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About fractional orders derivatives and its applications

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Dedication

We dedicate this humble work to the honorable parents, may God preserve them and grant them .success in their obedience and approval

And to our brothers and everyone who has merit over us, and we also dedicate this work to our esteemed friends, especially those who were a strong support for us after God the Mighty and Sublime, and especially our brothers: Al-Aidi Ibrahim and Muhammad Weld Allal from the sisterly Western Sahara, and friend of the study Hijazi AbdulMalik, and many others that are too .numerous to place. to remember them

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Abstract

We have focused within this Thesis on some new problems involving fractional differential equations and coupled systems involving Hilfer fractional-order derivatives, which are applied in real life. In particular, we firstly started to review some essential facts from fractional calculus, abstract differential equations, fixed-point techniques that are used to obtain our main results. For this purpose, the technique used is to reduce the study of our problem to the research of a fixed point of an integral operator properly constructed. We have also provided a relevant example to each of our considered problems to show the validity of conditions and justify the efficiency of our established results. Finally, we ended by a conclusion that summarized our strict scientific novelties and contributions to this thesis, as well as, some possible open problems to be investigated in the future as new directions.
Key words and phrases : Fractional differential equation, Coupled fractional differential system, Caputo fractional derivative, Riemann-Lioille fractional derivative, Hilfer fractional derivative, boundary value problems, fixed-point theorem.
AMS Subject Classification : 26A33, 34A08, 34B15.

Résumé

Nous nous sommes concentrés dans cette thèse sur de nouveaux problèmes impliquant des équations différentielles fractionnaires et des systèmes couplés impliquant des dérivées d'ordre fractionnaire de Hilfer, qui sont appliqués dans la vie réelle. En particulier, nous avons d'abord commencé à passer en revue quelques faits essentiels du calcul fractionnaire, des équations différentielles abstraites, des techniques de points fixe qui sont utilisées pour obtenir nos principaux résultats. Pour cela, la technique utilisée est de réduire l'étude de notre problème à la recherche d'un point fixe d'un opérateur intégral correctement construit. Nous avons également fourni un exemple pertinent à chacun de nos problèmes considérés pour montrer la validité des conditions et justifier l'efficacité de nos résultats établis. Enfin, nous avons terminé par une conclusion résumant nos nouveautés scientifiques strictes et nos contributions à cette thèse, ainsi que quelques problèmes ouverts possibles à étudier à l'avenir en tant que nouvelles directions.

الملخص

لقد ركزنا في هذه الرسالة على بعض المشكلات الجديدة التي تتضمن معادلات تفاضلية كسرية وأنظمة مقترنة تتضمن مشتقات ذات الترتيب الكسري هيلفر ، والتي يتم تطبيقها في الحياة الواقعية. على وجه الخصوص ، بدأنا أولاً في مراجعة بعض الحقائق الأساسية من حساب التفاضل والتكامل الكسري والمعادلات التفاضلية المجردة وتقنيات النقطة الثابتة المستخدمة للحصول على نتائجنا الرئيسية. لهذا الغرض ، فإن التقنية المستخدمة هي تحويل دراسة مشكلتنا إلى البحث عن نقطة ثابتة لمشغل متكامل تم إنشاؤه بشكل صحيح. لقد قدمنا أيضًا مثالًا ذا صلة بكل مشكلة من مشاكلنا المدروسة لإظهار صحة الشروط وتبرير كفاءة نتائجنا المحددة. أخيرًا ، انتهينا بخاتمة لخصت ابتكاراتنا العلمية الصارمة وإسهاماتنا في هذه الأطروحة ، بالإضافة إلى بعض المشكلات المفتوحة المحتملة التي سيتم التحقيق فيها في المستقبل كتوجهات جديدة.

List of symbols

We use the following notations throughout this thesis

Acronyms

- FC : Fractional calculus.
- FD : Fractional derivative.
- FDE : Fractional differential equation.
- FI : Fractional integral.
- IVP : Initial value problem.
- BVP : Boundary value problem.

Notation

- $\mathbb{N}$: Set of natural numbers.
- $\mathbb{R}$: Set of real numbers.
- $\mathbb{R}^n$: Space of n-dimensional real vectors.
- $\in$: belongs to.
- inf : Inferior.
- sup : Supremum.
- max : Maximum.
- $n!$: Factorial $(n), n \in \mathbb{N}$: The product of all the integers from1 to n.
- $\Gamma(\cdot)$: Gamma function.
- $\mathrm{B}(\cdot,\cdot)$: Beta function.
- $I_{a^+}^{\alpha}$: The Riemann-Liouville fractional integral of order $\alpha > 0$.
- ${}^{RL}D_{a^+}^{\alpha}$: The Riemann-Liouville fractional derivative of orde $\alpha > 0$.
- ${}^{C}D_{a^+}^{\alpha}$: The Caputo fractional derivative of orde $\alpha > 0$.
- $D_{a^+}^{\alpha,\beta}$: The Hilfer fractional derivative of orde $\alpha > 0$ and type β.

- $C(J,\mathbb{R})$: Space of continuous functions on J.
- $C^n(J,\mathbb{R})$: Space of n time continuously differentiable functions on J.
- $C_\gamma(J,\mathbb{R})$: Weighted space of continuous functions on J.
- $AC(J,\mathbb{R})$: Space of absolutely continuous functions on J.
- $L^1(J,\mathbb{R})$: space of Lebesgue integrable functions on J.
- $L^p(J,\mathbb{R})$: space of measurable functions u with $|u|^p$ belongs to $L^1(J,\mathbb{R})$.
- $L^\infty(J,\mathbb{R})$: space of functions u that are essentially bounded on J .

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Introduction

When we introduce the notion of derivative, we quickly realize that we can apply the concept of derivative to the derivative function itself, and by the same to introduce the second derivative. Then the successive derivatives of integer order. Integration, the inverse operator of the derivative, can optionally be like a derivative of order "minus one". One can also ask whether these derivatives of successive order have an equivalent of fractional order.

Fractional derivative theory is a subject almost as old as classical calculus as we know it today, its origins going back to the end of the 17^{th} century [55], the time when Newton and Leibniz developed the foundations of differential and integral calculus. In particular, Leibniz introduced the symbol $\frac{d^n f}{dt^n}$ to designate the $n^{\varepsilon me}$ derived from a function f. When he announced in a letter to the Hospital (apparently with the implicit assumption that $n \in \mathbb{N}$), the Hospital replied : What does $\frac{d^n f}{dt^n}$ si $n = \frac{1}{2}$. This letter from the Hospital, written in 1695, is now accepted as the first incident of what we call fractional derivation, and the fact that the Hospital asked specifically for $n = \frac{1}{2}$, i.e. a fraction (rational number) actually gave rise to the name of this part of mathematics.

One could think that this search for fractional derivative is a question of pure mathematics without interest for the engineer, however a simple example of fluid mechanics shows how the derivative of order one-half appears quite naturally when one wants to explain a flow of heat exiting laterally from a fluid flow as a function of the time evolution of the internal source.

A particular interest in fractional derivation is linked to the mechanical modeling of rubbers and cahoutchous, in short all kinds of materials which retain the memory of past deformations and whose behavior is said to be viscoelastic. Indeed, the fractional derivation is naturally introduced there, in other words the non-integer derivatives have a memory effect that they share with several materials such as viscoelastic or polymeric materials. This fact is also one of the reasons why fractional calculus has recently been of great interest. The use of the memory effect of fractional derivatives in the construction of simple material models comes with a high cost with regard to numerical resolution. While using a discretization algorithm for non-integer derivatives one has to take into account its non-local structure which generally means high information storage and high complexity of the algorithm.

Many attempts to solve equations involving different types of non-integer order operators can be found in the literature. A list of mathematicians who made important contributions to fractional calculus up to the middle of the 20^{th} century includes : PS. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1873), B. Riemann (1847), H. Holmgren (1865-67), A.K. Grunwald (1867-1872), A.V. Letnikov (1868 -1872), H. Laurent

(1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892-1912) S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. L'evy (1923), A. Marchaud (1927), H.T. Davis (1924-1936), A. Zygmund (1935-1945) E.R. Amour (1938-1996), A. Erd'elyi (1939- 1965), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949), ...ect.

Different approaches have been used for this notion of derivation : - The limit of rate of increase of a function is generalized in the form of Grunwald-Letnikov, very useful numerically. - The integration, inverse operator, via the Liouville integral formula, leads to the Riemann-Liouville and Caputo formulas. - Finally the Fourier and Laplace transformations associate the fractional derivation with a multiplication by $(i\omega)^{\alpha}$ or p^{a} with non-integer α.

But these different definitions have for a long time seemed not always to give the same results. This apparent inconsistency could be dissipated in the new framework proposed by Laurent Schwartz's theory of distributions [60].

However, this theory can be considered as a new subject, for only a little more than thirty years it has been the subject of specialized conferences. For the first lecture, credit goes to B. Ross who organized the first lecture on fractional calculus and its applications at the University of New Haven in June 1974, and he edited the proceedings. For the first monograph credit is given to K.B. Oldham and J. Spanier, who published a book devoted to fractional calculus in 1974 after a joint collaboration, begun in 1968.

The study of fractional problems is topical and several methods are applied to solve these problems. However, methods based on the fixed point principle play a major role. Fixed point theorems are the basic mathematical tools, showing the existence of solutions in various kinds of equations. Fixed point theory is central to nonlinear analysis since it provides the tools to have existence theorems in many different nonlinear problems.

The development of fixed point theory, which is the cardinal branch of nonlinear analysis has given great effects on the advancement of nonlinear analysis, regarded as an autonomous branch of mathematics, nonlinear analysis has was developed in the 1950s by mathematicians like Felix Browder as a combination of functional analysis and variational analysis.

This method is associated with the names of famous mathematicians such as Cauchy, Liouville, Lipschitz and above all, Picard. In fact, the precursors of approximate fixed point theory are explicit in the work of Picard. However, it is the Polish mathematician Stefan Banach, who is credited with the placement of an abstract idea. The principle of contracting application is one of the few constructive theorems of mathematical analysis. It is a tool of great importance given the extent of its a priori fields of application, in the study of nonlinear equations which play a crucial role in both mathematics and applied sciences. The principle is the theorem of the fixed point of Banach or that of Picard which ensures the existence of a unique fixed point for a contracting application of a complete metric space in itself. The fixed point is the limit of an iterative process defined from an image repetition by this contracting application of an arbitrary initial point in this space. This concept was first proven by Banach in 1922 and then developed by several mathematicians, including Brouwer and Schauder in 1930 and Krasnoselskii in 1955. Schauder's fixed point theorem, which is by the way, an extension of that of Brouwer in infinite dimension is more topological than that of Banach and affirms that a continuous map on a compact convex one admits a fixed point which is not necessarily unique. It is therefore not necessary to establish increases on the function but simply its continuity. Nevertheless, the setting in motion of variational methods has focused these last decades, for the most part the interest of mathematicians to the detriment of topological methods.

Indeed, these variational methods draw their effectiveness from their orientation towards applications. Their major asset is the fact that they involve generalized functions and reflexive Sobolev spaces which constitute a chain of connection between the weak solutions and the classical solutions. However, there are situations where variational methods prove to be laborious or even ineffective. And here again, topological methods will be essential, hence their fundamental importance.

It can be noted here that most of the work on fractional calculus is devoted to the solvability of boundary value problems generated by linear fractional differential equations at the base of the special functions [39, 53, 58]. Recently, other results dealing with the existence, uniqueness and multiplicity of solutions or positive solutions of nonlinear fractional problems by the use of nonlinear analysis techniques such as fixed point theorems have appeared.

