1(\|x\|)+\frac{(|v|+|a_2|)\|\phi_2\|_{L^1}}{|a_1v|}\psi_2(\|x\|)+\frac{\|\phi_3\|_{L^1}}{|v|}\psi_3(\|x\|)\Big\} \\
&\leq [\varphi(\|x\|)+M]\left(A_g\psi_1(\|x\|)+A_k\psi_3(\|x\|)+A_h\psi_2(\|x\|)\right).
\end{aligned}
$$

Therefore,

$$\|\mathscr{T}x\|\leq[\varphi(\|x\|)+M]\left(A_g\psi_1(\|x\|)+A_h\psi_2(\|x\|)+A_k\psi_3(\|x\|)\right).$$

By assumption (C_4), we deduce that the operator $\mathscr{T}$ maps the ball $B_r\subset C(J)$ into itself. Moreover, let us observe that from the last estimates, we obtain

$$
\begin{aligned}
&\|\mathscr{F}B_r\|\leq\varphi(r)+M, \\
&\|\mathscr{P}B_r\|\leq A_g\psi_1(r)+A_h\psi_2(r)+A_k\psi_3(r).
\end{aligned}
\tag{3.21}
$$

Step 3 : The operators $\mathscr{F}$ and $\mathscr{P}$ are continuous on the ball B_r.
In fact, firstly we prove that the operator $\mathscr{F}$ is continuous on the ball B_r. To do this, we take

a sequence $\{x_n\} \subset B_r$ and $x \in B_r$ such that $\|x_n - x\| \to 0$ as $n \to \infty$, and we have to prove that $\|\mathscr{F}x_n - \mathscr{F}x\| \to 0$ as $n \to \infty$. In fact, for all $t \in J$, using the assumption (C_2), we have

$$\begin{aligned}|(\mathscr{F}x_n)(t) - (\mathscr{F}x)(t)| &= |f(t,x_n(t)) - f(t,x(t))| \leq \varphi(|x_n(t) - x(t)|) \\ &\leq \varphi(\|x_n - x\|) \leq \|x_n - x\|.\end{aligned}$$

So we get

$$\|\mathscr{F}x_n - \mathscr{F}x\| \leq \|x_n - x\|.$$

Thus, from the above inequality we obtaind

$$\lim_{n\to\infty} \|\mathscr{F}x_n - \mathscr{F}x\| = 0.$$

Therefore, the operator $\mathscr{F}$ is continuous on the ball B_r.

Next, we prove that the operator $\mathscr{P}$ is continuous on the ball B_r. To do this, fix $\varepsilon > 0$ and take arbitrary $x, y \in B_r$ such that $\|x - y\| \leq \varepsilon$. Then for $t \in J$, we get

$$\begin{aligned}&(\mathscr{P}x)(t) - (\mathscr{P}y)(t) \\ &= \int_0^t \frac{g(s,x(s)) - g(s,y(s))}{\Gamma(\alpha)(t-s)^{1-\alpha}}\,\mathrm{d}s - \frac{a_2 t}{v}\int_0^1 \frac{g(s,x(s)) - g(s,y(s))}{\Gamma(\alpha)(1-s)^{1-\alpha}}\,\mathrm{d}s \\ &+ \frac{b_2 t}{v}\int_0^1 \frac{g(s,x(s)) - g(s,y(s))}{\Gamma(\alpha-\sigma)(1-s)^{1+\sigma-\alpha}}\,\mathrm{d}s + \frac{(v - a_2 t)}{a_1 v}\int_0^1 \big(h(s,x(s)) - h(s,y(s))\big)\,\mathrm{d}s \\ &+ \frac{t}{v}\int_0^1 \big(k(s,x(s)) - k(s,y(s))\big)\,\mathrm{d}s.\end{aligned}$$

So,

$$\begin{aligned}&|(\mathscr{P}x)(t) - (\mathscr{P}y)(t)| \\ &\leq \omega_g(r,\varepsilon)\left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \frac{|a_2|}{|v|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s + \frac{|b_2|}{|v|}\int_0^1 \frac{(1-s)^{\alpha-\sigma-1}}{\Gamma(\alpha-\sigma)}\,\mathrm{d}s\right) \\ &+ \frac{|v| + |a_2|}{|a_1 v|}\omega_h(r,\varepsilon) + \frac{1}{|v|}\omega_k(r,\varepsilon) \\ &\leq \left(\frac{|v| + |a_2|}{\Gamma(\alpha+1)} + \frac{|b_2|}{|v|\Gamma(\alpha-\sigma+1)}\right)\omega_g(r,\varepsilon) + \frac{|v| + |a_2|}{|a_1 v|}\omega_h(r,\varepsilon) + \frac{1}{|v|}\omega_k(r,\varepsilon),\end{aligned}$$

where

$$\begin{aligned}\omega_g(r,\varepsilon) &= \sup\{|g(t,u) - g(t,v)| : t \in J, u, v \in [-r;r], |u - v| \leq \varepsilon\}, \\ \omega_h(r,\varepsilon) &= \sup\{|h(t,u) - h(t,v)| : t \in J, u, v \in [-r;r], |u - v| \leq \varepsilon\}, \\ \omega_k(r,\varepsilon) &= \sup\{|k(t,u) - k(t,v)| : t \in J, u, v \in [-r;r], |u - v| \leq \varepsilon\}.\end{aligned}$$

Therefore,

$$\|\mathscr{P}x - \mathscr{P}y\| \leq \left(\frac{|v| + |a_2|}{\Gamma(\alpha+1)} + \frac{|b_2|}{|v|\Gamma(\alpha-\sigma+1)}\right)\omega_g(r,\varepsilon) + \frac{|v| + |a_2|}{|a_1 v|}\omega_h(r,\varepsilon) + \frac{1}{|v|}\omega_k(r,\varepsilon)$$

Since, we know that the function $g(t,x), h(t,x)$ and $k(t,x)$ are uniformly continuous on the compact $J \times [-r;r]$, we conclude that $\omega_g(r,\varepsilon), \omega_h(r,\varepsilon)$ and $\omega_k(r,\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, the above inequality gives us

$$\lim_{\varepsilon\to 0} \|\mathscr{P}x - \mathscr{P}y\| = 0.$$

So, the operator $\mathscr{P}$ is continuous in B_r. Hence we conclude that $\mathscr{T}$ is continuous operator on B_r.

Step 4 : Estimates of $\omega_0(\mathscr{F}X)$ and $\omega_0(\mathscr{P}X)$ for $\emptyset \neq X \subset B_r$.
Firstly, we estimate $\omega_0(\mathscr{F}X)$. Let $\varepsilon > 0$ be fixed, $x \in X$ and $t_1, t_2 \in J$ with $|t_1 - t_2| \leq \varepsilon$. Then, in view of assumption (C_2), we have

$$\begin{aligned}|(\mathscr{F}x)(t_1) - (\mathscr{F}x)(t_1)| &= |f(t_1, x(t_1)) - f(t_2, x(t_2))| \\ &\leq |f(t_1, x(t_1)) - f(t_1, x(t_2))| + |f(t_1, x(t_2)) - f(t_2, x(t_2))| \\ &\leq \varphi(|x(t_1) - x(t_2)|) + \omega(f, \varepsilon) \\ &\leq \varphi\big(\omega(x, \varepsilon)\big) + \omega(f, \varepsilon),\end{aligned}$$

where

$$\omega(f, \varepsilon) = \sup\big\{|f(t_1, x) - f(t_2, x)| : t_1, t_2 \in J, |t_1 - t_2| \leq \varepsilon,\, x \in [-r\,; r]\big\}.$$

So,

$$\omega(\mathscr{F}X, \varepsilon) \leq \varphi\big(\omega(X, \varepsilon)\big) + \omega(f, \varepsilon).$$

Observe that the function $f(t, x)$ is uniformly continuous on the set $J \times [-r\,; r]$. Hence, we deduce that $\omega(f, \varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from the above inequality, we conclude

$$\omega_0(\mathscr{F}X) \leq \varphi\big(\omega_0(X)\big). \tag{3.22}$$

Next, we estimate $\omega_0(\mathscr{P}X)$. Fix $\varepsilon > 0$, and we take $x \in X$ and $t_1, t_2 \in J$ with $t_2 - t_1 \leq \varepsilon$ and we can assume that $t_1 \leq t_2$. Then, in view of assumptions (C_3), we get

$$\begin{aligned}&|\mathscr{P}x(t_2) - \mathscr{P}x(t_1)| \\ &\leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} |g(s, x(s))|\,\mathrm{d}s + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |g(s, x(s))|\,\mathrm{d}s \\ &\quad + \frac{t_2 - t_1}{v} \bigg\{ |a_2| \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} |g(s, x(s))|\,\mathrm{d}s + |b_2| \int_0^1 \frac{(1 - s)^{\alpha-\sigma-1}}{\Gamma(\alpha - \sigma)} |g(s, x(s))|\,\mathrm{d}s \\ &\qquad + |a_2| \int_0^1 h(s, x(s))\,\mathrm{d}s + \int_0^1 k(s, x(s))\,\mathrm{d}s \bigg\} \\ &\leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} |g(s, x(s))|\,\mathrm{d}s + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |g(s, x(s))|\,\mathrm{d}s \\ &\quad + \frac{t_2 - t_1}{v} \bigg\{ |a_2| \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} |g(s, x(s))|\,\mathrm{d}s + |b_2| \int_0^1 \frac{(1 - s)^{\alpha-\sigma-1}}{\Gamma(\alpha - \sigma)} |g(s, x(s))|\,\mathrm{d}s \\ &\qquad + |a_2| \|\phi_2\|_{L^1} \psi_2(\|x\|) + \|\phi_3\|_{L^1} \psi_3(\|x\|) \}.\end{aligned}$$

Since $g \in C(J \times \mathbb{R}, \mathbb{R})$, g will be bounded on the compact $J \times [-r\,; r]$ and we put $N = \sup\{|g(s, x)| : s \in J,\, x \in [-r\,; r]\}$. From the last estimate and Lemma 3.5, we get

$$\begin{aligned}&|\mathscr{P}x(t_2) - \mathscr{P}x(t_1)| \\ &\leq \frac{N(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha + 1)} + \frac{t_2 - t_1}{v} \left\{ \frac{|a_2| N}{\Gamma(\alpha + 1)} + \frac{|b_2| N}{\Gamma(\alpha - \sigma + 1)} + |a_2| \|\phi_2\|_{L^1} \psi_2(r) + \|\phi_3\|_{L^1} \psi_3(r) \right\} \\ &\leq \frac{N(t_2 - t_1)}{\Gamma(\alpha)} + \frac{t_2 - t_1}{v} \left\{ \frac{|a_2| N}{\Gamma(\alpha + 1)} + \frac{|b_2| N}{\Gamma(\alpha - \sigma + 1)} + |a_2| \|\phi_2\|_{L^1} \psi_2(r) + \|\phi_3\|_{L^1} \psi_3(r) \right\} \\ &\leq \frac{N\varepsilon}{\Gamma(\alpha)} + \frac{\varepsilon}{v} \left\{ \frac{|a_2| N}{\Gamma(\alpha + 1)} + \frac{|b_2| N}{\Gamma(\alpha - \sigma + 1)} + |a_2| \|\phi_2\|_{L^1} \psi_2(r) + \|\phi_3\|_{L^1} \psi_3(r) \right\}.\end{aligned}$$

Therefore,

$$\omega(\mathscr{P}x,\varepsilon) \leq \frac{N\varepsilon}{\Gamma(\alpha)} + \frac{\varepsilon}{v}\left\{\frac{|a_2|N}{\Gamma(\alpha+1)} + \frac{|b_2|N}{\Gamma(\alpha-\sigma+1)} + |a_2|\|\phi_2\|_{L^1}\psi_2(r) + \|\phi_3\|_{L^1}\psi_3(r)\right\}.$$

Taking $\varepsilon \to 0$, we get

$$\omega_0(\mathscr{P}X) = 0. \tag{3.23}$$

Step 5 : An estimate of $\omega_0(\mathscr{T}X)$ for $\emptyset \neq X \subset B_r$.
From Lemma 1.75 and the estimates (3.21), (3.22) and (3.23), we have

$$\begin{aligned}\omega_0(\mathscr{T}X) = \omega_0(\mathscr{F}X.\mathscr{P}X) &\leq \|\mathscr{F}X\|\omega_0(\mathscr{P}X) + \|\mathscr{P}X\|\omega_0(\mathscr{F}X)\\ &\leq \|\mathscr{F}B_{r_0}\|\omega_0(\mathscr{P}X) + \|\mathscr{P}B_{r_0}\|\omega_0(\mathscr{F}X)\\ &\leq (A_g\psi_1(r) + A_h\psi_2(r) + A_k\psi_3(r))\,\varphi(\omega_0(X)).\end{aligned}$$

Under the assumption (C_3), we know that

$$A_g\psi_1(r) + A_h\psi_2(r) + A_k\psi_3(r) \leq 1.$$

Hence,

$$\omega_0(\mathscr{T}X) \leq \varphi(\omega_0(X)).$$

Then by Theorem 1.83, we deduce that the operator $\mathscr{T}$ has at least one fixed point in the ball B_r. Consequently, the problem (3.5) has at least one solution in B_r. This completes the proof.