On the other hand, in this thesis, we are interested in talking about some essential types on derivatives of fractional order. And as a second goal is to contribute to the study of the existence and uniqueness of solutions of some differential equations of fractional order. The technique used is to reduce the study of our problem to the search for a fixed point of a suitably constructed integral operator.

The boundary value problems (BVPs) acquainted by FDE have been broadly concentrated throughout the most recent years. Especially, the investigation of solutions of FDEs is the key and critical subject of applied mathematics research. Many interesting and fascinating results have been considered with respect to the existence, uniqueness, and stability of solutions via some fixed point theorems [13, 14, 15, 16, 17, 18, 19, 20, 21]. Before we start off this study, we need to make it very clear that we are only going to scratch the surface of the topic of boundary value problems. There is enough material in the topic of boundary value problems that we could devote a whole class to it.

In the context of this study, we have organized this thesis as follows :

In **Chapter** 1, we give a Casual but technically precise overview of definitions, notations, lemmas and notions of fractional calculus, fixed point theorems that are used throughout this thesis.

Chapter 2, is reserved to expose results of existence and uniqueness of solutions concerning a boundary problem for a fractional differential equation of the Hilfer type. Our results are based on some standard fixed point theorem (Banach, Schaefer and Leary-Schauder). More specifically, in Section 2.2 we are interested in the existence and uniqueness of solutions for the nonlinear Langevin and Sturm-Liouville fractional differential equation

$$\begin{cases} {}^{H}\mathscr{D}_{a^+}^{\alpha_1,\beta}\left[p(t)\,{}^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta}-q(t)\right]u(t)=\mathscr{F}\left(t,u(t)\right),t\in \mathrm{J}=[a,b] \\ \\ u(a)=0, \quad p(b)\,{}^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta}u(t)-q(b)u(b)=0, \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta. \end{cases} \tag{1}$$

where ${}^{H}\mathscr{D}_{a^+}^{\rho,\beta;\psi}$ is the Hilfer FD of order $\rho\in(0,1)$ with $\rho\in\{\alpha_1,\alpha_2\}$ and type $\beta\in[0,1]$. Also, $p\in C(\mathrm{J},R\backslash\{0\})$, $q\in C(\mathrm{J},R)$ and $\mathscr{F}:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ is continuous.

In Section 2.3, we give similar result to the following boundary value problem defined by system of generalized Sturm-Liouville and Langevin Hilfer fractional differential equations :

$$\begin{cases} {}^{H}\mathscr{D}_{a^+}^{\alpha_1,\beta_1;\psi}\left[p_1(t)^H\mathscr{D}_{a^+}^{\alpha_2,\beta_1} - q_1(t)\right]u(t) = \mathscr{F}(t,u(t),v(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ,t \in \mathrm{J} := [a,b], \\ {}^{H}\mathscr{D}_{a^+}^{\alpha_3,\beta_2}\left[p_2(t)^H\mathscr{D}_{a^+}^{\alpha_4,\beta_2} - q_2(t)\right]v(t) = \mathscr{G}(t,u(t),v(t)), \end{cases} \tag{2}$$

supplemented with the boundary conditions of the form :

$$\begin{cases} u(a) = 0, \quad p_1(b)^H\mathscr{D}_{a^+}^{\alpha_2,\beta}u(t) - q_1(b)u(b) = 0, \quad ,\gamma = (\alpha_1+\alpha_2)(1-\beta_1)+\beta_1, \\ v(a) = 0, \quad p_2(b)^H\mathscr{D}_{a^+}^{\alpha_4,\beta}v(t) - q_2(b)v(b) = 0, \quad \gamma = (\alpha_3+\alpha_4)(1-\beta_2)+\beta_2. \end{cases} \tag{3}$$

where ${}^{H}\mathscr{D}_{a^+}{}_{a^+}^{\rho,\beta}$ is the ψ-Hilfer FD of order $\rho \in (0,1)$ with $\rho \in \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ and type $\beta_i \in [0,1]$, $i = 1,2$. Also, $p_i \in C(\mathrm{J},R\backslash\{0\})$, $q_i \in C(\mathrm{J},R)$, $i = 1,2$ and $\mathscr{F},\mathscr{G} : \mathrm{J}\times\mathbb{R} \to \mathbb{R}$ are continuous.

Finally, to illustrate the theoretical results, an example is given at the end of each section.

In **Chapter** 3, Conclusion and perspectives : We end with a general conclusion summarizing the main results achieved with an outlook for upcoming researches.

Chapitre 1

Preliminaries and Basic concepts

In this chapter, we present some basic theories which concern useful functions which are used in the other chapters. We give here the definitions of the Gamma functions. These functions play a very important role in the theory of problems with fractional derivative and this chapter will be devoted to elementary definitions for integrals and fractional derivatives in the sense of Riemann-Liouville and Caputo, as well as some definitions of functional spaces and theorem of fixed point.

1.1 Functional spaces

Let $\mathbb{E}$ Banach space endowed with the norm $\|.\|$ and let $J := [a,b]$ the compact interval of $\mathbb{R}$. we present the following functional spaces :

1.1.1 Space of Continuous Functions

Definition 1.1. Let $C(J,\mathbb{E})$ is the Banach space of continuous functions $u : J \longrightarrow \mathbb{E}$ have the valued in $\mathbb{E}$, equipped with the norm

$$\|u\|_\infty = \sup_{t\in J} |u(t)|.$$

Analogoustly, $C^n(J,\mathbb{E})$ the Banach space of functions $u : J \longrightarrow \mathbb{E}$ where u is n time continuously differentiable on J.

1.1.2 Spaces of Integrable Functions L^p

Let $L^1(J,\mathbb{E})$ be the Banach space of measurable functions $u : J \to \mathbb{E}$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_J |x(t)|\, dt.$$

and we denote $L^p(J,\mathbb{E})$ the space of Lebesgue integrable functions on J where $|u|^p$ belongs to $L^1(J,\mathbb{E})$, endowed with the norm

$$\|u\|_{L^p} = \left[\int_0^T |u(t)|^p \mathrm{dt}\right]^{\frac{1}{p}}.$$

In particular, if $p=\infty$, $L^\infty(J,\mathbb{E})$ is the space of all functions u that are essentially bounded on J with essential supremum

$$\|u\|_{L^\infty}=\operatorname*{ess\,sup}_{t\in J}|u(t)|=\inf\{c\geq 0:|u(t)|\leq c \text{ for a.e.} t\}.$$

1.1.3 Spaces of Absolutely Continuous Functions

Definition 1.2. A function $u\colon J\to\mathbb{E}$ is said absolutly continuous on J if for all $\varepsilon>0$ there exists a number $\delta>0$ such that ; for all fnite partition $[a_i,b_i]_{i=1}^n$ in J then $\sum_{k=1}^n(b_k-a_k)<\delta$ implies that $\sum_{k=1}^n|f(b_k)-f(a_k)|<\varepsilon$

We denote by $AC(J,\mathbb{E})$ (or $AC^1(J,\mathbb{E})$) the space of all absolutely continuous functions defined on J. It is known that $AC(J,\mathbb{E})$ coincides with the space of primitives of Lebesgue summable functions :

$$u\in AC(J,\mathbb{E})\Leftrightarrow u(t)=c+\int_0^t\psi(s)\mathrm{d}s,\quad \psi\in L^1(J,\mathbb{R}),\tag{1.1}$$

and therefore an absolutely continuous function u has a summable derivative $u'(t)=\psi(t)$ almost everywhere on J. Thus (1.1) yields

$$u'(t)=\psi(t)\text{ and }\ c=u(0).$$

Definition 1.3. For $n\in\mathbb{N}^*$ we denote by $AC^n(J,\mathbb{E})$ the space of functions $u:J\longrightarrow\mathbb{E}$ which have continuous derivatives up to order $n-1$ on J such that $u^{(n-1)}$ belongs to $AC(J,\mathbb{E})$:

$$\begin{aligned}AC^n(J,\mathbb{E})&=\left\{u\in C^{n-1}(J,\mathbb{E}):u^{(n-1)}\in AC(J,\mathbb{E})\right\}\\&=\left\{u\in C^{n-1}(J,\mathbb{R}):u^{(n)}\in L^1(J,\mathbb{E})\right\}.\end{aligned}$$

The space $AC^n(J,\mathbb{E})$ consists of those and only those functions u which can be represented in the form

$$u(t)=\frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}\psi(s)\mathrm{d}s+\sum_{k=0}^{n-1}c_kt^k,\tag{1.2}$$

where $\psi\in L^1(J,\mathbb{R}),c_j\ (k=1,\ldots,n-1)\in\mathbb{R}$.
It follows from (1.2) that

$$\psi(t)=u^{(n)}(t)\text{ and }c_k=\frac{u^{(k)}(0)}{k!},\ (k=1,\ldots,n-1).$$

and

$$AC^n_\delta([a,b],E)=\left\{h:[a,b]\to\mathbb{R}:\delta^{n-1}h(t)\in AC([a,b],E)\right\}.$$

where $\delta=t\frac{d}{dt}$ is the Hadamard derivative.

1.1.4 Weighted spaces of Continuous Functions $C_{1-\gamma}(J,\mathbb{E})$

Definition 1.4. [?] Let J be a finite interval and $0\leq\gamma<1$, we introduce the weighted space $C_\gamma(J,\mathbb{E})$ of continuous functions f on J

$$C_\gamma(J,\mathbb{E})=\{f:(0,T]\to\mathbb{E}:(t-a)^\gamma f(t)\in C(J,\mathbb{E})\}.$$

In the space $C_\gamma(J,\mathbb{E})$, we define the norm

$$\|f\|_{C_\gamma} = \|(t-a)^\gamma f(t)\|_C$$

In particular,

$$C_0(J,\mathbb{E}) = C(J,\mathbb{E}).$$

Definition 1.5. [30, 34] Let $0<\alpha<1$, $0\leq\beta\leq 1$, the weighted space $C_\gamma^{\alpha,\beta}(J,\mathbb{E})$ is defined by

$$C_\gamma^{\alpha,\beta}(J,\mathbb{E}) = \{f : J\to\mathbb{R} : D_{0^+}^{\alpha,\beta} f \in C_\gamma(J,\mathbb{E})\}, \gamma=\alpha+\beta-\alpha\beta$$

and

$$C_\gamma^1(J,\mathbb{E}) = \{f : J\to\mathbb{E} : f'\in C_\gamma(J,\mathbb{E})\}, \gamma=\alpha+\beta-\alpha\beta$$

with the norm

$$\|f\|_{C_\gamma^1} = \|f\|_C + \|f'\|_{C_\gamma}, \tag{1.3}$$

Clearly,

$$D_{0^+}^{\alpha,\beta} f = I_{0^+}^{\beta(1-\alpha)} D_{0^+}^{\gamma} f$$

and

$$C_\gamma^\gamma(J,\mathbb{E}) \subset C_\gamma^{\alpha,\beta}(J,\mathbb{E}), \gamma=\alpha+\beta-\alpha\beta, 0<\alpha<1, 0\leq\beta\leq 1$$

, See [30, 34].

Moreover, $C_\gamma(J,\mathbb{E})$ is complete metric space of all continuous functions mapping J into $\mathbb{E}$ with the metric d defined by

$$d(u_1,u_2) = \|u_1-u_2\|_{C_\gamma(J,E)} := \max_{t\in J} |(t-a)^\gamma[u_1(t)-u_2(t)]|$$

For details see [30, 34].

1.2 Basic Results From Fractional Calculus Theory

There are some definitions in fractional calculus which are very widely used and have importance in proving various results of fractional calculus. In this section, we recall some definitions of fractional integral and fractional differential operators that include all we use throughout this thesis.

1.2.1 Fractional Integrals

For the convenience of the reader, we can use the following notations which will be needed later.

Definition 1.6.

1. Denoted by D, we mean the operator that maps a differentiable function onto its derivative, i.e., $Df(t) := f'(t)$.
2. Denoted by $I_a f$, we mean the operator that maps a function f, supposed to be Lebesgue integrable on the compact interval $[a,b]$, onto its primitive centered at a, i.e.,

$$I_a f(t) = \int_a^t f(s)\mathrm{ds}, \text{ for } a\leq t\leq b.$$

3. For $n \in \mathbb{N}$, we use the symbols D^n and I_a^n to indicate the n-fold iterates of D and I_a, respectively, that is, we put $D^1 := D, I_a^1 := I_a$, and $D^n := DD^{n-1}$ and $I_a^n := I_a I_a^{n-1}$, for $n \geq 2$.

A successive integration of $f(t)$ for n-times shows that

$$I_a^n f(t) = \int_a^t \mathrm{dt}_1 \int_a^{t_1} \mathrm{dt}_2 \int_a^{t_2} \mathrm{dt}_3 \cdots \int_a^{t_{n-1}} f(t_n) \mathrm{dt}_\mathrm{n}, \text{ for } a \leq t.$$

Cauchy provided a closed-form formula for n successive integrations :

$$I_a^n f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s) \mathrm{ds}, \quad \text{for } a \leq t. \tag{1.4}$$

Focusing on the formula (1.4) we see it made an inspiration to define the Riemann-Liouville fractional integral, we generalize this formula by letting n take values other than the non-negative integers and noting at the same time that the factorial function is a special case of the Gamma function $\Gamma(\cdot)$.

Definition 1.7 ([39]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b])$ is defined by

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{ds}, \quad (t > a, \ \alpha > 0).$$

Moreover, for $\alpha = 0$, we set $I_{a^+}^0 f := f$. It is obvious that the Riemann-Liouville fractional integral coincides with the classical definition of I_a^n in the case $n \in \mathbb{N}$.

Example 1.8. The Riemann-Liouville fractional integral of the power function $(t-a)^{\beta}, \alpha > 0, \beta > -1$. By definition,

$$I_{a^+}^{\alpha} (t-a)^{\beta} = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (s-a)^{\beta} \mathrm{ds}.$$

Using the change of variables $s = a + (t-a)\tau, \quad \tau \in [0,1]$, we get,

$$\begin{aligned} I_{a^+}^{\alpha} (t-a)^{\beta} &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} \tau^{\beta} \, d\tau \\ &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} B(\alpha, \beta+1) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta}. \end{aligned}$$

For $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$, the relation becomes :

$$I_a^{\frac{1}{2}} (t-a)^{\frac{3}{2}} = \frac{\Gamma(5/2)}{\Gamma(3)} (t-a)^2 = \frac{3\sqrt{\pi}}{8} (t-a)^2.$$

Lemma 1.9 ([39]). *The following basic properties of the Riemann-Liouville integrals hold :*

1. *The integral operator $I_{a^+}^{\alpha}$ is linear ;*

2. *The semigroup property of the fractional integration operator $I_{a^+}^{\alpha}$ is given by the following result*
$$I_{a^+}^{\alpha}(I_{a^+}^{\beta}f(t)) = I_{a^+}^{\alpha+\beta}f(t), \quad \alpha,\beta > 0,$$
holds at every point if $f \in C([a,b])$ and holds almost everywhere if $f \in L^1([a,b])$,

3. *Commutativity*
$$I_{a^+}^{\alpha}(I_{a^+}^{\beta}f(t)) = I_{a^+}^{\beta}(I_{a^+}^{\alpha}f(t)), \quad \alpha,\beta > 0;$$

4. *The fractional integration operator $I_{a^+}^{\alpha}$ is bounded in $L^p[a,b]$ $(1 \le p \le \infty)$;*
$$\|I_{a^+}^{\alpha}f\|_{L^p} \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p}.$$

Proof. Let $\alpha > 0, \beta > 0$ et $f \in C([a,b))$
the proof is obtained by direct calculation using the Beta function. Indeed :

1. proposal.1 : Let f and g be two continuous functions and (A,B) two real numbers, we have :
$$\begin{aligned}
I_{a^+}^{\alpha}[Af(t)+Bg(t)] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}[Af(s)+Bg(s)]ds \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t A(t-s)^{\alpha-1}f(s)+B(t-s)^{\alpha-1}g(s)ds \\
&= \frac{1}{\Gamma(\alpha)}[A\int_a^t (t-s)^{\alpha-1}f(s)ds+B\int_a^t (t-s)^{\alpha-1}g(s)ds] \\
&= A\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds+B\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)ds \\
&= AI_a^{\alpha}f(t)+BI_a^{\alpha}g(t)
\end{aligned}$$

2. proposal.2 and 3 :
$$\begin{aligned}
I_{a^+}^{\alpha}I_{a^+}^{\beta}f(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}I_{a^+}^{\beta}f(t)dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^t (x-t)^{\alpha-1}(t-s)^{\beta-1}f(s)dsdt \qquad (1.5)\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x f(s)\int_s^x (x-t)^{\alpha-1}(t-s)^{\beta-1}dtds.
\end{aligned}$$
We take $t-s = u(x-s)$,then $dt = (x-s)du$ and
$$\begin{aligned}
I_{a^+}^{\alpha}I_{a^+}^{\beta}f(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x f(s)\int_0^1 (1-u)^{\alpha-1}(x-s)^{\alpha-1}u^{\beta-1}(x-s)^{\beta-1}duds, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-s)^{\alpha+\beta-1}f(s)ds\int_0^1 u^{\beta-1}(1-u)^{\alpha-1}, \\
&= \frac{\mathrm{B}(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-s)^{\alpha+\beta-1}f(s)ds.
\end{aligned}$$

By replacing the last formula in (2.24), we have :

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^x (x-s)^{\alpha+\beta-1} f(s) ds = \mathrm{I}^{\alpha+\beta} f(x).$$

Hence. Using the above property, we have

$$I_{a^+}^{\alpha} \left(I_{a^+}^{\beta} f \right)(t) = I_{a^+}^{\alpha+\beta} f(t) = I_{a^+}^{\beta+\alpha} f(t) = I_{a^+}^{\beta} \left(I_{+a}^{\alpha} f \right)(t)$$

3. proposal.4 : By introducing the definition 1.7 then using Fubini's theorem, we find :

$$\begin{aligned}
\int_a^b \left| \left(I_a^{\alpha} f \right)(x) \right| dx &\leq \frac{1}{\Gamma(\alpha)} \int_a^b \int_a^x (x-t)^{\alpha-1} |f(t)| dt dx \\
&\leq \frac{1}{\Gamma(\alpha)} \int_a^b |f(t)| \int_t^b (x-t)^{\alpha-1} dx dt \\
&\leq \frac{1}{\Gamma(\alpha+1)} \int_a^b |f(t)| (b-t)^{\alpha} dt \\
&\leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \int_a^b |f(t)| dt
\end{aligned}$$

□

1.2.2 Fractional Derivatives

Unlike classical Newtonian derivatives, the fractional derivative is defined by an integral part. whereas the derivative of the fractional-order are generalizations of the integer of the integers. There are several mathematical definitions for deriving fractional-order. We will introduce the five famous methods of the fractional derivative : fractional derivative of Riemann-Liouville (1847), Caputo (1967) and finally, the Hilfer method (2000).

The Riemann-Liouville fractional derivative

Definition 1.10 ([39, 53])**.** The Riemann-Liouville fractional derivative of order α of a function $f \in L^1([a,b])$ is defined by

$${}^{RL}D_{a^+}^{\alpha} f(t) = D^n I_{a^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{\mathrm{d}t^n} \int_a^t (t-s)^{n-\alpha-1} f(s) \mathrm{ds},\ a < t < b,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Example 1.11. Riemann-Liouville fractional derivative the power function $(t-a)^{\beta}, \alpha >$

$0, \beta > -1$

$$\begin{aligned}
{}^{RL}D^{\alpha}_{a^+}(t-a)^{\beta} = D^n I^{n-\alpha}_{a^+}(t-a)^{\beta} &= \frac{d^n}{\mathrm{d}t^n}\left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+n-\alpha)}(t-a)^{\beta+n-\alpha}\right] \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+n-\alpha)}\frac{d^n}{\mathrm{d}t^n}(t-a)^{\beta+n-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+n-\alpha)}\frac{\Gamma(\beta+n-\alpha+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.
\end{aligned}$$

If we take $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$, then

$$ {}^{RL}D^{\frac{1}{2}}_{a}(t-a)^{\frac{3}{2}} = \frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{2}+1\right)}(t-a)^{\frac{3}{2}-\frac{1}{2}} = \frac{3\sqrt{\pi}}{4}(t-a)$$

Remark 1.12. If we let $\beta = 0$ in the previous example, we see that the Riemann-Liouville fractional derivative of a constant is not 0. In fact,

$$ {}^{RL}D^{\alpha}_{a^+}1(t) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Remark 1.13. On the other hand, for $j = 1, 2, \cdots, [\alpha]+1$,

$$ {}^{RL}D^{\alpha}_{a^+}(t-a)^{\alpha-j}(t) = 0.$$

We could say that $(t-a)^{\alpha-j}$ plays the same role in Riemann-Liouville fractional differentiation as a constant does in classical integer-ordered differentiation.

As a result, we have the following fact :

Lemma 1.14 ([39, 53])**.** $\alpha > 0$, *and* $n = [\alpha]+1$ *then*

$$ {}^{RL}D^{\alpha}_{a^+}f(t) = 0 \Leftrightarrow f(t) = \sum_{j=1}^{n} c_j (t-a)^{\alpha-j},$$

where c_j $(j = 1, \ldots, n)$ *are arbitrary constants.*

The next result describes ${}^{RL}D^{\alpha}_{a^+}$ in the space $AC^n([a,b])$.

Lemma 1.15 ([53, 39])**.** *Let* $\alpha > 0$, *and* $n = [\alpha]+1$. *If* $f \in AC^n([a,b])$, *then the fractional derivative* ${}^{RL}D^{\alpha}_{a^+}$ *exists almost everywhere on* $[a,b]$ *and can be represented in the form :*

$$ {}^{RL}D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^n(s)ds + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}.$$

The following lemma shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1.16 ([39, 53])**.** *If $\alpha > 0$ and $f \in L^p([a,b])$ $(1 \leq p \leq \infty)$, then the following equalities*

$$^{RL}D^{\alpha}_{a^+} I^{\alpha}_{a^+} f(t) = f(t), \tag{1.6}$$

hold almost everywhere on $[a,b]$.

we derive the following composition relations between fractional differentiation and fractional integration operators.

Theorem 1.17 ([39, 53])**.** *Let $\alpha, \beta > 0$ such that $n-1 \leq \alpha < n$, $m-1 \leq \beta < m$ $(n,m \in \mathbb{N}^*)$, then we have*

(1) If $\alpha > \beta > 0$ and $f \in L^p([a,b])$ $(1 \leq p \leq \infty)$ then

$$^{RL}D^{\beta}_{a^+}(I^{\alpha}_{a^+} f)(t) = I^{\alpha-\beta}_{a^+} f(t),$$

hold almost everywhere on $[a,b]$.