□

3.2.2 Example

Consider the following fractional hybrid problem

$$\begin{cases} {}^cD^{\frac{3}{2}}_{0^+}\left(\frac{x(t)}{1+\ln(1+|x(t)|)}\right) = \frac{t^3}{3}\sin|x(t)|, \quad t\in J=[0;1]\\ \left(\frac{x(0)}{1+\ln(1+|x(0)|)}\right) + b_1\,{}^cD^{\frac{1}{2}}_{0^+}\left(\frac{x(0)}{1+\ln(1+|x(0)|)}\right) = \int_0^1 \frac{s^2}{2}\sin x(s)\,\mathrm{d}s\\ \frac{1}{2}\left(\frac{x(1)}{1+\ln(1+|x(1)|)}\right) + \frac{\sqrt{\pi}}{4}\,{}^cD^{\frac{1}{2}}_{0^+}\left(\frac{x(1)}{1+\ln(1+|x(1)|)}\right) = \int_0^1 \frac{s}{2}x(s)\cos x(s)\,\mathrm{d}s \end{cases} \tag{3.24}$$

Corresponding to the problem (3.5), we have that $\alpha = 3/2$, $\sigma = 1/2$, $a_1 = 1 \neq 0$, $a_2 = 1/2$, $b_2 = \sqrt{\pi}/4$, $b_1$is arbitrary, $f(t,x) = 1+\ln(1+|x|)$, $g(t,x) = \frac{t^3}{3}\sin x$, $h(t,x) = \frac{t^2}{2}\sin x$, $k(t,x) = \frac{t}{2}x\cos x$. Further $v = M = 1$.
It is clear that the functions f,g,h and k satisfy (C_1) of Theorem 3.6.

On the other hand, for any $t\in J$ and $x_1,x_2\in\mathbb{R}$ we can assume that $|x_1| < |x_2|$. Then

$$\begin{aligned}|f(t,x_2) - f(t,x_1)| &= |\ln(1+|x_2|) - \ln(1+|x_1|)|\\ &\leq \ln\left(\frac{1+|x_2|}{1+|x_1|}\right) = \ln\left(1+\frac{|x_2|-|x_1|}{1+|x_1|}\right)\\ &\leq \ln(1+(|x_2|-|x_1|)) \leq \ln(1+|x_2-x_1|).\end{aligned}$$

Therefore, assumption (C_2) of the Theorem 3.6 is satisfied, with $\varphi(t) = \ln(1+t)$.

Moreover, for any $t \in J$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
|g(t,x)| &= \frac{t^3}{3}|\sin x| \leq \frac{t^3}{3}|x|, \\
|h(t,x)| &= \frac{t^2}{2}|\sin x| \leq \frac{t^2}{2}|x|, \\
|k(t,x)| &= \frac{t}{4}|x\cos x(s)| \leq \frac{t}{2}|x|.
\end{aligned}$$

We can get that the condition (C_3) of the Theorem 3.6 holds ; that is

$$\psi_1(x) = \psi_2(x) = \psi_3(x) = x,$$

and

$$\phi_1(t) = \frac{t^3}{3}, \quad \phi_2(t) = \frac{t^2}{2}, \quad \phi_3(t) = \frac{t}{2}.$$

By simple calculations we get

$$A_g + A_h + A_k = \frac{13\pi + 6}{24\pi}$$

The inequality appearing in (C_4) of Theorem 3.6 has the expression

$$r \leq \exp\left(\frac{1 - A_g - A_h - A_k}{A_g + A_h + A_k}\right) - 1 = 1.3771,$$

and

$$r \leq \frac{1}{A_g + A_h + A_k} = 1.6097.$$

So, assumption (C_4) of the Theorem 3.6 is satisfied for all $0 < r \leq 1.3771$.

Thus, all the assumption of Theorem 3.6 are satisfied, and consequently problem (3.24) has at least one solution in $C(J)$.

Chapitre 4

Solvability of Fractional Multi-Point BVP with Nonlinear Growth at Resonance [1]

4.1 Introduction

In this chapter, by using Mawhin's continuation theorem, we establish some sufficient conditions for the existence of at least one solution for the following fractional multi-point boundary value problems (BVPs) at resonance

$$\begin{cases} \left(\phi(t)^c D^{\alpha}_{0^+} u(t)\right)' = f(t, u(t), u'(t), u''(t), {}^c D^{\alpha}_{0^+} u(t)), \quad t \in I = [0;1], \\ u(0) = 0,\ {}^c D^{\alpha}_{0^+} u(0) = 0,\ u''(0) = \sum_{i=1}^{m} a_i u''(\xi_i),\ u'(1) = \sum_{j=1}^{l} b_j u'(\eta_j), \end{cases} \tag{4.1}$$

where ${}^c D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $2 < \alpha \leq 3$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_l < 1$, $a_i, b_j \in \mathbb{R}$, $(i = 1, \ldots, m,\ j = 1, \ldots, l)$, $\phi(t) \in C^1[0;1]$, $\mu = \min_{t \in [0;1]} \phi(t) > 0$. The nonlinearity is such that

(H_0) $f\colon [0;1] \times \mathbb{R}^4 \to \mathbb{R}$ is a Carathéodory function, that is,

- (i) for each $x \in \mathbb{R}^4$, the function $t \to f(t,x)$ is Lebesgue measurable ;
- (ii) for almost every $t \in [0;1]$, the function $t \to f(t,x)$ is continuous on $\mathbb{R}^4$;
- (iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1[0;1]$ such that, for a.e. $t \in [0;1]$ and every $|x| \leq r$, we have $|f(t,x)| \leq \varphi_r(t)$.

The resonant conditions of (4.1) are as follows

(H_1) $\sum_{i=1}^{m} a_i = 1, \quad \sum_{j=1}^{l} b_j = 1, \quad \sum_{j=1}^{l} b_j \eta_j = 1.$

This means that the linear operator $Lu = \left(\phi\, {}^c D^{\alpha}_{0^+} u\right)'$ corresponding to (4.1) has a nontrivial solutions or, in a functional framework, L is not invertible i.e. $\dim \operatorname{Ker} L \geq 1$.

In order to make sure that the linear operator Q (to be specified later on) is well defined, we assume in addition, that

1. **Z. Baitiche**, K. Guerbati, M. Benchohra and Yong Zhou, Solvability of Fractional Multi-Point BVP with Nonlinear Growth at Resonance, *Journal of Contemporary Mathematical Analysis.*

(H_2) There exist $p,q \in \mathbb{Z}^+, q \geq p+1$ such that $\Delta(p,q) = d_{11}d_{22} - d_{12}d_{21} \neq 0$, where

$$d_{11} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^p(\xi_i - s)^{\alpha-3}}{p\phi(s)} \mathrm{d}s, \quad d_{21} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^q(\xi_i - s)^{\alpha-3}}{q\phi(s)} \mathrm{d}s,$$

$$d_{12} = \int_0^1 \frac{s^p(1-s)^{\alpha-2}}{p\phi(s)} \mathrm{d}s - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^p(\eta_j - s)^{\alpha-2}}{p\phi(s)} \mathrm{d}s.$$

$$d_{22} = \int_0^1 \frac{s^q(1-s)^{\alpha-2}}{q\phi(s)} \mathrm{d}s - \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^q(\eta_j - s)^{\alpha-2}}{q\phi(s)} \mathrm{d}s.$$

The existence of solutions for fractional boundary-value problems at resonance case has been extensively studied by many authors ; see [22, 23, 24, 25, 47, 66, 67, 68, 71, 72, 77, 78, 119] and the references therein. It is considerable that there are many papers that have dealt with the solutions of multi-point boundary value problems of fractional differential equations at resonance [23, 24, 47, 71].

In this chapter, we study (4.1) at resonance which allow f to have nonlinear growth.

4.2 Main Results

For our purpose, the adequate functional space is :

$$X = \{u : {}^cD^{\alpha}_{0^+}u \in C[0;1], u \,\text{satisfies boundary value conditions of (4.1)}\}.$$

Equipped with norm

$$\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|{}^cD^{\alpha}_{0^+}u\|_\infty, \ \text{ where } \|u\|_\infty = \max_{t\in[0;1]} |u(t)|.$$

By means of the functional analysis theory, we can prove that $(X, \|\cdot\|_X)$ is Banach space.

Let $Y = L^1[0;1]$ be the Lebesgue space of real measurable functions $t \to y(t)$ defined on $[0;1]$ and such that $t \to |y(t)|$ is Lebesgue integrable. Y is Banach space with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$. Define L to be the linear operator from $\mathrm{dom}\,L \bigcap X$ to Y :

$$Lu = (\phi\, {}^cD^{\alpha}_{0^+}u)', \quad u \in \mathrm{dom}\,L. \tag{4.2}$$

where

$$\mathrm{dom}\,L = \{u \in X \mid {}^cD^{\alpha}_{0^+}u(t) \,\text{is absolutely continuous on } [0;1]\}$$

and define the operator $N \colon X \to Y$ as :

$$Nu(t) = f(t, u(t), u'(t), u''(t), {}^cD^{\alpha}_{0^+}u(t)), \quad t \in [0;1].$$

Then the boundary value problem (4.1) can be written in abstract form as :

$$Lu = Nu, \quad u \in \mathrm{dom}\,L.$$

To study the compactness of operator N, we need the following Lemma.

Lemma 4.1. *$U \subset X$ is a relatively compact set in X if and only if U is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$ such that for every $u \in U$*

$$\|u\|_X = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty + \|{}^cD^\alpha_{0^+}u\|_\infty \leq M.$$

and equicontinuous means that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$|u^{(i)}(t_1) - u^{(i)}(t_2)| < \varepsilon, \quad \forall u \in U,\ \forall t_1, t_2 \in I,\ |t_1 - t_2| < \delta,\ \forall i \in \{0,1,2\}.$$

and

$$|{}^cD^\alpha_{0^+}u(t_1) - {}^cD^\alpha_{0^+}u(t_2)| < \varepsilon, \quad \forall u \in U,\ \forall t_1, t_2 \in I,\ |t_1 - t_2| < \delta.$$

Let $T_1, T_2 \colon Y \to Y$ be two linear operators defined as follows

$$T_1(y) = \sum_{i=1}^m a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$$T_2(y) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - \sum_{j=1}^l b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s.$$

Lemma 4.2. *Let L be the operator defined by (4.2). Then*

$$\operatorname{Ker} L = \{u \mid u(t) = c_1 t + c_2 t^2,\ c_1, c_2 \in \mathbb{R}\},$$

$$\text{and} \ \operatorname{Im} L = \{y \in Y \mid T_1(y) = T_2(y) = 0\}. \tag{4.3}$$

Proof. Firstly, $u \in \operatorname{Ker} L$ if and only if $\left(\phi(t)\,{}^cD^\alpha_{0^+}u(t)\right)' = 0$ which from condition ${}^cD^\alpha_{0^+}u(0) = 0$ has ${}^cD^\alpha_{0^+}u(t) = 0$; this with $u(0) = 0$ yield $u(t) = c_1 t + c_2 t^2$. So we just characterize $\operatorname{Im} L$. Given $y \in \operatorname{Im} L$, there exists $u \in \operatorname{dom} L$ such that $\left(\phi(t)\,{}^cD^\alpha_{0^+}u(t)\right)' = y(t)$ we have

$${}^cD^\alpha_{0^+}u(t) = \frac{1}{\phi(t)} \int_0^t y(s)\,\mathrm{d}s.$$

By Lemma 1.44 and $u(0) = 0$, we find

$$u(t) = c_1 t + c_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s, \quad c_1, c_2 \in \mathbb{R}, \tag{4.4}$$

By $u''(0) = \sum_{i=1}^m a_i u''(\xi_i)$ and $\sum_{i=1}^l a_i = 1$. We obtain

$$\sum_{i=1}^l a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s = 0.$$

From the conditions $u'(1) = \sum_{j=1}^l b_j u'(\eta_j)$ and $\sum_{j=1}^l b_j = \sum_{j=1}^l b_j \eta_j = 1$. We get

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - \sum_{j=1}^l b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s = 0.$$

On the other hand, we let

$$u(t)=c_1t+c_2t^2+\int_0^t\frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

where c_1,c_2 are arbitrary constants. It is clear that $u(0)=0$, in view of Lemma 1.42 and 1.43, we obtain ${}^cD^{\alpha}_{0^+}u(0)=0$ and $\left(\phi(t){}^cD^{\alpha}_{0^+}u(t)\right)'=y(t)$ for all $t\in[0;1]$. If $T_1(y)=T_2(y)=0$ holds, we can calculate the following equations

$$u''(0)-\sum_{i=1}^{m}a_iu''(\xi_i)=\frac{T_1(y)}{\Gamma(\alpha-2)}=0,\quad u'(1)-\sum_{j=1}^{l}b_ju'(\eta_j)=\frac{T_2(y)}{\Gamma(\alpha-1)}=0.$$

So, $u\in\operatorname{dom}L$ and $y\in\operatorname{Im}L$. Then, we complete the proof.

□

Lemma 4.3. *L is a Fredholm operator of index zero, and the inverse linear operator $K_p=L_p^{-1}\colon\ \operatorname{Im}L\to\operatorname{dom}L\cap\operatorname{Ker}P$ is defined by*

$$(K_py)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s. \tag{4.5}$$

It satisfies,

$$\|K_py\|_X\le\rho_1\|y\|_{L^1}. \tag{4.6}$$

where

$$\rho_1=\frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)}+1\right) \tag{4.7}$$

Proof. Consider continuous linear mapping $Q\colon Y\to Y$ defined by

$$Qy=Q_1(y).t^{p-1}+Q_2(y).t^{q-1} \tag{4.8}$$

where p,q are given by (H_2) and

$$Q_1(y)=\frac{1}{\Delta(p,q)}(d_{22}T_1(y)-d_{21}T_2(y)),$$

$$Q_2(y)=\frac{1}{\Delta(p,q)}(-d_{12}T_1(y)+d_{11}T_2(y)),$$

We will prove that $\operatorname{Ker}Q=\operatorname{Im}L$. Obviously, $\operatorname{Im}L\subset\operatorname{Ker}Q$. As well, if $y\in\operatorname{Ker}Q$, then

$$\begin{cases} d_{22}T_1(y)-d_{21}T_2(y)=0.\\ -d_{12}T_1(y)+d_{11}T_2(y)=0.\end{cases} \tag{4.9}$$

The determinant of coefficiency for (4.9) is $\Delta(p,q)\neq0$. we get $T_1(y)=T_2(y)=0$ and that implies $y\in\operatorname{Im}L$. So, $\operatorname{Ker}Q\subset\operatorname{Im}L$. Now, we prove $Q^2y=Qy$, $y\in Y$. For $y\in Y$, we have

$$\begin{aligned}Q_1(Q_1(y).t^{p-1})&=\frac{1}{\Delta(p,q)}\left[d_{22}T_1(Q_1(y).t^{p-1})-d_{21}T_2(Q_1(y).t^{p-1}))\right]\\&=\frac{1}{\Delta(p,q)}(d_{22}d_{11}-d_{21}d_{12})Q_1y\\&=Q_1y,\end{aligned}$$

$$\begin{aligned}
Q_1(Q_2(y).t^{q-1}) &= \frac{1}{\Delta(p,q)}\left[d_{22}T_1(Q_2(y).t^{q-1}) - d_{21}T_2(Q_2(y).t^{q-1})\right] \\
&= \frac{1}{\Delta(p,q)}(d_{22}d_{21} - d_{21}d_{22})Q_2y \\
&= 0,
\end{aligned}$$

Similarly, we obtain

$$Q_2(Q_1(y).t^{p-1}) = 0, \quad Q_2(Q_2(y).t^{q-1}) = Q_2y.$$

Therefore, we get

$$\begin{aligned}
Q^2y &= Q_1(Q_1(y).t^{p-1}).t^{p-1} + Q_1(Q_2(y).t^{q-1}).t^{p-1} \\
&\quad + Q_2(Q_1(y).t^{p-1}).t^{q-1} + Q_2(Q_2(y).t^{q-1}).t^{q-1} \\
&= Q_1(y).t^{p-1} + Q_2(y).t^{q-1} \\
&= Qy.
\end{aligned}$$

That implies the operator Q is a projector.

Take $y \in Y$ in the form $y = (y - Qy) + Qy$. Then, $(y - Qy) \in \operatorname{Ker} Q = \operatorname{Im} L$ and $Qy \in \operatorname{Im} Q$. Thus $Y = \operatorname{Im} Q + \operatorname{Im} L$. And for any $y \in \operatorname{Im} Q \cap \operatorname{Im} L$, from $y \in \operatorname{Im} Q$, there exists constants $c_1, c_2 \in \mathbb{R}$ such that $y(t) = c_1.t^{p-1} + c_2.t^{q-1}$, from $y \in \operatorname{Im} L$, we obtain

$$\begin{cases} d_{11}c_1 + d_{21}c_2 = 0, \\ d_{12}c_1 + d_{22}c_2 = 0. \end{cases} \tag{4.10}$$

The determinant of coefficiency for (4.10) is $\Delta(p,q) \neq 0$. Therefore (4.10) has an unique solution $c_1 = c_2 = 0$, which implies $\operatorname{Im} Q \cap \operatorname{Im} L = 0$. Then, we have

$$Y = \operatorname{Im} Q \oplus \operatorname{Ker} Q = \operatorname{Im} Q \oplus \operatorname{Im} L. \tag{4.11}$$

Thus, $\dim \operatorname{Ker} L = 2 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Ker} Q = \operatorname{codim} \operatorname{Im} L$, this means that L is a Fredholm operator of index zero.

Let a mapping $P\colon X \to X$ be defined by

$$Pu(t) = u'(0)t + \frac{u''(0)}{2}t^2. \tag{4.12}$$

We note that P is a linear continuous projector and $\operatorname{Im} P = \operatorname{Ker} L$. It follows from $u = (u - Pu) + Pu$ that $X = \operatorname{Ker} P + \operatorname{Ker} L$. By simple calculation, we obtain that $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$. Hence

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P. \tag{4.13}$$

Define $K_p\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ as follows :

$$(K_py)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s.$$

Now, we will prove that K_p is the inverse of $L\mid_{\mathrm{dom}\,L\cap\mathrm{Ker}\,P}$. In fact, for $u\in\mathrm{dom}\,L\cap\mathrm{Ker}\,P$, we have

$$(K_pL)u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s(\phi\,{}^cD^{\alpha}_{0^+}u)'(r)\,\mathrm{d}r\,\mathrm{d}s=I^{\alpha}_{0^+}\,{}^cD^{\alpha}_{0^+}u(t)$$
$$=u(t)+u(0)+u'(0)t+\frac{u''(0)}{2}t^2,$$

In view of $u\in\mathrm{dom}\,L\cap\mathrm{Ker}\,P$, $u(0)=0$ and $Pu=0$. Thus

$$(K_pL)u(t)=u(t), \tag{4.14}$$

and for $y\in\mathrm{Im}\,L$, we find

$$(LK_p)y(t)=L(K_py)(t)=\left[\phi(t)\,{}^cD^{\alpha}_{0^+}I^{\alpha}_{0^+}\left(\frac{I^1_{0^+}y}{\phi}\right)(t)\right]'=y(t).$$

Thus, $K_p=(L\mid_{\mathrm{dom}\,L\cap\mathrm{Ker}\,P})^{-1}$. Again for each $y\in\mathrm{Im}\,L$, and from Lemmas 1.43, 1.34 and 1.35, we have

$$\begin{aligned}\|K_py\|_X&=\sum_{i=0}^{2}\max_{t\in[0;1]}|(K_py)^{(i)}(t)|+\max_{t\in[0;1]}|{}^cD^{\alpha}_{0^+}(K_py)(t)|\\&=\sum_{i=0}^{2}\max_{t\in[0;1]}\left|I^{\alpha-i}_{0^+}\left(\frac{I^1_{0^+}y}{\phi}\right)(t)\right|+\max_{t\in[0;1]}\left|\left(\frac{I^1_{0^+}y}{\phi}\right)(t)\right|\\&\le\sum_{i=0}^{2}\max_{t\in[0;1]}\left|\frac{I^{\alpha+1-i}_{0^+}y(t)}{\mu}\right|+\max_{t\in[0;1]}\left|\frac{I^1_{0^+}y(t)}{\mu}\right|\\&\le\sum_{i=0}^{2}\frac{\|y\|_{L^1}}{\mu\Gamma(\alpha+1-i)}+\frac{\|y\|_{L^1}}{\mu}\\&\le\rho_1\|y\|_{L^1}.\end{aligned}$$

□

Lemma 4.4. *Suppose that Ω is an open bounded subset of X such that $\mathrm{dom}\,L\bigcap\overline{\Omega}\neq\emptyset$. Then N is L-compact on $\overline{\Omega}$.*

Proof. It is clear that $QN(\overline{\Omega})$ and $K_p(Id-Q)N(\overline{\Omega})$ are bounded, due to the fact that f realize the Carathéodory conditions. Using the Lebesgue dominated convergence theorem 1.55, we can easily find that QN and $K_{P,Q}N=K_p(Id-Q)N:\overline{\Omega}\to X$ are continuous. By the hypothesis (iii) on the function f, there exists a constant $M>0$, such that $|(Id-Q)N(u(t))|\le M$, for all

$u \in \Omega$, $t \in [0;1]$. For $i = 0,1,2$, $0 \leq t_1 \leq t_2 \leq 1$, and $u \in \Omega$, we have

$$\begin{aligned}
&\left|(K_{P,Q}Nu)^{(i)}(t_2) - (K_{P,Q}Nu)^{(i)}(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha-i)}\left|\int_0^{t_2} \frac{(t_2-s)^{\alpha-i-1}}{\phi(s)} \int_0^s (Id-Q)Nu(r)\,\mathrm{d}r\,\mathrm{d}s \right.\\
&\qquad \left. - \int_0^{t_1} \frac{(t_1-s)^{\alpha-i-1}}{\phi(s)} \int_0^s (Id-Q)Nu(r)\,\mathrm{d}r\,\mathrm{d}s\right| \\
&\leq \frac{M}{\mu\Gamma(\alpha-i)}\left\{\int_0^{t_1} (t_2-s)^{\alpha-i-1} - (t_1-s)^{\alpha-i-1}\,\mathrm{d}s + \int_{t_1}^{t_2} (t_2-s)^{\alpha-i-1}\,\mathrm{d}s\right\} \\
&= \frac{M}{\mu\Gamma(\alpha+1-i)}(t_2^{\alpha-i} - t_1^{\alpha-i}).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|{}^cD_{0^+}^{\alpha} K_{P,Q}Nu(t_2) - {}^cD_{0^+}^{\alpha} K_{P,Q}Nu(t_1)\right| \\
&= \left|\frac{1}{\phi(t_2)}\int_0^{t_2} (Id-Q)Nu(s)\,\mathrm{d}s - \frac{1}{\phi(t_1)}\int_0^{t_1} (Id-Q)Nu(s)\,\mathrm{d}s\right| \\
&= \left|\left(\frac{1}{\phi(t_2)} - \frac{1}{\phi(t_1)}\right)\int_0^{t_1} (Id-Q)Nu(s)\,\mathrm{d}s + \frac{1}{\phi(t_2)}\int_{t_1}^{t_2} (Id-Q)Nu(s)\,\mathrm{d}s\right| \\
&\leq \frac{M}{\mu^2}|\phi(t_2) - \phi(t_1)| + \frac{M}{\mu}(t_2 - t_1).
\end{aligned}$$

Since t^{α}, $t^{\alpha-1}$, $t^{\alpha-2}$ and $\phi(t)$ are uniformly continuous on $[0;1]$, we get that $K_p(Id-Q)N\colon \overline{\Omega} \to X$ is compact. The Lemma is then proved.