(2) If $\beta \geq \alpha > 0$, and if the fractional derivative $^{RL}D^{\beta-\alpha}_{a^+}$ exist, then

$$^{RL}D^{\beta}_{a^+}(I^{\alpha}_{a^+} f(t) = {}^{RL}D^{\beta-\alpha}_{a^+} f(t).$$

(3) If $f \in L^1[a,b]$ and $I^{n-\alpha}_{a^+} f \in AC^n[a,b]$, then the equality

$$I^{\alpha}_{a^+} {}^{RL}D^{\alpha}_{a^+} f(t) = f(t) - \sum_{k=1}^{n} \frac{D^{n-k}(I^{n-\alpha}_{a^+} f)(a)}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k},$$

hold almost everywhere on $[a,b]$.

(4) Let $\alpha > 0, k \in \mathbb{N}^*$, if the fractional derivatives $^{RL}D^{\alpha}_{a^+}$ et $^{RL}D^{k+\alpha}_{a^+}$ exists, then

$$D^{k\,RL}D^{\alpha}_{a^+} f(t) = {}^{RL}D^{k+\alpha}_{a^+} f(t), \quad \textit{for all } \; t \in [a,b].$$

(5) Let $f \in L^1[a,b]$, if $I^{m-\alpha}_{a^+} f \in AC^m[a,b]$ et $\alpha+\beta < n$, then we have

$$^{RL}D^{\alpha}_{a^+}({}^{RL}D^{\alpha}_{a^+} f(t)) = ({}^{RL}D^{\alpha+\beta}_{a^+} f)(t) - \sum_{k=1}^{m} \frac{D^{m-k}(I^{m-\beta}_{a^+} f)(a)}{\Gamma(1-\beta-k)}(t-a)^{-k-\alpha}, \textit{for all } \; t \in [a,b].$$

The Caputo Fractional Derivative

Definition 1.18 ([39, 53])**.** For a function $f \in AC^n([a,b])$, the Caputo fractional derivative of order α defined by

$$\begin{aligned} ^{c}D^{\alpha}_{a^+} f(t) &= I^{n-\alpha}_{a^+} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \end{aligned}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Example 1.19. The Caputo derivative of the power function $(t-a)^{\beta}, \alpha > 0, \beta > 1, n = [\alpha]+1$, then the following relation hold

$$
{}^{c}D^{\alpha}_{a^+}(t-a)^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, & (\beta \in \mathbb{N} \text{ and } \beta \geq n \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > n-1), \\ \\ 0, & \beta \in \{0, \ldots, n-1\}. \end{cases} \tag{1.7}
$$

If we take $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$, we obtain

$$
\begin{aligned}
{}^{C}D^{\frac{1}{2}}_{a}(t-a)^{\frac{3}{2}} &= \left(I^{1-\frac{1}{2}}_{a^+}\right)\left(\frac{d}{dt}(t-a)^{\frac{3}{2}}\right) \\
&= \left(I^{\frac{1}{2}}_{a^+}\right)\left(\frac{d}{dt}(t-a)^{\frac{2}{2}}\right)
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
{}^{C}D^{\frac{1}{2}}_{a^+}(t-a)^{\frac{3}{2}} &= \left(I^{\frac{1}{2}}_{a^+}\right)\left(\frac{\Gamma\left(\frac{3}{2}+1\right)}{\Gamma\left(\frac{3}{2}+1-1\right)}(t-a)^{\frac{3}{2}-1}\right) \\
&= \frac{3}{2}\left(I^{\frac{1}{2}}_{a^+}\right)(t-a)^{\frac{1}{2}} \\
&= \frac{3}{2}\frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1+\frac{1}{2}\right)}(t-a)^{\frac{1}{2}+\frac{1}{2}} \\
&= \frac{3\sqrt{\pi}}{4}(t-a)
\end{aligned}
$$

Remark 1.20. We see that consistent with classical integer-ordered derivatives, for any constant C

$$
{}^{c}D^{\alpha}_{a^+}C = 0.
$$

We also recognize from (1.7) that :

Lemma 1.21 ([39, 53])**.** *Let $\alpha > \beta > 0$, and $f \in L^1([a,b])$. Then we have :*

(1) *The Caputo fractional derivative is linear ;*

(2) ${}^{c}D^{\alpha}_{a^+}I^{\alpha}_{a^+}f(t) = f(t)$ *;*

(3) ${}^{c}D^{\beta}_{a^+}I^{\alpha}_{a^+}f(t) = I^{\alpha-\beta}_{a^+}f(t)$.

The relation between the derivative of Caputo and that of Riemann-Liouville is given by :

Remark 1.22. We note that if $f \in AC^n([a,b])$, then Lemma 1.15 is equivalent to saying that

$$
{}^{RL}D^{\alpha}_{a^+}f(t) = {}^{c}D^{\alpha}_{a^+}f(t) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}.
$$

Clearly, we see that if $f^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n-1$ then we have

$$
{}^{c}D^{\alpha}_{a^+}f(t) = {}^{RL}D^{\alpha}_{a^+}f(t).
$$

Hilfer Fractional Derivative

In [34], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [35, 36, 37]).

Definition 1.23. [34] The Hilfer fractional derivative $D_{a^+}^{\alpha,\beta}$ of order α $(n-1<\alpha<n)$ and type β $(0\leq\beta\leq 1)$ is defined by

$$D_{a^+}^{\alpha,\beta} = I_{a^+}^{\beta(n-\alpha)} D^n I_{a^+}^{(1-\beta)(n-\alpha)} f(t) \tag{1.8}$$

where $I_{a^+}^{\alpha}$ and $D_{a^+}^{\alpha}$ are Riemann-Liouville fractional integral and derivative defined by (1.7) and (1.10), respectively.

Theorem 1.24. *Let $y\in C_\gamma[0,1], 0<\alpha<1$, and $0\leq\gamma<1$. Then*

$$D_{a^+}^{\alpha} I_{a^+}^{\alpha} f(t) = f(t), \forall t\in(a,b].$$

Moreover, if $f\in C_\gamma[a,b]$ and $I_{a^+}^{1-\beta(1-\alpha)} f\in C_\gamma^1[a,b]$, then

$$D_{a^+}^{\alpha,\beta} I_{a^+}^{\alpha} f(t) = f(t), \textit{ for a.e. } t\in(a,b].$$

Theorem 1.25. *Let $\alpha,\beta\geq a$ and $f\in C_\gamma^1[a,b], a<\alpha<1$, and $a\leq\gamma<1$. Then*

$$I_{a^+}^{\alpha} I_{a^+}^{\beta} f(t) = I_{a^+}^{\alpha+\beta} f(t)$$

Lemma 1.26. *Let $\alpha\geq a$, and $\sigma>a$. Then*

$$I_{a+}^{\alpha} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)} t^{\alpha+\sigma-1}, t>a$$

and

$$D_{a^+}^{\alpha} t^{\alpha-1} = a, \quad a<\alpha<1$$

Lemma 1.27. *Let $a<\alpha<1, a\leq\gamma\leq 1$, if $f\in C_\gamma[a,b]$ and $I_{a^+}^{1-\alpha} f\in C_\gamma^1[a,b]$, we have*

$$I_{a^+}^{\alpha} D_{a^+}^{\alpha} f(t) = f(t) - \frac{I_{a^+}^{1-\alpha} f(a)}{\Gamma(\gamma)} t^{\alpha-1}, \textit{ for all } t\in(a,b].$$

The relation between the Hilfer derivative and the Riemann-Liouville and Caputo fractional derivatives

Hilfer fractional derivative interpolates between the Riemann-Liouville and Caputo fractional derivatives since

— if we set $\beta=0$, $0<\alpha<1$ and $a=0$, in The formula of fractional derivative in the sense of Hilfer (1.8) we obtain the classical fractional derivative in the sense of Riemann-Liouville :

$$D_{0^+}^{\alpha,0} h(t) = \frac{d}{dt} I_{0^+}^{(1-\alpha)} h(t) =^{RL} D_{0_+}^{\alpha} h(t) \tag{1.9}$$

— if we set $\beta=1$, $0<\alpha<1$ and $a=0$, in The formula of fractional derivative in the sense of Hilfer (1.8) we obtain the classical fractional derivative in the sense of Caputo :

$$D_{a^+}^{\alpha,1} h(t) = I_{a^+}^{(1-\alpha)} \frac{d}{dt} h(t) =^{c} D_{0+}^{\alpha} h(t). \tag{1.10}$$

1.3 Basic results from nonlinear functional analysis and Fixed points theory

1.3.1 Topics of functional analysis

In this subsection, we present main preliminaries of functional analysis that will be used in this thesis.

Definition 1.28. (Normed Space) Let X be a vector space over a real field. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that
(i) $\|x\| \geq 0, x \in X$,
(ii) $\|x\| = 0$ if and only if $x = 0$,
(iii) $\|ax\| = |a|\|x\|, a \in \mathbb{R}$,
(iv) $\|x+y\| \leq \|x\| + \|y\|, x, y \in X$.
We call the pair $(X, \|\cdot\|)$ as a normed space. For simplicity, we use the symbol X as a normed space.

Definition 1.29. A sequence $\{x_n\}$ in a normed space X is said to be Cauchy if for every $\varepsilon > 0$, there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$, for every $m, n > N$. The space X is said to be complete if every Cauchy sequence in X converges in X.

Definition 1.30. (Banach Space) A Banach space is a complete normed space, that is if every Cauchy sequence in X converges in X.

Definition 1.31. A set $\mathscr{M}$ in a Banach space X is compact if for each sequence $\{x_n\} \subset \mathscr{M}$ has a subsequence with limit in $\mathscr{M}$. A set $\mathscr{M}$ in $(\mathbb{R}^n, \|\cdot\|)$ is compact if and only if it is closed and bounded. The space $C([a,b],\mathbb{R})$, consisting of all continuous functions $f : [a,b] \to \mathbb{R}$, is Banach space with norm $\|f\| = \sup\{|f(t)|, t \in [a,b]\}$.

Definition 1.32. A relatively compact subspace (or relatively compact subset) $\mathscr{M}$ of a normed space X is a subset whose closure is compact.

Every subset of a compact space is relatively compact since the closed subset of compact space is compact.

Definition 1.33. A subset $\mathscr{M}$ of $C([a,b],\mathbb{R})$, is equicontinuous provided for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \varepsilon$, for $x, y \in [a,b], f \in \mathscr{M}$.

Definition 1.34. Let $\{f_n\}$ be a sequence of functions with $f_n : [a,b] \to \mathbb{R}$, then $\{f_n\}$ is uniformly bounded on $[a,b]$ if there exists $c > 0$ such that $|f_n(t)| \leq c$ for all $n \in \mathbb{N}, t \in [a,b]$.

Theorem 1.35. *(Arzela-Ascoli Theorem) Let $\mathscr{M}$ be an open bounded subset of a Banach spaces X. A subset of $C(\mathscr{M},\mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous.*

Definition 1.36. For a normed space X, the operator $T : X \to X$ is said to satisfy the Lipschitz condition, if there exists a positive real constant k such that for all x and y in X,

$$\|Tx - Ty\| \leq k\|x - y\|.$$

Remark 1.37. In Definition 1.3.9, if $0 < k < 1$, the operator Tis called a contraction mapping on the normed space X.

Theorem 1.38. *(Dominated Convergent Theorem) Let $\{f_n\}$ be a sequence of real valued measurable functions on a measure space S.Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that $|f_n(x)| \leq g$, for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable and* $\lim_{n\to\infty} \int_S |f_n - f| \, ds = 0$.