□

Theorem 4.5. *In addition to $(H_0)-(H_2)$, suppose that the following conditions hold :*

(H_3) *There exists a Carathéodory function $\Phi\colon [0;1]\times(\mathbb{R}_+)^4 \to \mathbb{R}_+$ nondecreasing with respect to the last four arguments such that*

$$|f(t,x_0,x_1,x_2,x_3)| \leq \Phi(t,|x_0|,|x_1|,|x_2|,|x_3|),$$

(H_4) $\lim_{r\to\infty}\sup \frac{1}{r}\int_0^1 |\Phi(s,r,r,r,r)|\,\mathrm{d}s < \frac{1}{\rho_1+\rho_2}$ *where ρ_1 is defined by (4.7) and*

$$\rho_2 = \frac{1}{\mu}\left(\frac{2}{\Gamma(\alpha)} + \frac{5}{\Gamma(\alpha-1)}\right).$$

(H_5) *There exists a constant $A > 0$ such that for $u \in \operatorname{dom} L \setminus \operatorname{Ker} L$, if $|u'(t)| > A$ or $|u''(t)| > A$ for all $t \in [0;1]$, then $T_1(Nu) \neq 0$ or $T_2(Nu) \neq 0$.*

(H_6) *There exists a constant $B > 0$ such that for any $c_1, c_2 \in \mathbb{R}$, if $|c_1| > B, |c_2| > B$, then either*

$$T_1N(c_1t + c_2t^2) + T_2N(c_1t + c_2t^2) < 0,$$

or

$$T_1N(c_1t + c_2t^2) + T_2N(c_1t + c_2t^2) > 0.$$

Then, the problem (4.1) has at least one solution.

Remark 4.6. A sufficient condition for (H_3) be satisfied is the existence of functions $\theta_i(t) \in Y$, $i = 0, \ldots, 5$ and a constant $\nu \in (0;1)$ such that for all $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $t \in [0;1]$ the nonlinearity f verifies one of the following growth conditions :

$$|f(t, x_0, x_1, x_2, x_3)| \leq \sum_{i=0}^{3} \theta_i(t)|x_i| + \theta_4(t)|x_0|^\nu + \theta_5(t),$$

$$|f(t, x_0, x_1, x_2, x_3)| \leq \sum_{i=0}^{3} \theta_i(t)|x_i| + \theta_4(t)|x_1|^\nu + \theta_5(t),$$

$$|f(t, x_0, x_1, x_2, x_3)| \leq \sum_{i=0}^{3} \theta_i(t)|x_i| + \theta_4(t)|x_2|^\nu + \theta_5(t),$$

$$|f(t, x_0, x_1, x_2, x_3)| \leq \sum_{i=0}^{3} \theta_i(t)|x_i| + \theta_4(t)|x_3|^\nu + \theta_5(t).$$

In this case, (H_4) reduces to
(H_4^*) $\sum_{i=0}^{3} \|\theta_i\|_{L^1} < \frac{1}{\rho_1 + \rho_2}$.

Proof. Consider the set

$$\Omega_1 = \{u \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lu = \lambda N u, \, \lambda \in [0;1]\}$$

Then for $u \in \Omega_1$, $Lu = \lambda N u$, thus $\lambda \neq 0$, $Nu \in \operatorname{Im} L = \operatorname{Ker} Q \subset Y$, hence, $Q(Nu) = 0$ that is, $T_1(Nu) = T_2(Nu) = 0$. From (H_5) we have that the exists $t_1, t_2 \in [0;1]$ such that, $|u'(t_1)| \leq A, |u''(t_2)| \leq A$.

If $t_1 = t_2 = 0$, we have that $|u'(0)| \leq A, |u''(0)| \leq A$. Otherwise, by $Lu = \lambda N u$, we obtain

$$u(t) = u'(0)t + \frac{u''(0)}{2} t^2 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s Nu(r)\, \mathrm{d}r\, \mathrm{d}s.$$

If $t_2 \neq 0$, then

$$u''(t_2) = u''(0) + \frac{\lambda}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s Nu(r)\, \mathrm{d}r\, \mathrm{d}s,$$

together with $|u''(t_2)| \leq A$, we have

$$\begin{aligned} |u''(0)| &\leq |u''(t_2)| + \frac{1}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s |Nu(r)|\, \mathrm{d}r\, \mathrm{d}s \\ &\leq A + \frac{\|Nu\|_{L^1}}{\mu \Gamma(\alpha-1)}. \end{aligned}$$

Consequently

$$|u''(0)| \leq A + \frac{1}{\mu \Gamma(\alpha-1)} \|Nu\|_{L^1}. \tag{4.15}$$

If $t_1 \neq 0$, thus

$$u'(t_1) = u'(0) + u''(0)t_1 + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s Nu(r)\,\mathrm{d}r\,\mathrm{d}s,$$

according to (4.15) and $|u'(t_1)| \leq A$, we get

$$\begin{aligned}|u'(0)| &\leq |u'(t_1)| + |u''(0)| + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s |Nu(r)|\,\mathrm{d}r\,\mathrm{d}s \\ &\leq 2A + \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right)\|Nu\|_{L^1}.\end{aligned}$$

So

$$|u'(0)| \leq 2A + \frac{1}{\mu}\left(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right)\|Nu\|_{L^1}. \tag{4.16}$$

Again for $u \in \Omega_1$, we get

$$\|Pu\|_X = \sum_{i=0}^{2} \max_{t\in[0;1]} |(Pu)^{(i)}(t)| + \max_{t\in[0;1]} |{}^cD_{0^+}^{\alpha}(Pu)(t)| \leq 2|u'(0)| + 3|u''(0)|.$$

From (4.15) and (4.16), we obtain

$$\|Pu\|_X \leq 7A + \rho_2\|Nu\|_{L^1}. \tag{4.17}$$

Again for all $u \in \Omega_1$, we have $(Id-P)u \in \operatorname{dom} L \cap \operatorname{Ker} P$; thus by (4.14) and (4.6), we find

$$\|(Id-P)u\|_X = \|K_pL(Id-P)u\|_X \leq \rho_1\|L(Id-P)u\|_{L^1} = \rho_1\|Lu\|_{L^1} \leq \rho_1\|Nu\|_{L^1}. \tag{4.18}$$

From (4.17) and (4.18), we obtain

$$\|u\|_X \leq \|Pu\|_X + \|(Id-P)u\|_X \leq 7A + (\rho_1+\rho_2)\|Nu\|_{L^1}. \tag{4.19}$$

On the other hand from (H_3), we have

$$\begin{aligned}\|Nu\|_{L^1} &= \int_0^1 \left|f(s,u(s),u'(s),u''(s),{}^cD_{0^+}^{\alpha}u(s))\right|\mathrm{d}s \\ &\leq \int_0^1 \left|\Phi(s,u(s),u'(s),u''(s),{}^cD_{0^+}^{\alpha}u(s))\right|\mathrm{d}s \\ &\leq \int_0^1 |\Phi(s,\|u\|_X,\|u\|_X,\|u\|_X,\|u\|_X)|\,\mathrm{d}s\end{aligned} \tag{4.20}$$

Because the function Φ is Carathéodory, then the function $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ given by $\Psi(r) = \frac{1}{r}\int_0^1 |\Phi(s,r,r,r,r)|\,\mathrm{d}s$ is well defined. Let $\ell = \lim_{r\to\infty}\sup \Psi(r)$. By (H_4) $0 < \ell < \frac{1}{\rho_1+\rho_2}$, then for each $0 < \varepsilon < \frac{1}{\rho_1+\rho_2} - \ell$, there exists r_ε such that

$$r \geq r_\varepsilon \Longrightarrow \Psi(r) < \ell + \varepsilon.$$

If $\|u\|_X \geq r_\varepsilon$, then $\Psi(\|u\|_X) < \frac{1}{\rho_1+\rho_2}$ and thus (4.20) implies that

$$\|Nu\|_{L^1} \leq (\ell+\varepsilon)\|u\|_X. \tag{4.21}$$

Therefore, (4.19) and (4.21), it yield

$$r_\varepsilon \leq \|u\|_X \leq \frac{7A}{1-(\rho_1+\rho_2)(\ell+\varepsilon)}.$$

Consequently

$$\|u\|_X \leq \max\left\{r_\varepsilon, \frac{7A}{1-(\ell+\varepsilon)(\rho_1+\rho_2)}\right\} = \frac{7A}{1-(\ell+\varepsilon)(\rho_1+\rho_2)}. \tag{4.22}$$

Since (4.22) is valid for all $0<\varepsilon<\frac{1}{\rho_1+\rho_2}-\ell$, then

$$\|u\|_X \leq \frac{7A}{1-\ell(\rho_1+\rho_2)}.$$

So, Ω_1 is bounded. Let

$$\Omega_2 = \{u \in \operatorname{Ker} L \mid Nu \in \operatorname{Im} L\}$$

For $u \in \Omega_2$, then $u \in \operatorname{Ker} L = \left\{u \mid u(t) = c_1 t + c_2 t^2, c_1, c_2 \in \mathbb{R}\right\}$, and $Q(Nu) = 0$, that is, $T_1 N(c_1 t + c_2 t^2) = T_2 N(c_1 t + c_2 t^2) = 0$. From condition (H_6), we get $|c_1| \leq B, |c_2| \leq B$. Hence, Ω_2 is bounded. Let

$$\Omega_3 = \{u \in \operatorname{Ker} L \mid -\lambda Ju + (1-\lambda)QNu = 0,\ \lambda \in [0;1]\}$$

if the first part of (H_6) holds.
Or we'll set

$$\Omega_3 = \{u \in \operatorname{Ker} L \mid \lambda Ju + (1-\lambda)QNu = 0,\ \lambda \in [0;1]\}$$

if the second part of (H_6) holds.
Where $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by

$$J(c_1 t + c_2 t^2) = \omega_1 . t^{p-1} + \omega_2 . t^{q-1}, \quad c_1, c_2 \in \mathbb{R}. \tag{4.23}$$

where

$$\omega_1 = \frac{1}{\Delta(p,q)}(d_{22}|c_1| - d_{21}|c_2|), \quad \omega_2 = \frac{1}{\Delta(p,q)}(-d_{12}|c_1| + d_{11}|c_2|).$$

Without loss of generality, we assume that the first part of (H_6) hold.
In fact $u \in \Omega_3$, means that $u = c_1 t + c_2 t^2$ and $-\lambda Ju + (1-\lambda)QNu = 0$. Then we obtain

$$-\lambda J(c_1 t + c_2 t^2) + (1-\lambda)QN(c_1 t + c_2 t^2) = 0. \tag{4.24}$$

If $\lambda = 0$, then $|c_1| \leq B, |c_2| \leq B$. If $\lambda = 1$, then

$$\begin{cases} d_{22}|c_1| - d_{21}|c_2| = 0, \\ -d_{12}|c_1| + d_{11}|c_2| = 0. \end{cases} \tag{4.25}$$

The determinant of coefficiency for (4.25) is $\Delta(p,q) \neq 0$. Thus, (4.25) only have zero solutions, that is $c_1 = c_2 = 0$.
Otherwise, if $\lambda \neq 0$ and $\lambda \neq 1$, again from (4.23), (4.24) becomes

$$\lambda(\omega_1.t^{p-1} + \omega_2.t^{q-1}) = (1-\lambda)\left(Q_1 N(c_1 t + c_2 t^2).t^{p-1} + Q_2 N(c_1 t + c_2 t^2).t^{q-1}\right).$$

Hence,

$$\begin{cases} \lambda\omega_1 = (1-\lambda)Q_1(c_1 t + c_2 t^2), \\ \lambda\omega_2 = (1-\lambda)Q_2(c_1 t + c_2 t^2). \end{cases}$$

Thus,

$$\begin{cases} \lambda|c_1| = (1-\lambda)T_1 N(c_1 t + c_2 t^2), \\ \lambda|c_2| = (1-\lambda)T_2 N(c_1 t + c_2 t^2). \end{cases}$$

Then, we get

$$\lambda(|c_1| + |c_2|) = (1-\lambda)\left(T_1 N(c_1 t + c_2 t^2) + T_2 N(c_1 t + c_2 t^2)\right) < 0.$$

By the first part of (H_6), we have $|c_1| \leq B, |c_2| \leq B$. Here, Ω_3 is bounded.

Now, we shall prove that all the conditions of Theorem 1.106 are satisfied. Let Ω to be a bounded open set of X containing $\bigcup_{i=1}^{3} \overline{\Omega}_i$. By Lemma 4.4, N is L-compact on $\overline{\Omega}$. Because Ω_1 and Ω_2 are bounded sets, then

(1) $Lu \neq \lambda Nu$ for each $(u,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0;1)$;

(2) $Nu \notin \operatorname{Im} L$ for each $u \in \operatorname{Ker} L \cap \partial\Omega$.

At least we will prove that (3) of Theorem 1.106 is satisfied. Let

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu$$

Because Ω_3 is bounded, then

$$H(u,\lambda) \neq 0, \quad \forall u \in \operatorname{Ker} L \bigcap \partial\Omega.$$

Appealing to the homotopy property of the degree, we obtain

$$\begin{aligned} \deg(QN\mid_{\operatorname{Ker} L}, \Omega \bigcap \operatorname{Ker} L, 0) &= \deg(H(\cdot,0), \Omega \bigcap \operatorname{Ker} L, 0) \\ &= \deg(H(\cdot,1), \Omega \bigcap \operatorname{Ker} L, 0) \\ &= \deg(\pm J, \Omega \bigcap \operatorname{Ker} L, 0) \neq 0. \end{aligned}$$

Then by Theorem 1.106, $Lu = Nu$ has at least one solution in $\operatorname{dom} L \bigcap \overline{\Omega}$, we conclude that the boundary value problem (4.1) has at least one solution in X. The proof is finished.

□

4.2.1 Example

To illustrate our main results, we will present an example.

Example 4.7. Let us consider the following fractional boundary value problem

$$\begin{aligned} &(\phi(t)^c D_{0^+}^{\frac{5}{2}} u(t))' = f(t, u(t), u'(t), u''(t), {}^c D_{0^+}^{\frac{5}{2}} u(t)),\ t \in [0;1] \\ &u(0) = {}^c D_{0^+}^{\alpha} u(0) = 0,\ u''(0) = -u''(\frac{1}{3}) + 2u''(\frac{1}{6}),\ u'(1) = -2u'(\frac{1}{4}) + 3u'(\frac{1}{2}). \end{aligned} \tag{4.26}$$

where $\phi(t) = e^{t-3}$ and

$$f(t,x_0,x_1,x_2,x_3) = x_2 + \cos x_3(1-\sin x_1) + \sqrt{|x_2|}.$$

Corresponding to the problem (4.1), we have that $\alpha = \frac{5}{2}$, $l = 2$, $m = 2$, $a_1 = -1$, $a_2 = 2$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{1}{6}$, $b_1 = -2$, $b_2 = 3$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\mu = \min_{t\in[0;1]} \phi(t) = e^{-3} > 0$. Then we get $a_1 + a_2 = b_1 + b_2 = 1$, $b_1\eta_1 + b_2\eta_2 = 1$. Thus the condition (H_1) holds. Also we find

$$T_1(y) = -\int_0^{\frac{1}{3}} \left(\frac{1}{3}-s\right)^{-\frac{1}{2}} e^{3-s}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s + 2\int_0^{\frac{1}{6}} \left(\frac{1}{6}-s\right)^{-\frac{1}{2}} e^{3-s}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$$\begin{aligned}T_2(y) = &\int_0^1 (1-s)^{\frac{1}{2}} e^{3-s}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - 2\int_0^{\frac{1}{4}} \left(\frac{1}{4}-s\right)^{\frac{1}{2}} e^{3-s}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s\\ &+3\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{1}{2}} e^{3-s}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

By calculations, we get

$$\Delta(1,2) = \begin{vmatrix} -761/993 & -301/982 \\ 1545/311 & 463/431 \end{vmatrix} = \frac{263}{376} \neq 0,$$

Therefore, the condition (H_2) holds. On the other hand, we have

$$|f(t,x_0,x_1,x_2,x_3)| \leq |x_2| + \sqrt{|x_2|} + 2.$$

We can get that the condition (H_3) holds, where

$$\theta_0(t) = \theta_1(t) = \theta_3(t) = 0,\ \theta_2(t) = 1,\ \theta_4(t) = \frac{1}{2},\ \theta_5(t) = 2,\ \nu = \frac{1}{2}.$$

Also we have

$$(\rho_1+\rho_2)\sum_{i=0}^{3}\|\theta_i\|_{L^1} = e^{-3}\left(\frac{1}{\Gamma(3.5)} + \frac{3}{\Gamma(2.5)} + \frac{6}{\Gamma(1.5)} + 1\right) = \frac{833}{1620} < 1.$$

Therefore, the condition (H_4^*) holds.

Let $A = 9$ and assume that $|u''(t)| > 9$ holds for all $t \in [0;1]$, by the continuity of $u''(t)$, we either have $u''(t) > 9$, for all $t \in [0;1]$, or $u''(t) < -9$, for all $t \in [0;1]$. If $u''(t) > 9$, for all $t \in [0;1]$ we obtain

$$\begin{aligned}T_2(y) = &\int_0^1 (1-s)^{\frac{1}{2}} e^{3-s}\int_0^s \left(u''(r) + \cos {}^cD_{0+}^{\alpha}u(r)(1-\sin u'(r)) + \sqrt{|u''(r)|}\right)\mathrm{d}r\,\mathrm{d}s\\ &-2\int_0^{\frac{1}{4}} \left(\frac{1}{4}-s\right)^{\frac{1}{2}} e^{3-s}\int_0^s \left(u''(r) + \cos {}^cD_{0+}^{\alpha}u(r)(1-\sin u'(r)) + \sqrt{|u''(r)|}\right)\mathrm{d}r\,\mathrm{d}s\\ &+3\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{1}{2}} e^{3-s}\int_0^s \left(u''(r) + \cos {}^cD_{0+}^{\alpha}u(r)(1-\sin u'(r)) + \sqrt{|u''(r)|}\right)\mathrm{d}r\,\mathrm{d}s.\\ &\geq 5\int_0^1 s(1-s)^{\frac{1}{2}} e^{3-s}\,\mathrm{d}s - 14\int_0^{\frac{1}{4}} s\left(\frac{1}{4}-s\right)^{\frac{1}{2}} e^{3-s}\,\mathrm{d}s + 15\int_0^{\frac{1}{2}} s\left(\frac{1}{2}-s\right)^{\frac{1}{2}} e^{3-s}\,\mathrm{d}s\\ &\geq \frac{7280}{257}\end{aligned}$$

If $u''(t) < -9$, for all $t \in [0;1]$ we obtain

$$\begin{aligned}
T_2(y) &= \int_0^1 (1-s)^{\frac{1}{2}} e^{3-s} \int_0^s \left(u''(r) + \cos {}^cD^{\alpha}_{0^+} u(r)(1-\sin u'(r)) + \sqrt{|u''(r)|}\right) \mathrm{d}r\,\mathrm{d}s \\
&\quad - 2\int_0^{\frac{1}{4}} \left(\frac{1}{4}-s\right)^{\frac{1}{2}} e^{3-s} \int_0^s \left(u''(r) + \cos {}^cD^{\alpha}_{0^+} u(r)(1-\sin u'(r)) + \sqrt{|u''(r)|}\right) \mathrm{d}r\,\mathrm{d}s \\
&\quad + 3\int_0^{\frac{1}{2}} \left(\frac{1}{2}-s\right)^{\frac{1}{2}} e^{3-s} \int_0^s \left(u''(r) + \cos {}^cD^{\alpha}_{0^+} u(r)(1-\sin u'(r)) + \sqrt{|u''(r)|}\right) \mathrm{d}r\,\mathrm{d}s. \\
&\le -4\int_0^1 s(1-s)^{\frac{1}{2}} e^{3-s}\,\mathrm{d}s + 14\int_0^{\frac{1}{4}} s\left(\frac{1}{4}-s\right)^{\frac{1}{2}} e^{3-s}\,\mathrm{d}s - 12\int_0^{\frac{1}{2}} s\left(\frac{1}{2}-s\right)^{\frac{1}{2}} e^{3-s}\,\mathrm{d}s \\
&\le -\frac{12329}{544}
\end{aligned}$$

So condition (H_5) is satisfied.

Let $B = 1$ and $c_1, c_2 \in \mathbb{R}$ be such that $|c_1| > 1$, $|c_2| > 1$, we have

$$T_1N(c_1t + c_2t^2) + T_2N(c_1t + c_2t^2) = (2|c_2| + \sqrt{2|c_2|})(d_{11} + d_{12}) < 0.$$

So, (H_6) hold.

Then, all the assumptions of Theorem 4.5 hold. Thus, the problem (4.26) has at last one solution.

Chapitre 5

Solvability for Multi-Point BVP of Nonlinear Fractional Differential Equations at Resonance with Three Dimensional Kernels [1]

5.1 Introduction

The present work in this chapter is concerned with a kind of fractional differential equation witch can be written as $Lx = Nx$, where L is a linear Fredholm operator of index zero and N is a nonlinear operator. It is well known that if the kernel of the linear part of the above equation contains only zero, the corresponding boundary value problem is called non-resonant, in this case L is invertible, the equation can be reduced to a fxed point problem for the $L^{-1}N$ operator. Otherwise, if L is a non-invertible, i.e. $\dim \operatorname{Ker} L \geq 1$, then the problem is said to be at resonance, and then the problem can be solved by using the coincidence degree theory. The higher value of $\dim \operatorname{Ker} L$, is the more difficult, it will solve the problem. More recently, many authors investigated the existence of solutions for fractional differential equations at resonance. For instance see [22, 23, 24, 25, 47, 66, 67, 68, 71, 72, 77, 78, 119] and the references therein.

The case of $\dim \operatorname{Ker} L = 1$ has been discussed by many authors [22, 24, 25, 47, 66, 67, 68, 72, 77, 78, 119]. In [24], Z. Bai and Zhang investigated the boundary value problem for a fractional differential equation with nonlinear growth with $\dim \operatorname{Ker} L = 1$:

$$\begin{cases} D^{\alpha}_{0^+} u(t) = f(t, u(t), D^{\alpha-1}_{0^+} u(t)), \quad t \in [0;1], \\ \qquad u(0) = 0, \quad u(1) = \sigma u(\eta), \end{cases}$$

where $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville derivative, $1 < \alpha \leq 2$, $f \colon [0;1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\sigma \in (0, \infty)$, $\eta \in (0;1)$ are given constants such that $\sigma \eta^{\alpha-1} = 1$.

Z. Hu et al. showed in [67] an existence of solutions a two-point boundary value problems

1. **Z. Baitiche**, M. Benbachir and K. Guerbati, Solvability for Multi-Point BVP of Nonlinear Fractional Differential Equations at Resonance with Three Dimensional Kernels, *Kragujevac Journal of Mathematics.*

for fractional differential equations at resonance with $\dim \operatorname{Ker} L = 1$:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t,u(t),u'(t)), \quad t \in [0;1], \\ \qquad u(0) = 0, \quad u(1) = u'(1), \end{cases}$$

where $D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $1 < \alpha \leq 2$, $f\colon [0;1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory condition.

L. Hu et al. have studied in [68] a two-point boundary value problems for fractional differential equations at resonance with $\dim \operatorname{Ker} L = 1$:

$$\begin{cases} \qquad D^{\alpha}_{0^+}u(t) = f(t,u(t),D^{\alpha-1}_{0^+}u(t),D^{\alpha-2}_{0^+}u(t),\dots,D^{\alpha-(N-1)}_{0^+}u(t)), \\ u(0) = D^{\alpha-2}_{0^+}u(0) = \cdots = D^{\alpha-(N-1)}_{0^+}u(0) = 0,\ D^{\alpha-1}_{0^+}u(0) = D^{\alpha-1}_{0^+}u(1), \end{cases}$$

where $0 < t < 1$, $N-1 < \alpha \leq N$, $D^{\alpha}_{0^+}$ is Riemann-Liouville fractional derivative, and $f\colon [0;1] \times \mathbb{R}^2 \to \mathbb{R}$, is continuous function.

Y. Chen and Tang studied in [47] the existence of solutions for the following fractional multi-point boundary value problems at resonance with $\dim \operatorname{Ker} L = 1$:

$$\begin{cases} \left(a(t)^c D^{\alpha}_{0^+}u(t)\right)' = f(t,u(t),u'(t),{}^cD^{\alpha}_{0^+}u(t)), \quad t \in J, \\ u(0) = 0, \quad {}^cD^{\alpha}_{0^+}u(0) = 0, \quad u(1) = \sum_{j=1}^{m-1} \sigma_j u(\xi_j), \end{cases}$$

where ${}^cD^{\alpha}_{0^+}$ is the Caputo fractional derivative, $1 < \alpha \leq 2$, $f\colon [0;1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions, $a(t) \in C^1[0;1]$, $\min_{t\in J} a(t) > 0$, $J = [0;1]$, $\sigma_j \in \mathbb{R}^*_+$, $\xi_j \in (0,1)$, $j = 1,\dots,m-1$, $m \in \mathbb{N}$, $m > 1$, and $\sum_{j=1}^{m-1} \sigma_j \xi_j = 1$.