1.3.2 Fixed-Point Theory

Fixed point theorems are the basic mathematical tools that help establish the existence of solutions of various kinds of equations. The fixed point method consists of transforming a given problem into a fixed point problem. The fixed points of the transformed problem are thus the solutions of the given problem. In this section, we recall the famous fixed point theorems that we will use to obtain varied existence results. We start with the definition of a fixed point.

Definition 1.39. Let f be an application of a set E in itself. We call fixed point of f any point $u \in E$ such that

$$f(u) = u.$$

1.3.3 Some Standard Fixed-Point Theorem

Banach's contraction principle

Banach's contraction principle, which guarantees the existence of a single fixed point of a contraction of a complete metric space with values in itself, is certainly the best known of the fixed point theorems. This theorem proved in 1922 by Stefan Banach is based essentially on the notions of Lipschitzian application and of contracting application.

Theorem 1.40. *[39, 62](Banach contraction principle)*
Let E be a complete metric space and let $F : E \to E$ be a contracting application, then F has a unique fixed point.

Schaefer's Fixed-Point Theorem

Lemma 1.41. *[39, 62] Let X be a Banach space. Assume that $T : X \to X$ is completely continuous operator and the set*

$$\Omega = \{x \in X : x = \mu T x, 0 < \mu < 1\}$$

is bounded, Then T has a fixed point in X.

Schauder Fixed-Point Theorem

This theorem extends the result of Brouwer's theorem to show the existence of a fixed point for a continuous function on a compact convex set in a Banach space. Schauder's fixed point theorem is more topological and asserts that a continuous map on a compact convex set admits a fixed point , which is not necessarily unique. And we have the following result :

Theorem 1.42. *[3] Let k be a non-empty, compact, convex subset in a Banach space E and let $T : k \to k$ be a continuous map. Then T admits a fixed point.*

Let us cite some generalizations of Schauder's fixed point theorem.

Leray-Schauder Nonlinear Alternative Fixed-Point Theorem

Theorem 1.43. *[3] (Leray-Schauder nonlinear alternative) Let K be a convex subset of a Banach space, and let U be an open subset of K with $0 \in U$. Then every completely continuous map $N : \overline{U} \to K$ has at least one of the following two properties :*

1. *N has a fixed point in $\overline{U}$;*
2. *there is an $x \in \partial U$ and $\lambda \in (0,1)$ with $x = \lambda N x$.*

Chapitre 2

Hilfer Fractional Langevin and Sturm-Liouville Differential Equation with Boundary Conditions

2.1 Introduction

In this chapter, we are concerned with the existence of solutions for certain classes of nonlinear fractional differential equations via Hilfer fractional derivative. First, In section 2.2, we investigate the problem of existence and uniqueness for a boundary value problem for the nonlinear Langevin and Sturm-Liouville fractional differential equation boundary conditions in Real spaces. The first results are provided by applying some notions of functional analysis. Next, In section 2.3, we study the existence of solutions for a boundary value problem defined by system of generalized Sturm-Liouville and Langevin Hilfer fractional differential equations involving boundary conditions. The used approach is based on some standard fixed point theorem. Finally, we provide an illustrative example at the end of each section in support of our existence theorems.

2.2 Boundary Value Problems for Hilfer Fractional Langevin and Sturm-Liouville differential equation

2.2.1 Introduction

In this subsection, we concentrate on the following boundary value problem of nonlinear Hilfer fractional differential equation

$$ {}^{H}\mathscr{D}_{a^{+}}^{\alpha_1,\beta}\left[p(t)^{H}\mathscr{D}_{a^{+}}^{\alpha_2,\beta}-q(t)\right]u(t)=\mathscr{F}\left(t,u(t)\right),t\in \mathrm{J}=[a,b], \tag{2.1}$$

supplemented with the boundary conditions of the form :

$$u(a)=0, \quad p(b)^{H}\mathscr{D}_{a^{+}}^{\alpha_2,\beta}u(t)-q(b)u(b)=0, \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta. \tag{2.2}$$

where ${}^{H}\mathscr{D}_{a^+}{}_{a^+}^{\rho,\beta;\psi}$ is the ψ-Hilfer FD of order $\rho \in (0,1)$ with $\rho \in \{\alpha_1, \alpha_2\}$ and type $\beta \in [0,1]$. Also, $p \in C(\mathrm{J}, R\backslash\{0\})$, $q \in C(\mathrm{J}, R)$ and $\mathscr{F} : \mathrm{J} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous.

In the present paper we initiate the study of boundary value problems like (2.1)-(2.2), in which we combine Hilfer fractional differential equations subject to the Katugampola fractional integral boundary conditions.

2.2.2 Existence of solutions

In this section we shall present and prove a preparatory lemma for boundary value problem of linear fractional differential equations with Hilfer derivative.

Definition 2.1. A function $u(t) \in C_{1-\gamma}(\mathrm{J}, \mathbb{R})$ is said to be a solution of (2.1)-(2.2) if u satisfies the equation ${}^{H}\mathscr{D}_{a^+}^{\alpha_1,\beta}\left[p(t)^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta} - q(t)\right]u(t) = \mathscr{F}(t, u(t))$ on J , and the conditions (2.2).

For the existence of solutions for the problem (2.1)-(2.2), we need the following auxiliary lemma.

Lemma 2.2. *Let $\mathscr{F} : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. A function u is a solution of the fractional integral equation*

$$\begin{aligned} u(t) =& \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)(t,u(t)) + \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t) \\ &- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1}, \end{aligned} \tag{2.3}$$

if and only if u is a solution of the fractional BVP

$${}^{H}\mathscr{D}_{a^+}^{\alpha_1,\beta}\left[p(t)^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta} - q(t)\right]u(t) = \mathscr{F}(t,u(t)), \tag{2.4}$$

and

$$u(a) = 0, \quad p(b)^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta}u(b) - q(b)u(b) = 0. \tag{2.5}$$

Proof. Taking the $\alpha_1^{\text{th}} - RL$ integral to FDE of (2.1), we obtain

$${}^{c}\mathscr{D}_{a^+}^{\alpha_2,\beta}u(t) = \frac{\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}(t,u(t)) + q(t)u(t) + \frac{k_1(t-a)^{\gamma-1}}{\Gamma(\gamma)}}{p(t)} \tag{2.6}$$

where $k_1 \in \mathbb{R}$. From the BCs of (2.2), we get

$$k_1 = -\Gamma(\gamma)(b-a)^{\gamma-1}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))$$

Taking the $\alpha_2^{th} - RL$ integral to FDE of (2.6), we get

$$u(t) = \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta},\mathscr{F}\right)(t,u(t)) + \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t)$$

$$-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1} + \frac{k_2}{\Gamma(\gamma)}(t-a)^{\gamma-1}.$$

where $k_2 \in \mathbb{R}$. Using the BCs of (2.2) gives

$$k_2 = 0$$

In this regard, if we apply the α_2^{th}-Hilfer FD and α_1^{th} -Hilfer FD to both sides of (2.3) and use lemma 2.3, then the problem (2.1)-(2.2) immediately is established.

□

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem $(2.1)-(2.2)$ by using a variety of fixed point theorems.

Existence and uniqueness result via Banach's fixed point theorem

Theorem 2.3. *Assume the following hypothesis :*
(H1) There exists a constant $L > 0$ such that

$$|\mathscr{F}(t,u) - \mathscr{F}(t,v)| \leq L|u - v|.$$

If

$$(L\Lambda_1 + \Lambda_2) < 1 \tag{2.7}$$

where

$$\Lambda_1 := \left\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)} - \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)} \frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)} \right\}, \quad \Lambda_2 := \left(\frac{q^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)} \right) \tag{2.8}$$

and

$$p^* = \sup_{t\in \mathrm{J}} p(t), \quad q^* = sup_{t\in \mathrm{J}} q(t). \tag{2.9}$$

Then the problem (2.1) has a unique solution on J.

Proof. Transform the problem $(2.1)-(2.2)$ into a fixed point problem for the operator G defined by

$$\begin{aligned} \mathrm{G}u(t) =& \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)(t,u(t)) + \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t) \\ &- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1}, \end{aligned} \tag{2.10}$$

Applying the Banach contraction mapping principle, we shall show that G is a contraction.

We put $\sup\limits_{t\in[a,b]} |\mathscr{F}(t,0)| = \mathscr{F}^* < \infty$, and choose

$$r \geq \frac{M\Lambda_1}{1-(L\Lambda_1+\Lambda_2)}.$$

To show that $\mathrm{G}B_r \subset B_r$, where $B_r = \left\{u \in \mathscr{C}_{1-\gamma} : \|u\|_{\mathscr{C}_{1-\gamma}} \leq r\right\}$, we have for any $u \in B_r$

$$
\begin{aligned}
|((\mathrm{G}u)(t))(t-a)^{1-\gamma}| &\leq \sup_{t\in \mathrm{J}}\Big\{(t-a)^{1-\gamma}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)(t,u(t)) + \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t) \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1}\Big\}\Big\} \\
&\leq (t-a)^{1-\gamma}\left\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(t,u(t))|\right) - \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)|\right)\right\} \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(b,u(b))|(t-a)^{\alpha_2}\Big\} \\
&\leq (b-a)^{1-\gamma}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(|\mathscr{F}(s,u(s))-\mathscr{F}(t,0)|+|\mathscr{F}(t,0)|)(t)\right) \\
&+ \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)|\right)\Big\} \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(|\mathscr{F}(s,u(s))-\mathscr{F}(t,o)|+|\mathscr{F}(t,0)|)(b)| \\
&\leq (Lr+\mathscr{F}^*)\Big\{(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(1)(t)\right) \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(1)(b)|\Big\} \\
&+ r(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(1)(t)\right) \\
&\leq (Lr+\mathscr{F}^*)\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)} - \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\} \\
&+ \left(\frac{rq^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)}\right) \\
&:= (Lr+\mathscr{F}^*)\Lambda_1 + r\Lambda_2 \leq r.
\end{aligned}
$$

which implies that $\mathrm{G}B_r \subset B_r$.

Now let $u,v \in C_{1-\gamma}(\mathrm{J},\mathbb{R})$. Then, for $t \in \mathrm{J}$, we have

$$\begin{aligned}
|((\mathrm{G}u)(t)-(\mathrm{G}v)(t))(t-a)^{1-\gamma}| \leq \sup_{t\in[a,b]} & \Big\{(t-a)^{1-\gamma}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(t)\right) \\
& +\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)-v(t)|\right)\Big\} \\
& -\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(b)\Big\} \\
& \leq \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(t-a)^{1-\gamma}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(t)\right) \\
& +\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(t-a)^{1-\gamma}|u(t)-v(t)|\right)\Big\} \\
& -\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(b)\Big\} \\
& \leq \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(t-a)^{1-\gamma}L|u(t)-v(t)|(t)\right) \\
& +\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(t-a)^{1-\gamma}|u(t)-v(t)|\right)\Big\} \\
& -\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma}L|u(t)-v(t)|(b)\Big\} \\
& \leq \left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)}-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\}L\|u-v\| \\
& +\left(\frac{q^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)}\right)\|u-v\|_{C_{1-\gamma}} \\
& := (L\Lambda_1+\Lambda_2)\|u-v\|_{C_{1-\gamma}}
\end{aligned}$$

Thus

$$\|((\mathrm{G}u)(t)-(\mathrm{G}v)(t))(t-a)^{1-\gamma}\|_\infty \leq (L\Lambda_1+\Lambda_2)\|u-v\|_{C_{1-\gamma}}.$$

We deduce that G is a contraction mapping. As a consequence of Banach contraction principle. the problem (2.1)-(2.2) has a unique solution on J. This completes the proof.