For the case of $\dim \operatorname{Ker} L = 2$, there are some results in [23, 71]. Bai and Zhang established in [23] the existence of at least one solution for the m-point boundary value problems for fractional differential equations at resonance with $\dim \operatorname{Ker} L = 2$,

$$\begin{cases} \qquad D^{\alpha}_{0^+}u(t) = f(t,u(t),D^{\alpha-2}_{0^+}u(t),D^{\alpha-1}_{0^+}u(t)), \quad t \in (0;1), \\ I^{\alpha-1}_{0^+}u(0) = 0, \quad D^{\alpha-1}_{0^+}u(0) = D^{3-\alpha}_{0^+}(\eta), \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \end{cases}$$

where $2 < \alpha < 3$, $0 < \eta \leq 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $m \geq 2$, $\sum_{i=1}^{m} \alpha_i \eta_i^{\alpha-1} = \sum_{i=1}^{m} \alpha_i \eta_i^{\alpha-2} = 1$. $D^{\alpha}_{0^+}$ and $I^{\alpha}_{0^+}$ are the standard Riemann-Liouville fractional derivative and fractional integral respectively and $f\colon [0;1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Carathéodory conditions. The results are obtained under the assumption that

$$R = \frac{1}{\alpha}\eta^{\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1 - \sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-2}\right] - \frac{1}{\alpha-1}\eta^{\alpha-1}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left[1 - \sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-1}\right] \neq 0.$$

W. Jiang showed in [71] an existence result for the boundary value problems of fractional differential equations at resonance with $\dim \operatorname{Ker} L = 2$:

$$\begin{cases} \qquad D^{\alpha}_{0^+}u(t) = f(t,u(t),D^{\alpha-1}_{0^+}u(t)), \quad \forall t \in J = [0;1], \\ u(0) = 0, \quad D^{\alpha-1}_{0^+}u(0) = \sum_{i=1}^{m} a_i D^{\alpha-1}_{0^+}(\xi_i), \quad D^{\alpha-2}_{0^+}u(0) = \sum_{j=1}^{n} b_j D^{\alpha-2}_{0^+}(\eta_j), \end{cases}$$

where $2 < \alpha < 3$, $D^{\alpha}_{0^+}$ is Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^{m} a_i = 1$, $\sum_{j=1}^{n} b_j = 1$, $\sum_{j=1}^{n} b_j\eta_j = 1$, $f\colon [0;1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions. The results are obtained under the assumption that

$$\frac{1}{3}\left[1 - \sum_{j=1}^{n} b_j\eta_j^3\right]\sum_{i=1}^{m} a_i\xi_i - \frac{1}{2}\left[1 - \sum_{j=1}^{n} b_j\eta_j^2\right]\sum_{i=1}^{m} a_i\xi_i^2 \neq 0.$$

Thus, motivated by the results mentioned, in this paper, we discuss the existence of solutions for the following multi-point boundary value problems by using Mawhin's continuation theorem :

$$\begin{cases} \left(\phi(t)^c D^{\alpha}_{0^+} u(t)\right)' = f(t,u(t),u'(t),u''(t),u'''(t),{}^c D^{\alpha}_{0^+} u(t)), \quad t \in I = [0;1] \\ u(0) = 0, \quad {}^c D^{\alpha}_{0^+} u(0) = 0, \quad u'''(0) = \sum_{i=1}^m a_i u'''(\xi_i), \\ u''(0) = \sum_{j=1}^l b_j u''(\eta_j), \quad u'(1) = \sum_{k=1}^n c_k u'(\rho_k), \end{cases} \tag{5.1}$$

where ${}^c D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $3 < \alpha \leq 4$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_l < 1$, $0 < \rho_1 < \cdots < \rho_n < 1$, $a_i, b_j, c_k \in \mathbb{R}$, $(i = 1,\ldots,m,\ j = 1,\ldots,l, k = 1,\ldots,n)$, $\phi(t) \in C^1[0;1]$, $\mu = \min_{t\in I} \phi(t) > 0$ and $f\colon [0;1] \times \mathbb{R}^5 \to \mathbb{R}$ is a Carathéodory function, that is,

(i) for each $x \in \mathbb{R}^5$, the function $t \to f(t,x)$ is Lebesgue measurable ;

(ii) for almost every $t \in [0;1]$, the function $t \to f(t,x)$ is continuous on $\mathbb{R}^5$;

(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1[0;1]$ such that, for a.e. $t \in [0;1]$ and every $|x| \leq r$, we have $|f(t,x)| \leq \varphi_r(t)$.

In this work, we will always suppose that the following condition hold :

(H_1) $\sum_{i=1}^m a_i = \sum_{j=1}^l b_j = \sum_{k=1}^n c_k = 1$, $\sum_{j=1}^l b_j \eta_j = 0$, $\sum_{k=1}^n c_k \rho_k = \sum_{k=1}^n c_k \rho_k^2 = 1$.

(H_2)

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} \neq 0,$$

where for $\nu = 1,2,3$, we define

$$d_{\nu 1} = \sum_{i=1}^m a_i \int_0^{\xi_i} \frac{s^\nu (\xi_i - s)^{\alpha-4}}{\nu \phi(s)} \mathrm{d}s, \quad d_{\nu 2} = \sum_{j=1}^l b_j \int_0^{\eta_j} \frac{s^\nu (\eta_j - s)^{\alpha-3}}{\nu \phi(s)} \mathrm{d}s,$$

$$d_{\nu 3} = \int_0^1 \frac{s^\nu (1-s)^{\alpha-2}}{\nu \phi(s)} \mathrm{d}s - \sum_{k=1}^n c_k \int_0^{\rho_k} \frac{s^\nu (\rho_k - s)^{\alpha-2}}{\nu \phi(s)} \mathrm{d}s.$$

5.2 Existence Results

In this section, we shall present and prove our main result.
Let $Y = L^1[0;1]$ with the norm $\|y\|_{L^1} = \int_0^1 |y(t)| dt$. Define

$$X = \{u : {}^c D^{\alpha}_{0^+} u \in C[0;1], u \, \text{satisfies boundary value conditions of (5.1)} \}.$$

Equipped with norm

$$\|u\|_X = \sum_{i=0}^3 \|u^{(i)}\|_\infty + \|{}^c D^{\alpha}_{0^+} u\|_\infty, \ \text{ where } \|u\|_\infty = \max_{t\in[0;1]} |u(t)|.$$

By means of the functional analysis theory, we can prove that $(X, \|\cdot\|_X)$ is Banach space.

Define the operator of differentiation $L\colon domL\bigcap X \to Y$:

$$Lu = (\phi\,^cD^{\alpha}_{0^+}u)', \quad u \in \mathrm{dom}\,L.$$

where

$$\mathrm{dom}\,L = \{u \in X \mid {}^cD^{\alpha}_{0^+}u(t)\, \text{is absolutely continuous on}\, [0;1]\}$$

and the Nemytskii operator $N\colon X \to Y$ as :

$$Nu(t) = f(t, u(t), u'(t), u''(t), u'''(t), {}^cD^{\alpha}_{0^+}u(t)), \quad t \in [0;1].$$

Thus, bvp (5.1) is equivalent to

$$Lu = Nu, \quad u \in \mathrm{dom}\,L.$$

Lemma 5.1. *$U \subset X$ is a relatively compact set in X if and only if U is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$ such that for every $u \in U$*

$$\|u\|_X = \sum_{i=0}^{3} \|u^{(i)}\|_\infty + \|{}^cD^{\alpha}_{0^+}u\|_\infty \le M.$$

and equicontinuous means that $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$|u^{(i)}(t_1) - u^{(i)}(t_2)| < \varepsilon, \quad \forall u \in U,\, \forall t_1, t_2 \in I,\, |t_1 - t_2| < \delta,\, \forall i \in \{0,1,2,3\}.$$

and

$$|{}^cD^{\alpha}_{0^+}u(t_1) - {}^cD^{\alpha}_{0^+}u(t_2)| < \varepsilon, \quad \forall u \in U,\, \forall t_1, t_2 \in I,\, |t_1 - t_2| < \delta.$$

With arguments similar to those of Lemma 4.2, we obtain the following lemma.

Lemma 5.2. *Let $y \in Y$, then $u \in X$ is the solution of the following fractional differential equation :*

$$\begin{cases} \left(\phi(t)\,^cD^{\alpha}_{0^+}u(t)\right)' = y(t), \quad t \in I = [0;1] \\ u(0) = 0, \quad {}^cD^{\alpha}_{0^+}u(0) = 0, \quad u'''(0) = \sum_{i=1}^{m} a_i u'''(\xi_i), \\ u''(0) = \sum_{j=1}^{l} b_j u''(\eta_j), \quad u'(1) = \sum_{k=1}^{n} c_k u'(\rho_k), \end{cases} \tag{5.2}$$

where u is given by :

$$u(t) = \sum_{i=1}^{3} \delta_i t^i + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}, \tag{5.3}$$

and

$$T_1(y) = T_2(y) = T_3(y) = 0. \tag{5.4}$$

Where T_1, T_2, $T_3\colon Y \to Y$ are three linear operators defined as follow :

$$T_1(y) = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-4}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$$T_2(y) = \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$$T_3(y) = \int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - \sum_{k=1}^{n} c_k \int_0^{\rho_k} \frac{(\rho_k - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s.$$

Lemma 5.3. *Assume* (H_1) *and* (H_2) *hold. Let* $\phi(t) \in C^1[0;1]$, $\mu = \min_{t\in[0;1]}\phi(t) > 0$, *then* $L\colon \mathit{dom}L \subset X \to Y$ *is a Fredholm operator of index zero, and the inverse linear operator* $K_p = L_p^{-1}\colon \operatorname{Im}L \to \operatorname{dom}L \cap \operatorname{Ker}P$ *is defined by*

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s. \tag{5.5}$$

It satisfies

$$\|K_p y\|_X \le \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|y\|_{L^1}. \tag{5.6}$$

Proof. It is clear that $\operatorname{Ker}L = \{u \mid u(t) = \sum_{k=1}^3 \delta_k t^k,\ \delta_1,\delta_2,\delta_3 \in \mathbb{R}\}$. Furthermore, Lemma 5.2 implies

$$\operatorname{Im}L = \{y \in Y \mid T_1(y) = T_2(y) = T_3(y) = 0\}. \tag{5.7}$$

Consider continuous linear mapping $Q\colon Y \to Y$ defined by

$$Qy = Q_1(y) + Q_2(y).t + Q_3(y).t^2 \tag{5.8}$$

where $Q_1, Q_2, Q_3\colon Y \to Y$ be three linear operators defined as follows

$$Q_1(y) = \frac{1}{\Delta}\sum_{i=1}^3 e_{1i}T_i(y), \quad Q_2(y) = \frac{1}{\Delta}\sum_{i=1}^3 e_{2i}T_i(y), \quad Q_3(y) = \frac{1}{\Delta}\sum_{i=1}^3 e_{3i}T_i(y),$$

$e_{ij}(i,j=1,2,3)$ are the algebraic complements of d_{ij}.

We will prove that $\operatorname{Ker}Q = \operatorname{Im}L$. Obviously, $\operatorname{Im}L \subset \operatorname{Ker}Q$. As well, if $y \in \operatorname{Ker}Q$, then

$$\begin{cases} e_{11}T_1(y) + e_{12}T_2(y) + e_{13}T_3(y) = 0. \\ e_{21}T_1(y) + e_{22}T_2(y) + e_{23}T_3(y) = 0. \\ e_{31}T_1(y) + e_{32}T_2(y) + e_{33}T_3(y) = 0. \end{cases} \tag{5.9}$$

The determinant of coefficiency for (5.9) is $\Delta^2 \neq 0$. we find $T_1(y) = T_2(y) = T_3(y) = 0$ and that implies $y \in \operatorname{Im}L$. So, $\operatorname{Ker}Q \subset \operatorname{Im}L$. By the definitions of Q_1, Q_2, and Q_3 we cancalculate the following equations hold :

$$Q_1^2(y) = Q_1 y, \quad Q_1(Q_2(y).t) = 0, \quad Q_1(Q_3(y).t^2) = 0.$$

$$Q_2(Q_1(y)) = 0, \quad Q_2(Q_2(y).t) = Q_2 y, \quad Q_2(Q_3(y).t^2) = 0.$$

$$Q_3(Q_1(y)) = 0, \quad Q_3(Q_2(y).t) = 0, \quad Q_3(Q_3(y).t^2) = Q_3 y.$$

Thus,

$$\begin{aligned} Q^2 g &= Q_1(Q_1(y)) + Q_1(Q_2(y).t) + Q_1(Q_3(y).t^2) \\ &\quad + Q_2(Q_1(y)).t + Q_2(Q_2(y).t).t + Q_2(Q_3(y).t^2).t \\ &\quad + Q_3(Q_1(y)).t^2 + Q_3(Q_2(y).t).t^2 + Q_3(Q_3(y).t^2).t^2 \\ &= Q_1(y) + Q_2(y).t + Q_3(y).t^2 \\ &= Qg. \end{aligned}$$

That implies the operator Q is a projector.

Take $y \in Y$ in the form $y = (y - Qy) + Qy$. Then, $(y - Qy) \in \operatorname{Ker} Q = \operatorname{Im} L$ and $Qy \in \operatorname{Im} Q$. Thus $Y = \operatorname{Im} Q + \operatorname{Im} L$. And for any $y \in \operatorname{Im} Q \cap \operatorname{Im} L$, from $y \in \operatorname{Im} Q$, there exists constants $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ such that $y(t) = \sum_{k=1}^{3} \delta_k t^{k-1}$, from $y \in \operatorname{Im} L$, we obtain

$$\begin{cases} d_{11}\delta_1 + d_{12}\delta_2 + d_{13}\delta_3 = 0, \\ d_{21}\delta_1 + d_{22}\delta_2 + d_{23}\delta_3 = 0, \\ d_{31}\delta_1 + d_{32}\delta_2 + d_{33}\delta_3 = 0. \end{cases} \tag{5.10}$$

The determinant of coefficiency for (5.10) is $\Delta \neq 0$. Therefore (5.10) has an unique solution $\delta_1 = \delta_2 = \delta_3 = 0$, which implies $\operatorname{Im} Q \cap \operatorname{Im} L = 0$. Then, we have

$$Y = \operatorname{Im} Q \oplus \operatorname{Ker} Q = \operatorname{Im} Q \oplus \operatorname{Im} L. \tag{5.11}$$

Thus, $\dim \operatorname{Ker} L = 3 = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Ker} Q = \operatorname{codim} \operatorname{Im} L$, this means that L is a Fredholm operator of index zero.

Let a mapping $P\colon X \to X$ be defined by

$$Pu(t) = \sum_{k=1}^{3} \frac{u^{(k)}(0)}{k!} t^k. \tag{5.12}$$

We note that P is a linear continuous projector and $\operatorname{Im} P = \operatorname{Ker} L$. It follows from $u = (u - Pu) + Pu$ that $X = \operatorname{Ker} P + \operatorname{Ker} L$. By simple calculation, we obtain that $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$. Hence

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P. \tag{5.13}$$

Define $K_p\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ as follows :

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s.$$

Note that

$$(K_p L)u(t) = u(t), \quad \forall u \in \operatorname{dom} L \cap \operatorname{Ker} P \tag{5.14}$$

and for $y \in \operatorname{Im} L$, we find

$$(LK_p)y(t) = L(K_p y)(t) = y(t).$$

Thus, $K_p = (L\,|_{\operatorname{dom} L \cap \operatorname{Ker} P})^{-1}$. Again for each $y \in \operatorname{Im} L$, and from Lemmas 1.43, 1.34 and 1.35, we have

$$\|K_p y\|_X \leq \frac{4 + \Gamma(\alpha - 2)}{\mu \Gamma(\alpha - 2)} \|y\|_{L^1}.$$

□

With arguments similar to those of Lemma 4.4, we obtain the following lemma.

Lemma 5.4. *Suppose that Ω is an open bounded subset of X such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. *Then N is L-compact on $\overline{\Omega}$.*

Theorem 5.5. *Let f be a Carathéodory function, $\phi(t) \in C^1[0,1]$, $\mu = \min_{t\in[0;1]} \phi(t) > 0$. (H_1) and (H_2) hold. In addition, assume that the following conditions all hold.*

(H_3) *There exist functions non-negative* $\theta_i(t) \in Y$, $i = 0, \ldots, 5$ *such that*

$$|f(t, x_0, x_1, x_2, x_3, x_4)| \leq \sum_{i=0}^{4} \theta_i(t)|x_i| + \theta_5(t),$$

where

$$\Lambda = \frac{22 + \Gamma(\alpha - 2)}{\mu\Gamma(\alpha - 2)} \sum_{i=0}^{4} \|\theta_i\|_{L^1} < 1.$$

(H_4) *There exists a constant* $M > 0$ *such that for* $u \in \operatorname{dom} L \setminus \operatorname{Ker} L$, *if* $|u^{'}(t)| > M$ *or* $|u^{''}(t)| > M$ *or* $|u^{'''}(t)| > M$ *for all* $t \in [0;1]$, *then* $T_1(Nu) \neq 0$ *or* $T_2(Nu) \neq 0$ *or* $T_3(Nu) \neq 0$.

(H_5) *There exists a constant* $M^* > 0$ *such that for any* $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$, *if* $|\delta_1| > M^*, |\delta_2| > M^*, |\delta_3| > M^*$, *then either*

$$\sum_{i=1}^{3} T_i N\Big(\sum_{k=1}^{3} \delta_k t^k\Big) < 0,$$

or

$$\sum_{i=1}^{3} T_i N\Big(\sum_{k=1}^{3} \delta_k t^k\Big) > 0.$$

Then, the problem (5.1) *has at least one solution.*

Proof. Consider the set

$$\Omega_1 = \{u \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid Lu = \lambda Nu,\ \lambda \in [0;1]\}$$

Then for $u \in \Omega_1$, $Lu = \lambda Nu$, thus $\lambda \neq 0$, $Nu \in \operatorname{Im} L = \operatorname{Ker} Q \subset Y$, hence, $Q(Nu) = 0$ that is, $T_1(Nu) = T_2(Nu) = T_3(Nu) = 0$. From (H_4) we have that the exists $t_1, t_2, t_3 \in [0;1]$, such that, $|u'(t_1)| \leq M, |u''(t_2)| \leq M, |u'''(t_3)| \leq M$.

If $t_1 = t_2 = t_3 = 0$, we have that $|u'(0)| \leq M, |u''(0)| \leq M, |u'''(0)| \leq M$. Otherwise, if $\max\{t_1, t_2, t_3\} \neq 0$, by $Lu = \lambda Nu$, we obtain

$$u(t) = \sum_{k=1}^{3} \frac{u^{(k)}(0)}{k!} t^k + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s Nu(r)\, \mathrm{d}r\, \mathrm{d}s.$$

Then,

$$u'''(t) = u'''(0) + \frac{\lambda}{\Gamma(\alpha - 3)} \int_0^t \frac{(t-s)^{\alpha-4}}{\phi(s)} \int_0^s Nu(r)\, \mathrm{d}r\, \mathrm{d}s,$$

If $t_3 \neq 0$, we get

$$u'''(t_3) = u'''(0) + \frac{\lambda}{\Gamma(\alpha - 3)} \int_0^{t_3} \frac{(t_3-s)^{\alpha-4}}{\phi(s)} \int_0^s Nu(r)\, \mathrm{d}r\, \mathrm{d}s,$$

together with $|u'''(t_3)| \leq M$, we have

$$\begin{aligned} |u'''(0)| &\leq |u'''(t_3)| + \frac{1}{\Gamma(\alpha - 3)} \int_0^{t_3} \frac{(t_3-s)^{\alpha-4}}{\phi(s)} \int_0^s |Nu(r)|\, \mathrm{d}r\, \mathrm{d}s \\ &\leq M + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha - 2)}. \end{aligned} \tag{5.15}$$

If $t_2 \neq 0$, then

$$u''(t_2) = u''(0) + u'''(0)t_2 + \frac{\lambda}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s Nu(r)\,\mathrm{d}r\,\mathrm{d}s,$$

from (5.15) and $|u''(t_2)| \leq M$, we find

$$\begin{aligned} |u''(0)| &\leq |u''(t_2)| + |u'''(0)| + \frac{1}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s |Nu(r)|\,\mathrm{d}r\,\mathrm{d}s \\ &\leq 2M + \frac{2\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \end{aligned} \tag{5.16}$$

If $t_1 \neq 0$, thus

$$u'(t_1) = u'(0) + u''(0)t_1 + \frac{u'''(0)}{2}t_1^2 + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s Nu(r)\,\mathrm{d}r\,\mathrm{d}s,$$

according to (5.15), (5.16) and $|u'(t_1)| \leq M$, we get

$$\begin{aligned} |u'(0)| &\leq |u'(t_1)| + |u''(0)| + |u'''(0)| + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s |Nu(r)|\,\mathrm{d}r\,\mathrm{d}s \\ &\leq 4M + \frac{4\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \end{aligned} \tag{5.17}$$

Again for $u \in \Omega_1$, we get

$$\begin{aligned} \|Pu\|_X &= \sum_{i=0}^{3} \max_{t\in[0;1]} |(Pu)^{(i)}(t)| + \max_{t\in[0;1]} |{}^cD_{0^+}^{\alpha}(Pu)(t)| \\ &\leq 2|u'(0)| + 3|u''(0)| + 4|u'''(0)|. \end{aligned}$$

From (5.15), (5.16) and (5.17), we obtain

$$\|Pu\|_X \leq 18M + \frac{18\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{5.18}$$

Again for all $u \in \Omega_1$, we have $(Id-P)u \in \operatorname{dom} L \cap \operatorname{Ker} P$; thus by (5.6) and (5.14) , we find

$$\begin{aligned} \|(Id-P)u\|_X = \|K_pL(Id-P)u\|_X &\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|L(Id-P)u\|_{L^1} = \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Lu\|_{L^1} \\ &\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Nu\|_{L^1}. \end{aligned} \tag{5.19}$$

From (5.18) and (5.19), we obtain

$$\|u\|_X \leq \|Pu\|_X + \|(Id-P)u\|_X \leq 18M + \frac{22+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Nu\|_{L^1}. \tag{5.20}$$

On the other hand from (H_4), we have

$$\begin{aligned}\|Nu\|_{L^1} &= \int_0^1 |(Nu)(s)|\,\mathrm{d}s = \int_0^1 \left|f(t,u(t),u'(t),u''(t),u'''(t),{}^cD^{\alpha}_{0^+}u(t))\right|\mathrm{d}s\\ &\le \sum_{i=0}^{3}\int_0^1 |\theta_i(s)|\cdot|u^{(i)}(s)|\,\mathrm{d}s + \int_0^1 |\theta_4(s)|\cdot|{}^cD^{\alpha}_{0^+}u(s)|\,\mathrm{d}s + \int_0^1 |\theta_5(s)|\,\mathrm{d}s \\ &\le \|u\|_X \sum_{i=0}^{4}\|\theta_i\|_{L^1} + \|\theta_5\|_{L^1}.\end{aligned} \tag{5.21}$$

Therefore, (5.20) and (5.21), it yield

$$\|u\|_X \le \frac{18\mu\Gamma(\alpha-2)M + (22+\Gamma(\alpha-2))\|\theta_5\|_{L^1}}{\mu(1-\Lambda)\Gamma(\alpha-2)}.$$

So, Ω_1 is bounded. Let

$$\Omega_2 = \{u \in \operatorname{Ker} L \mid Nu \in \operatorname{Im} L\}$$

For $u \in \Omega_2$, then $u \in \operatorname{Ker} L = \left\{u \mid u(t) = \sum_{k=1}^{3}\delta_k t^k,\ \delta_1,\delta_2,\delta_3 \in \mathbb{R}\right\}$, and $Q(Nu) = 0$, that is, $T_1N\left(\sum_{k=1}^{3}\delta_k t^k\right) = T_2N\left(\sum_{k=1}^{3}\delta_k t^k\right) = T_3N\left(\sum_{k=1}^{3}\delta_k t^k\right) = 0$. From condition (H_5), we get $|\delta_1| \le M^*, |\delta_2| \le M^*, |\delta_3| \le M^*$. Hence, Ω_2 is bounded. Let

$$\Omega_3 = \{u \in \operatorname{Ker} L \mid -\lambda Ju + (1-\lambda)QNu = 0,\ \lambda \in [0;1]\}$$

if the first part of (H_5) hold.
Or we'll set

$$\Omega_3 = \{u \in \operatorname{Ker} L \mid +\lambda Ju + (1-\lambda)QNu = 0,\ \lambda \in [0;1]\}$$

if the second part of (H_5) hold.