□

Existence result via Schaefer's fixed point theorem

Theorem 2.4. *Assume the hypotheses :*
(H2) : The function $f : \mathrm{J}\times\mathbb{R} \to \mathbb{R}$ *is continuous.*
(H3) There exists a constant $L_1 > 0$*, such that*

$$|(t-a)^{1-\gamma}\mathscr{F}(t,u)| \leq L_1, \text{ for a.e. } t \in \mathrm{J}, u \in \mathbb{R}.$$

Then, the problem (2.1)-(2.2) *has a least one solution in* J.

Proof. We shall use Schaefer's fixed point theorem to prove that G defined by (2.10) has a fixed point. The proof will be given in several steps.

Step 1 : G is continuous Let u_n be a sequence such that $u_n \to u$ in $C_{1-\gamma}(\mathrm{J},\mathbb{R})$. Then for each $t \in \mathrm{J}$,

$$\begin{aligned}
|((\mathrm{G}u_n)(t)-(\mathrm{G}u)(t))(t-a)^{1-\gamma}| \leq \sup_{t\in \mathrm{J}}\Bigg\{&(t-a)^{1-\gamma}\Bigg\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(s,u_n(s))-\mathscr{F}(s,u(s))|(t)\right)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u_n(t)-u(t)|\right)\Bigg\}\\
&-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma}|\mathscr{F}(s,u_n(s))-\mathscr{F}(s,u(s))|(b)\Bigg\}\\
&\leq \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(t-a)^{1-\gamma}|\mathscr{F}(s,u_n(s))-\mathscr{F}(s,u(s))|(t)\right)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(t-a)^{1-\gamma}|u_n(t)-u(t)|\right)\Bigg\}\\
&-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma}|\mathscr{F}(s,u_n(s))-\mathscr{F}(s,u(s))|(b)\Bigg\}\\
&:=\Lambda_1\|\mathscr{F}(s,u_n(s))-\mathscr{F}(s,u(s))\|_{C_{1-\gamma}}+\Lambda_2\|u_n-u\|_{C_{1-\gamma}}
\end{aligned}$$

Since f is continuous, so $\|((\mathrm{G}u_n)(t)-(\mathrm{G}u)(t))t^{1-\gamma}\|_\infty \to 0$ as $n\to\infty$.

Step 2 : G maps bounded sets into bounded sets in $C_{1-\gamma}(\mathrm{J},\mathbb{R})$. Indeed, it is enough to show that for any $r>0$, if we take $u\in B_r=\left\{u\in C_{1-\gamma}(\mathrm{J},\mathbb{R}),\|u\|_{C_{1-\gamma}}\leq r\right\}$, such that $\mathrm{G}u(t)$ is bounded. Indeed, from (H3), Then for $u\in B_r$ and for each $t\in\mathrm{J}$, we have

$$\begin{aligned}
|((\mathrm{G}u)(t))(t-a)^{1-\gamma}| &\leq \sup_{t\in\mathrm{J}}\Bigg\{(t-a)^{1-\gamma}\Bigg\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)(t,u(t))-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t)\\
&-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1}\Bigg\}\Bigg\}\\
&\leq (t-a)^{1-\gamma}\Bigg\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(t,u(t))|\right)-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)|\right)\Bigg\}\\
&-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(b,u(b))|(t-a)^{\alpha_2}\Bigg\}\\
&\leq L_1\Bigg\{(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(1)(t)\right)\\
&-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(1)(b)|\Bigg\}-r(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(1)(t)\right)\\
&\leq L_1\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)}-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\}-r\left(\frac{q^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)}\right)\\
&:=L_1\Lambda_1+r\Lambda_2.
\end{aligned}$$

Thus,

$$\|((\mathrm{G}u)(t))(t-a)^{1-\gamma}\|\leq L_1\Lambda_1+r\Lambda_2.$$

where Λ_1 and Λ_1 are defined in (2.19).

Step 3 : G maps bounded sets into equicontinuous sets of $C_{1-\gamma}(\mathrm{J},\mathbb{R})$.

Let $t_1, t_2 \in \mathrm{J}$, $t_1 < t_2$, B_r be a bounded set of $C_{1-\gamma}(\mathrm{J}, \mathbb{R})$ as in Step 2, and let $u \in B_r$. Then

$$
\begin{aligned}
&\|((t_2-a)^{1-\gamma}\mathrm{G}u(t_2)-(t_1-a)^{1-\gamma}\mathrm{G}u(t_1))\| \\
&\leq \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1}\left|(t_2-a)^{1-\gamma}\mathscr{F}(s,u(s))(t_2)-(t_1-a)^{1-\gamma}\mathscr{F}(s,v(s))(t_1)\right|\right) \\
&-\mathscr{I}_{a^+}^{\alpha_2}\left(\frac{q^*}{p^*}|(t_2-a)^{1-\gamma}u(t_2)-(t_1-a)^{1-\gamma}u(t_1)|\right) \\
&-\frac{\Gamma(\gamma)\Big((t_2-a)^{1-\gamma}-(t_1-a)^{1-\gamma}\Big)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(b)\Big\} \\
&\leq \frac{L_1}{p^*\Gamma(\alpha)}\Bigg|(t_2-a)^{1-\gamma}\int_1^{t_1}(t_2-s)^{\alpha_1+\alpha_2-1}(1)ds \\
&-(t_1-a)^{1-\gamma}\int_1^{t_1}(t_1-s)^{\alpha_1+\alpha_2-1}(1)ds\Bigg| \\
&+\frac{L_1}{p^*\Gamma(\alpha)}\left|(t_2-a)^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1+\alpha_2-1}(1)ds\right| \\
&+\frac{rq^*}{p^*\Gamma(\alpha_1+\alpha_2)}\Bigg|(t_2-a)^{1-\gamma}\int_1^{t_1}(t_2-s)^{\alpha_2-1}(1)ds \\
&-(t_1-a)^{1-\gamma}\int_1^{t_1}(t_1-s)^{\alpha_2-1}(1)ds\Bigg| \\
&+\frac{rq^*}{p^*\Gamma(\alpha_2)}\left|(t_2-a)^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}(1)ds\right| \\
&-\frac{\Gamma(\gamma)\Big((t_2-a)^{1-\gamma}-(t_1-a)^{1-\gamma}\Big)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(b)\Big\} \\
&\leq \frac{p^*L_1}{\Gamma(\alpha_1+\alpha_2+1)}\big((t_2-a)^{\alpha_1+\alpha_2-\gamma+1}-(t_1-a)^{\alpha_1+\alpha_2-\gamma+1}\big) \\
&+\frac{p^*r}{q^*\Gamma(\alpha_2+1)}\big((t_2-a)^{\alpha_2-\gamma+1}-(t_1-a)^{\alpha_2-\gamma+1}\big) \\
&-\frac{\Gamma(\gamma)\Big((t_2-a)^{1-\gamma}-(t_1-a)^{1-\gamma}\Big)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s))-\mathscr{F}(s,v(s))|(b)\Big\}.
\end{aligned}
$$

which implies $\|\mathrm{G}u(t_2)-\mathrm{G}u(t_1)\|_{\mathscr{C}_{1-\gamma}} \to 0$ as $t_1 \to t_2$, As consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that G is continuous and completely continuous.

Step 4 : A priori bounds.
Now it remains to show that the set $\Omega = \{u \in C(\mathrm{J}, \mathbb{R}) : u = \mu\mathrm{G}(u) \text{ for some } 0 < \mu < 1\}$ is bounded.

For such $u \in \Omega$. Thus, for each $t \in \mathrm{J}$, we have

$$u(t) \leq \mu \Bigg\{ \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{1}{p} \mathscr{I}_{a^+}^{\alpha_1,\beta} \mathscr{F} \right)(t,u(t)) - \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{q}{p} u \right)(t) \\ - \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)} \frac{(b-a)^{1-\gamma}}{p(t)} \mathscr{I}_{a^+}^{\alpha_1} \mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1} \Bigg\}.$$

For $\mu \in [0,1]$, let u be such that for each $t \in \mathrm{J}$

$$\begin{aligned} \|(\mathrm{G}u(t))t^{1-\gamma}\| \leq L_1 \Bigg\{ & \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{1}{p} \mathscr{I}_{a^+}^{\alpha_1,\beta} \mathscr{F} \right)(t,u(t)) \\ & - \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)} \frac{(b-a)^{1-\gamma}}{p(t)} \mathscr{I}_{a^+}^{\alpha_1} \mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1} \Bigg\} \\ & - r \Bigg\{ \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{q}{p} u \right)(t) \Bigg\} \\ & := L_1\Lambda_1 - r\Lambda_2. \end{aligned}$$

Thus

$$\|(\mathrm{G}u(t))(t-a)^{1-\gamma}\| \leq \infty$$

This implies that the set Ω is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that G has a fixed point which is a solution on J of the problem (2.1)-(2.2).

□

Existence result via the Leray-Schauder nonlinear alternative

Theorem 2.5. *Assume the following hypotheses :*
(H4) There exist $\omega \in L^1(\mathrm{J},\mathbb{R}^+)$ *and* $\Phi : [0,\infty) \to (0,\infty)$ *continuous and nondecreasing such that*

$$|f(t,u)| \leq \omega(t)\Phi(\|u\|), \text{ for a.e. } t \in \mathrm{J} \text{ and each } u \in \mathbb{R}.$$

(H5) There exists a constant $\varepsilon > 0$ *such that*

$$\frac{\varepsilon(1+\Lambda_2)}{\|\omega\|\Phi(\varepsilon)\Lambda_1} > 1.$$

Then the boundary value problem (2.1)-(2.2) *has at least one solution on* J.

Proof. We shall use the Leray-Schauder theorem to prove that G defined by (2.10) has a fixed point. As shown in Theorem 2.4, we see that the operator G is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem G is completely continuous.
Let u be such that for each $t \in \mathrm{J}$, we take the equation $u = \lambda \mathrm{G}u$ for $\lambda \in (0,1)$ and let u be a solution. After that, the following is obtained.

$$\begin{aligned}
|u(t)(t-a)^{1-\gamma}| &\leq \lambda\left\{(t-a)^{1-\gamma}\left\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)(t,u(t))-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t)\right.\right.\\
&\left.\left.-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1}\right\}\right\}\\
&\leq (t-a)^{1-\gamma}\left\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(t,u(t))|\right)-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)|\right)\right\}\\
&\left.-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(b,u(b))|(t-a)^{\alpha_2}\right\}\\
&\leq \Phi(\|u\|)\|\omega\|\left\{(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(1)(t)\right)\right.\\
&\left.-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(1)(b)|\right\}-r(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(1)(t)\right)\\
&\leq \Phi(\|u\|)\|\omega\|\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)}-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\}-\|u\|\left(\frac{q^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)}\right)\\
&:= \Phi(\|u\|)\|\omega\|\Lambda_1-\|u\|\Lambda_2.
\end{aligned}$$

which leads to

$$\frac{\|u\|(1+\Lambda_2)}{\|\omega\|\Phi(\|u\|)\Lambda_1}\leq 1.$$

In view of (H5), there exists ε such that $\|u\|\neq\varepsilon$. Let us set $U=\{u\in C_{1-\gamma}(\mathrm{J},\mathbb{R}):\|u\|_{1-\gamma}<\varepsilon\}$.
Obviously, the operator $\mathrm{G}:\overline{U}\to C_{1-\gamma}(\mathrm{J},\mathbb{R})$ is completely continuous. From the choice of U, there is no $u\in\partial U$ such that $u=\lambda\mathrm{G}(u)$ for some $\lambda\in(0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, G has a fixed point $u\in U$ which is a solution of the (2.1)-(2.2). The proof is completed.