Where $J\colon \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by

$$J\Big(\sum_{k=1}^{3}\delta_k t^k\Big) = \omega_1 + \omega_2.t + \omega_3.t^2, \quad \delta_1,\delta_2,\delta_3 \in \mathbb{R}. \tag{5.22}$$

where

$$\omega_1 = \frac{1}{\Delta}\sum_{i=1}^{3} e_{1i}|\delta_i|, \quad \omega_2 = \frac{1}{\Delta}\sum_{i=1}^{3} e_{2i}|\delta_i|, \quad \omega_3 = \frac{1}{\Delta}\sum_{i=1}^{3} e_{3i}|\delta_i|.$$

Without loss of generality, we assume that the first part of (H_5) hold.
In fact $u \in \Omega_3$, means that $u = \sum_{k=1}^{3}\delta_k t^k$ and $-\lambda Ju + (1-\lambda)QNu = 0$. Then we obtain

$$-\lambda J\Big(\sum_{k=1}^{3}\delta_k t^k\Big) + (1-\lambda)QN\Big(\sum_{k=1}^{3}\delta_k t^k\Big) = 0. \tag{5.23}$$

If $\lambda = 0$, then $|\delta_1| \le M^*, |\delta_2| \le M^*, |\delta_3| \le M^*$. If $\lambda = 1$, then

$$\begin{cases} e_{11}|\delta_1| + e_{12}|\delta_2| + e_{13}|\delta_3| = 0\\ e_{21}|\delta_1| + e_{22}|\delta_2| + e_{23}|\delta_3| = 0\\ e_{31}|\delta_1| + e_{32}|\delta_2| + e_{33}|\delta_3| = 0\end{cases} \tag{5.24}$$

The determinant of coefficiency for (5.24) is $\Delta^2 \neq 0$. Thus, (5.24) only have zero solutions, that is $\delta_1 = \delta_2 = \delta_3 = 0$.
Otherwise, if $\lambda \neq 0$ and $\lambda \neq 1$, again from (5.22), (5.23) becomes

$$\lambda(\omega_1 + \omega_2.t + \omega_3.t^2) = (1-\lambda)\left[Q_1 N(\sum_{k=1}^{3}\delta_k t^k) + Q_2 N(\sum_{k=1}^{3}\delta_k t^k).t + Q_3 N(\sum_{k=1}^{3}\delta_k t^k).t^2\right].$$

Hence,

$$\lambda\omega_i = (1-\lambda)Q_i N(\sum_{k=1}^{3}\delta_k t^k), \quad \text{for } i = 1,2,3.$$

Thus,

$$\lambda|\delta_i| = (1-\lambda)T_i N(\sum_{k=1}^{3}\delta_k t^k), \quad \text{for } i = 1,2,3.$$

Then, we get

$$\lambda\sum_{i=1}^{3}|\delta_i| = (1-\lambda)\sum_{i=1}^{3}T_i N(\sum_{k=1}^{3}\delta_k t^k) < 0.$$

By the first part of (H_5), we have $|\delta_1| \leq M^*, |\delta_2| \leq M^*, |\delta_3| \leq M^*$. Here, Ω_3 is bounded.

Now, we shall prove that all the conditions of Theorem 1.106 are satisfied. Let Ω to be a bounded open set of X containing $\bigcup_{i=1}^{3}\overline{\Omega}_i$. By Lemma 5.4, N is L-compact on $\overline{\Omega}$. Because Ω_1 and Ω_2 are bounded sets, then

(1) $Lu \neq \lambda Nu$ for each $(u,\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0;1)$;

(2) $Nu \notin \operatorname{Im} L$ for each $u \in \operatorname{Ker} L \cap \partial\Omega$.

At least we will prove that (3) of Theorem 1.106 is satisfied. Let

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu$$

Because Ω_3 is bounded, then

$$H(u,\lambda) \neq 0, \quad \forall u \in \operatorname{Ker} L \bigcap \partial\Omega.$$

Appealing to the homotopy property of the degree, we obtain

$$\begin{aligned}\deg\left(QN\mid_{\operatorname{Ker} L}, \Omega\bigcap \operatorname{Ker} L, 0\right) &= deg\left(H(\cdot,0), \Omega\bigcap \operatorname{Ker} L, 0\right)\\ &= \deg\left(H(\cdot,1), \Omega\bigcap \operatorname{Ker} L, 0\right)\\ &= \deg\left(\pm J, \Omega\bigcap \operatorname{Ker} L, 0\right) \neq 0.\end{aligned}$$

Then by Theorem 1.106, $Lu = Nu$ has at least one solution in $\operatorname{dom} L \bigcap \overline{\Omega}$, we conclude that the boundary value problem (5.1) has at least one solution in X. The proof is finished.

□

Remark 5.6. It is very important to note that the condition $\Delta \neq 0$ is not necessary since L still Fredholm even if this condition is dropped. Indeed the role of Q in Mawhin's theory is purely auxiliary and conditions like that usually arise from the authors of hundreds of paper choosing $\operatorname{Im} Q$ just simply being $\operatorname{Ker} L$. Avoiding such an assumption is just a matter of choosing Q differently, for more details see [71, 77, 78].

5.2.1 Example

To illustrate our main results, we will present an example.

Example 5.7. let us consider the following fractional boundary value problem

$$\begin{cases} (\phi(t)^c D_{0^+}^{\frac{7}{2}} u(t))' = f(t, u(t), u'(t), u''(t), u'''(t), {}^c D_{0^+}^{\frac{7}{2}} u(t)),\ t \in [0;1] \\ u(0) = 0, \quad {}^c D_{0^+}^{\alpha} u(0) = 0, \quad u'''(0) = -u'''\left(\frac{1}{6}\right) + 2u'''\left(\frac{1}{5}\right), \\ u''(0) = 4u''\left(\frac{1}{4}\right) - 3u''\left(\frac{1}{3}\right), u'(1) = u'\left(\frac{1}{4}\right) - 3u'\left(\frac{1}{2}\right) + 3u'\left(\frac{3}{4}\right). \end{cases} \tag{5.25}$$

where $\phi(t) = e^{-12t}$ and

$$100e^{12} f(t, x_0, x_1, x_2, x_3, x_4) = \frac{|x_3|}{1+(x_3)^2} + \cos x_4 (1 - \sin x_1)(1 - x_1) + \frac{2}{\pi} \arctan(x_0 x_4).$$

Corresponding to the problem (5.1), we have that $\alpha = \frac{7}{2}, l = 2, m = 2, n = 3, a_1 = -1, a_2 = 2, \xi_1 = \frac{1}{6}, \xi_2 = \frac{1}{5}, b_1 = 4, b_2 = -3, \eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{3}, c_1 = 1, c_2 = -3, c_3 = 3, \rho_1 = \frac{1}{4}, \rho_2 = \frac{1}{2}, \rho_3 = \frac{3}{4}, \mu = e^{-12}$. Then we get
$a_1 + a_2 = b_1 + b_2 = c_1 + c_2 + c_3 = 1, b_1\eta_1 + b_2\eta_2 = 0, c_1\rho_1 + c_2\rho_2 + c_3\rho_3 = c_1\rho_1^2 + c_2\rho_2^2 + c_3\rho_3^2 = 1$. Thus the condition (H_1) holds. Also we find

$$T_1(y) = -\int_0^{\frac{1}{6}} e^{12s} \left(\frac{1}{6} - s\right)^{-\frac{1}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s + 2\int_0^{\frac{1}{5}} e^{12s} \left(\frac{1}{5} - s\right)^{-\frac{1}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$$T_2(y) = 4\int_0^{\frac{1}{4}} e^{12s} \left(\frac{1}{4} - s\right)^{\frac{1}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - 3\int_0^{\frac{1}{3}} e^{12s} \left(\frac{1}{3} - s\right)^{\frac{1}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s,$$

$$\begin{aligned} T_3(y) = &\int_0^1 e^{12s}(1-s)^{\frac{3}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - \int_0^{\frac{1}{4}} e^{12s} \left(\frac{1}{4} - s\right)^{\frac{3}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s \\ &+ 3\int_0^{\frac{1}{2}} e^{12s} \left(\frac{1}{2} - s\right)^{\frac{3}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s - 3\int_0^{\frac{3}{4}} e^{12s} \left(\frac{3}{4} - s\right)^{\frac{3}{2}} \int_0^s y(r)\,\mathrm{d}r\,\mathrm{d}s. \end{aligned}$$

By calculations, we get

$$\Delta = \begin{vmatrix} \frac{1881}{1420} & \frac{207}{1669} & \frac{143}{9103} \\ -\frac{920}{1803} & -\frac{484}{6725} & -\frac{277}{20262} \\ \frac{15770}{51} & \frac{6489}{50} & \frac{5427}{74} \end{vmatrix} = -\frac{655}{539}.$$

Therefore, the condition (H_2) holds. On the other hand, we have

$$|f(t, x_0, x_1, x_2, x_3, x_4)| \le 0.01e^{-12}|x_3| + 0.05e^{-12}.$$

We can get that the condition (H_3) holds, where

$$\theta_0(t) = \theta_1(t) = \theta_2(t) = \theta_4(t) = 0,\ \theta_3(t) = 0.01e^{-12},\ \theta_5(t) = 0.05e^{-12},$$

and

$$\Lambda = \frac{838}{3245} < 1.$$

Let $M = 1$ and assume that $|u'''(t)| > 1$ holds for all $t \in [0;1]$, we obtain

$$\begin{aligned}
T_3(y) &> 0.01e^{-12}\int_0^1 e^{12s}(1-s)^{\frac{3}{2}} s\,\mathrm{d}s - 0.06e^{-12}\int_0^{\frac{1}{4}} e^{12s}\left(\frac{1}{4}-s\right)^{\frac{3}{2}} s\,\mathrm{d}s \\
&\quad + 0.03e^{-12}\int_0^{\frac{1}{2}} e^{12s}\left(\frac{1}{2}-s\right)^{\frac{3}{2}} s\,\mathrm{d}s - 0.18e^{-12}\int_0^{\frac{3}{4}} e^{12s}\left(\frac{3}{4}-s\right)^{\frac{3}{2}} s\,\mathrm{d}s. \\
&= \frac{43818}{2900}e^{-12} > 0
\end{aligned}$$

So, condition (H_4) is satisfied.

Let $M^* = 1$ and $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ be such that $|\delta_1| > 1$, $|\delta_2| > 1$, $|\delta_3| > 1$, we have

$$\begin{aligned}
& N(\delta_1 t + \delta_2 t^2 + \delta_3 t^3) \\
& = 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2} + 0.01e^{-12}\cos {}^cD_{0^+}^{\frac{7}{2}}(\delta_1 t + \delta_2 t^2 + \delta_3 t^3)(1-\sin(\delta_1 + 2\delta_2 t + 3\delta_3 t^2)) \\
& \times (1-\sin(2\delta_2 + 6\delta_3 t)) + \frac{0.02e^{-12}}{\pi}\arctan((\delta_1 t + \delta_2 t^2 + \delta_3 t^3)^c D_{0^+}^{\frac{7}{2}}(\delta_1 t + \delta_2 t^2 + \delta_3 t^3)) \\
& = 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}.
\end{aligned}$$

Hence,

$$T_i N(\sum_{k=1}^{3} \delta_k t^k) = 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2} d_{1i}, \quad \text{for } i = 1,2,3.$$

Thus,

$$\sum_{i=1}^{3} T_i N(\sum_{k=1}^{3} \delta_k t^k) = 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}(d_{11}+d_{12}+d_{13}) > 0.$$

So, (H_5) hold. Then, all the assumptions of Theorem 5.5 hold. Thus, the problem (5.25) has at last one solution.

Conclusion and perspectives

The proposed applications show that coincidence degree theory and the techniques of measures of non-compactness play an important role in the study of these problems and this concerns the existence of solutions, but not uniqueness.

In the future, we intend to study some questions related to the existence and uniqueness of solutions of some boundary problems on bounded or unbounded domains in the case of resonance and non-resonance. For this, the application of certain other methods can be associated with the techniques measure of noncompactness, degree of nondensifiability ([57, 58]), monotone iterative technique ([82]) and the coincidence degre.

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