□

2.2.3 Example

Example 2.6. We consider the problem for Hilfer fractional differential equations of the form

$$\begin{cases} {}^H\mathscr{D}_{a^+}^{\alpha_1,\beta}\left[\frac{\sqrt{t}}{\sqrt{t+1}}{}^H\mathscr{D}_{a^+}^{\alpha_2,\beta}-q(t)\right]u(t)=\frac{t}{12}\sin u(t), t\in\mathrm{J}=[0.2,1],\\ \\ u(0.2)=0,\quad p(b){}^H\mathscr{D}_{a^+}^{\alpha_2,\beta}u(1)-q(b)u(1)=0. \end{cases}\tag{2.11}$$

Here

$$a=0.2,\qquad b=1,\qquad \alpha_1=\frac{7}{8},\qquad \alpha_2=\frac{10}{11},$$

$$\beta=\frac{1}{7},\qquad \mathrm{p}(t)=\frac{\sqrt{t}}{\sqrt{t+1}},\qquad \mathrm{q}(t)=\frac{t}{10}$$

$$\gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta=\left(\frac{7}{8}+\frac{10}{11}\right)\left(1-\frac{1}{7}\right)+\frac{1}{7}\simeq 1.672077$$

With
$$\mathscr{F}(t,u) = \frac{t}{12}\sin|u(t)|, \quad t \in [0.2,1].$$
Clearly, the function $\mathscr{F}$ is continuous.
For each $u \in \mathbb{R}^+$ and $t \in [0.2,1]$, we have

$$|\mathscr{F}(t,u(t))| \leq \left|\frac{t}{12}\sin\right||u(t)| \leq \frac{t}{12} \leq \frac{1}{12}.$$

and
$$|\mathscr{F}(t,u(t)) - \mathscr{F}(t,v(t))| \leq \frac{1}{12}|u-v|.$$
Hence, the hypothesis (H1) is satisfied with $L = \frac{1}{12}$. Further,

$$\mathrm{p}^* = \sup_{t\in \mathrm{J}} \mathrm{p}(t) = \sup_{t\in \mathrm{J}} \frac{\sqrt{t}}{\sqrt{t+1}} = \frac{\sqrt{1}}{\sqrt{2}} \simeq 0.7071,$$
$$\mathrm{q}^* = \sup_{t\in \mathrm{J}} \mathrm{q}(t) = \sup_{t\in \mathrm{J}} \frac{t}{(4t)^2} = \frac{1}{(0.8)^2} = 1.5625.$$

$$\Lambda_1 := \left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)} - \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\} = -0.3768, \tag{2.12}$$

and
$$\Lambda_2 := \frac{q^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)} = 0.6835, \tag{2.13}$$
From (2.12) and (2.13), we get

$$L\Lambda_1 + \Lambda_2 \simeq 0.6521 < 1.$$

Therefore, by the conclusion of Theorem (2.3), It follows that the problem (2.11) has a unique solution defined on $[0.2,1]$.

2.3 Boundary value problem defined by coupled system of generalized Sturm-Liouville and Langevin Hilfer fractional differential equations

2.3.1 Introduction

The topic of boundary value problems of fractional differential equations supplemented with a variety of boundary conditions has attracted a significant attention in recent years. In particular, the literature on fractional order BVPs involving nonlocal and integral boundary conditions is now much enriched, for instance, see [73, 78] and the references cited therein.

Coupled systems of fractional-order differential equations have also been extensively studied due to their occurrence in several diverse disciplines, for example, nonlocal thermo elasticity. And constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [75], disease models [68, 74] ecological models [72], synchronization of chaotic systems [69, 70, 80] etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers [15, 66, 67, 71, 76, 77, 79, 61].

In this section, we concentrate on the following boundary value problem of nonlinear coupled system of generalized Sturm-Liouville and Langevin Hilfer fractional differential equations

$$\begin{cases} {}^{H}\mathscr{D}_{a^+}^{\alpha_1,\beta_1;\psi}\left[p_1(t)^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta_1}-q_1(t)\right]u(t)=\mathscr{F}\left(t,u(t),v(t)\right), & \\ & ,t\in \mathrm{J}:=[a,b], \\ {}^{H}\mathscr{D}_{a^+}^{\alpha_3,\beta_2}\left[p_2(t)^{H}\mathscr{D}_{a^+}^{\alpha_4,\beta_2}-q_2(t)\right]v(t)=\mathscr{G}\left(t,u(t),v(t)\right), & \end{cases} \tag{2.14}$$

supplemented with the boundary conditions of the form :

$$\begin{cases} u(a)=0, \quad p_1(b)^{H}\mathscr{D}_{a^+}^{\alpha_2,\beta}u(t)-q_1(b)u(b)=0, \quad ,\gamma=(\alpha_1+\alpha_2)(1-\beta_1)+\beta_1, \\ v(a)=0, \quad p_2(b)^{H}\mathscr{D}_{a^+}^{\alpha_4,\beta}v(t)-q_2(b)v(b)=0, \quad \gamma=(\alpha_3+\alpha_4)(1-\beta_2)+\beta_2. \end{cases} \tag{2.15}$$

where ${}^{H}\mathscr{D}_{a^+}{}_{a^+}^{\rho,\beta}$ is the ψ-Hilfer FD of order $\rho\in(0,1)$ with $\rho\in\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ and type $\beta_i\in[0,1]$, $i=1,2$. Also, $p_i\in C(\mathrm{J},R\backslash\{0\})$, $q_i\in C(\mathrm{J},R)$, $i=1,2$ and $\mathscr{F},\mathscr{G}:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ are continuous.

Let $\mathscr{C}_{1-\gamma}=C_{1-\gamma}(\mathrm{J},\mathbb{R})$ be a Banach space of all continuous functions defined on $[a,b]$ and endowed with the supremum norm. We define an operator $\mathrm{G}:\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}\to\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}$ by

$$\mathrm{G}w=\mathrm{G}\left(u,v\right)=\left(\mathrm{G}_1u,\mathrm{G}_2v\right),$$

with

$$\|\mathrm{G}w\|=\|\mathrm{G}_1u\|+\|\mathrm{G}_2v\|.$$

2.3.2 Existence of solutions

In this subsection we shall present and prove a preparatory lemma for boundary value problem of linear coupled system of generalized Sturm-Liouville and Langevin fractional differential equations with Hilfer derivative.

Definition 2.7. A coupled functions $(u(t),v(t)) \in \mathscr{C}_{1-\gamma} \times \mathscr{C}_{1-\gamma}$ are said to be a solution of (2.14)-(2.15) if u satisfies the equation (2.14) on J , and the conditions (2.15).

For the existence of solutions for the problem (2.14)-(2.15), we need the following auxiliary lemma.

Lemma 2.8. *Let $\mathscr{F},\mathscr{G} : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ be two continuous functions. The coupled functions (u,v) are a solution of the fractional integral equations*

$$\begin{aligned} u(t) =& \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{1}{p_1}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}\right)(t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{q_1}{p_1}u\right)(t) \\ &- \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\frac{(b-a)^{1-\gamma_1}}{p_1(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b),v(b))(t-a)^{\gamma_1+\alpha_2-1}, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} v(t) =& \mathscr{I}_{a^+}^{\alpha_4}\left(\frac{1}{p_2}\mathscr{I}_{a^+}^{\alpha_3}\mathscr{G}\right)(t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_4,\beta}\left(\frac{q_2}{p_2}v\right)(t) \\ &- \frac{\gamma_2(\gamma_2)}{\gamma_2(\alpha_4+\gamma_2)}\frac{(b-a)^{1-\gamma_2}}{p_2(t)}\mathscr{I}_{a^+}^{\alpha_3}\mathscr{G}(b,u(b),v(b))(t-a)^{\gamma_2+\alpha_4-1}, \end{aligned} \tag{2.17}$$

if and only if (u,v) is a solution of the fractional BVP (2.14)-(2.15).

Proof. The proof is similar to the Lemma 2.2 above.

□

Before introducing the main results, we investigate Equations 2.16 and 2.17 under the following assumptions :
(As1) : The function $\mathscr{F},\mathscr{G} : \mathrm{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.
(As2) There exists constants $L_1, L_2 > 0$ such that

$$|\mathscr{F}(t,u,v) - \mathscr{F}(t,u_1,v_1)| \leq L_1\big(|u-u_1| + |v-v_1|\big),$$

$$|\mathscr{G}(t,u,v) - \mathscr{G}(t,u_1,v_1)| \leq L_2\big(|u-u_1| + |v-v_1|\big).$$

(As3) There exists constants $M_1, M_2 > 0$, such that

$$\begin{aligned} |(t-a)^{1-\gamma}\mathscr{F}(t,u,v)| &\leq M_1, \\ |(t-a)^{1-\gamma}\mathscr{G}(t,u,v)| &\leq M_2, \end{aligned}$$

for $t \in \mathrm{J}$ and $(u,v) \in \mathbb{R} \times \mathbb{R}$.
(As4) There exists $\omega_1, \omega_2 \in L^1(\mathrm{J},\mathbb{R}^+)$ and $\Phi_1, \Phi_2 : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\begin{aligned} |\mathscr{F}(t,u,v)| &\leq \omega_1(t)\Phi_1(\|(u,v)\|), \\ |\mathscr{G}(t,u,v)| &\leq \omega_2(t)\Phi_2(\|(u,v)\|), \end{aligned}$$

for $t \in \mathrm{J}$ and $(u,v) \in \mathbb{R} \times \mathbb{R}$.
(As5) There exists a constant $\bar{\varepsilon} > 0$ such that

$$\frac{\bar{\varepsilon}\big(1+\Lambda_4\big)\big(1+\Lambda_6\big)}{\big(1+\Lambda_6\big)\|\omega_1\|\Phi_1(\bar{\varepsilon})\Lambda_3 + \big(1+\Lambda_4\big)\|\omega_2\|\Phi_2(\bar{\varepsilon})\Lambda_5} \leq 1.$$

For the sake of convenience, we setting the following symbols.

$$p_i^* = \inf_{t\in \mathrm{J}} p_i(t), \quad q_i^* = sup_{t\in \mathrm{J}} q_i(t), \quad i = 1,2.$$

$$\sup_{t\in \mathrm{J}} |\mathscr{F}(t,0,0)| = \mathscr{F}_0, \quad \sup_{t\in \mathrm{J}} |\mathscr{G}(t,0,0)| = \mathscr{G}_0,$$

$$\Lambda_3 := \left\{ \frac{(b-a)^{\alpha_1+\alpha_2}}{p_1^*\Gamma(\alpha_1+\alpha_2+1)} - \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)} \frac{(b-a)^{\alpha_1+\alpha_2}}{p_1^*\Gamma(\alpha_1+1)} \right\}, \quad \Lambda_4 := \left(\frac{q_1^*(b-a)^{\alpha_2}}{p_1^*\Gamma(\alpha_2+1)} \right) \tag{2.18}$$

$$\Lambda_5 := \left\{ \frac{(b-a)^{\alpha_3+\alpha_4}}{p_2^*\Gamma(\alpha_3+\alpha_4+1)} - \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_4+\gamma_2)} \frac{(b-a)^{\alpha_3+\alpha_4}}{p_2^*\Gamma(\alpha_3+1)} \right\}, \quad \Lambda_6 := \left(\frac{q_2^*(b-a)^{\alpha_4}}{p_2^*\Gamma(\alpha_4+1)} \right) \tag{2.19}$$

Existence and uniqueness result via Banach's fixed point theorem

Theorem 2.9. *Assume the hypothesis (As1) hold and If*

$$\left(\Delta_1 + \Delta_2\right) < 1. \tag{2.20}$$

Then the problem (2.14) has a unique solution on J.

Proof. Transform the problem (2.14) – (2.15) into a fixed point problem for the operator $\mathrm{G} : \mathscr{C}_{1-\gamma} \times \mathscr{C}_{1-\gamma} \to \mathscr{C}_{1-\gamma} \times \mathscr{C}_{1-\gamma}$ defined by

$$\mathrm{G}(u,v) = (\mathrm{G}_1 u, \mathrm{G}_2 v), \tag{2.21}$$

where

$$\begin{aligned} \mathrm{G}_1 u(t) =& \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{1}{p_1} \mathscr{I}_{a^+}^{\alpha_1,\beta} \mathscr{F} \right)(t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{q_1}{p_1} u \right)(t) \\ &- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)} \frac{(b-a)^{1-\gamma}}{p_1(t)} \mathscr{I}_{a^+}^{\alpha_1} \mathscr{F}(b,u(b),v(b))(t-a)^{\gamma+\alpha_2-1}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \mathrm{G}_2 v(t) =& \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{1}{p_2} \mathscr{I}_{a^+}^{\alpha_1,\beta} \mathscr{G} \right)(t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_2,\beta} \left(\frac{q_2}{p_2} v \right)(t) \\ &- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)} \frac{(b-a)^{1-\gamma}}{p_2(t)} \mathscr{I}_{a^+}^{\alpha_1} \mathscr{G}(b,u(b),v(b))(t-a)^{\gamma+\alpha_2-1}, \end{aligned} \tag{2.23}$$

Applying the Banach contraction mapping principle, we shall show that G is a contraction.

We choose

$$r \geq \frac{\mathscr{F}_0\Lambda_3 + \mathscr{G}_0\Lambda_5}{1 - L_{\mathscr{F}}\Lambda_3 - L_{\mathscr{G}}\Lambda_5 + \Lambda_4 + \Lambda_6}.$$

To show that $\mathrm{G}B_r \subset B_r$, where $B_r = \left\{(u,v) \in \mathscr{C}_{1-\gamma} \times \mathscr{C}_{1-\gamma} : \|(u,v)\|_{\mathscr{C}_{1-\gamma}} \leq r\right\}$, we have for any $u \in B_r$

$$\begin{aligned}
\left|\left((\mathrm{G}_1 u)(t)\right)(t-a)^{1-\gamma}\right| &\leq \sup_{t\in \mathrm{J}}\Bigg\{(t-a)^{1-\gamma}\Bigg\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1}{p_1}u\right)(t) \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p_1(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b),v(b))(t-a)^{\gamma+\alpha_2-1}\Bigg\}\Bigg\} \\
&\leq (t-a)^{1-\gamma}\Bigg\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(t,u(t),v(t))|\right) - \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1^*}{p_1^*}|u(t)|\right)\Bigg\} \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(b,u(b),v(b))|(t-a)^{\alpha_2}\Bigg\} \\
&\leq (b-a)^{1-\gamma}\Bigg\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(t,0,0)| + |\mathscr{F}(t,0,0)|)(t)\right) \\
&- \mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)|\right)\Bigg\} \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(t,0,0)| + |\mathscr{F}(t,0,0)|)(b)| \\
&\leq (L_{\mathscr{F}}r + \mathscr{F}_0)\Bigg\{(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(1)(t)\right) \\
&- \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(1)(b)|\Bigg\} \\
&- r(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1^*}{p_1^*}(1)(t)\right) \\
&\leq (L_{\mathscr{F}}r + \mathscr{F}_0)\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p_1^*\Gamma(\alpha_1+\alpha_2+1)} - \frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\} \\
&- \left(\frac{rq_1^*(b-a)^{\alpha_2}}{p_1^*\Gamma(\alpha_2+1)}\right) \\
&:= (L_{\mathscr{F}}r + \mathscr{F}_0)\Lambda_3 - r\Lambda_4.
\end{aligned}$$

for all $t \in \mathrm{J}$. Operating the supremum-norm over t, we get that

$$\|((\mathrm{G}_1 u)(t))\| \leq (L_{\mathscr{F}}r + \mathscr{F}_0)\Lambda_3 - r\Lambda_4 \tag{2.24}$$

Similarly, proceeding with the analogous arguments, we obtain

$$\|((\mathrm{G}_2 v)(t))\| \leq (L_{\mathscr{G}}r + \mathscr{G}_0)\Lambda_5 - r\Lambda_6 \tag{2.25}$$

Adding the inequalities (2.24) and (2.26), we obtain

$$\|((\mathrm{G}(u,v)(t))\| \leq \left(\left[(L_{\mathscr{F}}r + \mathscr{F}_0)\Lambda_3 - r\Lambda_4\right] + \left[(L_{\mathscr{G}}r + \mathscr{G}_0)\Lambda_5 - r\Lambda_6\right]\right) \leq r. \tag{2.26}$$

which implies that $\mathrm{G}B_r \subset B_r$.

Now let $(u,v),(u_1,v_1)\in\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}$. Then, for $t\in$ J, we have

$$
\begin{aligned}
&\Big|\big((\mathrm{G}_1u)(t)-(\mathrm{G}_1v)(t)\big)(t-a)^{1-\gamma_1}\Big|\\
&\le\sup_{t\in \mathrm{J}}\Big\{(t-a)^{1-\gamma_1}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(s,u(s),v(s))-\mathscr{F}(s,u(s)_1,v_1(s))|(t)\right)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1^*}{p_1^*}|u(t)-u_1(t)|\right)\Big\}\\
&-\frac{\gamma_1(\gamma_1)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma_1}|\mathscr{F}(s,u(s),v(s))-\mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\}\\
&\le\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(t-a)^{1-\gamma_1}|\mathscr{F}(s,u(s),v(s))-\mathscr{F}(s,u(s)_1,v_1(s))|(t)\right)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1^*}{p_1^*}(t-a)^{1-\gamma_1}|u(t)-u_1(t)|\right)\Big\}\\
&-\frac{\gamma_1(\gamma_1)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma_1}|\mathscr{F}(s,u(s),v(s))-\mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\}\\
&\le\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(t-a)^{1-\gamma_1}L_{\mathscr{F}}\big(|u(t)-u_1(t)|+|v(t)-v_1(t)|\big)\right)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1^*}{p_1^*}(t-a)^{1-\gamma_1}|u(t)-u_1(t)|\right)\Big\}\\
&-\frac{\gamma_1(\gamma_1)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma_1}L_{\mathscr{F}}\big(|u(t)-u_1(t)|+|v(t)-v_1(t)|\big)(b)\Big\}\\
&\le\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p_1^*\gamma_1(\alpha_1+\alpha_2+1)}-\frac{\gamma_1(\gamma_1)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p_1^*\gamma_1(\alpha_1+1)}\right\}L_{\mathscr{F}}\|u-u_1\|\\
&-\left(\frac{q_1^*(b-a)^{\alpha_2}}{p_1^*\gamma_1(\alpha_2+1)}\right)\left(\|u-u_1\|_{C_{1-\gamma_1}}+\|v-v_1\|_{C_{1-\gamma_1}}\right)\\
&:=(L_{\mathscr{F}}\Lambda_3+\Lambda_4)\|u-u_1\|_{C_{1-\gamma_1}}+L_{\mathscr{F}}\Lambda_3\|v-v_1\|_{C_{1-\gamma_1}}.
\end{aligned}
$$

Thus

$$
\|((\mathrm{G}_1u)(t)-(\mathrm{G}_1v)(t))\|_{C_{1-\gamma}}\le(L_{\mathscr{F}}\Lambda_3+\Lambda_4)\|u-u_1\|_{C_{1-\gamma}}+L_{\mathscr{F}}\Lambda_3\|v-v_1\|_{C_{1-\gamma}}. \tag{2.27}
$$

Similarly, we obtain

$$
\|((\mathrm{G}_2v)(t)-(\mathrm{G}_2v_1)(t))\|_{C_{1-\gamma}}\le(L_{\mathscr{G}}\Lambda_5+\Lambda_6)\|v-v_1\|_{C_{1-\gamma}}+L_{\mathscr{G}}\Lambda_5\|u-u_1\|_{C_{1-\gamma}}. \tag{2.28}
$$

Adding the inequalities (2.27) and (2.28), we obtain

$$
\begin{aligned}
\|(\mathrm{G}(u,v))(t)-(\mathrm{G}(u_1,v_1))(t)\|_{C_{1-\gamma}}&\le\Big((L_{\mathscr{F}}\Lambda_3-\Lambda_4)+L_{\mathscr{G}}\Lambda_5\Big)\|u-u_1\|_{C_{1-\gamma}}\\
&+\Big(L_{\mathscr{F}}\Lambda_3+(L_{\mathscr{G}}\Lambda_5+\Lambda_6)\Big)\|v-v_1\|_{C_{1-\gamma}}\\
&\le\Delta_1\|u-u_1\|_{C_{1-\gamma}}+\Delta_2\|v-v_1\|_{C_{1-\gamma}}\\
&\le\big(\Delta_1+\Delta_2\big)\|w-\bar{w}\|_{C_{1-\gamma}}.
\end{aligned}\tag{2.29}
$$

We deduce that G is a contraction mapping. As a consequence of Banach contraction principle. the problem (2.14)-(2.15) has a unique solution on J. This completes the proof.

□

Existence result via Schaefer's fixed point theorem

Theorem 2.10. *Suppose that the conditions (As1) and (As3) hold. Then, the problem* (2.14)-(2.15) *has a least one solution in* J.

Proof. We shall use Schaefer's fixed point theorem to prove that G defined by (2.10) has a fixed point. The proof will be given in several steps.

Step 1 : G is continuous Let (u_n,v_n) be a sequence such that $(u_n,v_n)\to(u,v)$ in $\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}$. Then for each $t\in$ J,

$$\begin{aligned}
&\Big|\big((\mathrm{G}u_n)(t)-(\mathrm{G}u)(t)\big)(t-a)^{1-\gamma_1}\Big|\\
&\le \sup_{t\in \mathrm{J}}\Big\{(t-a)^{1-\gamma_1}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\Big(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))|(t)\Big)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\Big(\frac{q_1^*}{p_1^*}|u_n(t)-u(t)|\Big)\Big\}\\
&-\frac{\gamma_1(\gamma_1)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma_1}|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))|(b)\Big\}\\
&\le \mathscr{I}_{a^+}^{\alpha_2,\beta}\Big(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(t-a)^{1-\gamma_1}|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))|(t)\Big)\\
&-\mathscr{I}_{a^+}^{\alpha_2,\beta}\Big(\frac{q_1^*}{p_1^*}(t-a)^{1-\gamma_1}|u_n(t)-u(t)|\Big)\Big\}\\
&-\frac{\gamma_1(\gamma_1)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}(t-a)^{1-\gamma_1}|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))|(b)\Big\}\\
&:=\Lambda_3\|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))\|_{C_{1-\gamma_1}}+\Lambda_4\|u_n-u\|_{C_{1-\gamma_1}}
\end{aligned}$$

Thus

$$\|((\mathrm{G}_1u_n)(t)-(\mathrm{G}_1u)(t))\|_{C_{1-\gamma}}\le\Lambda_3\|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))\|_{C_{1-\gamma}}+\Lambda_4\|u_n-u\|_{C_{1-\gamma}}. \tag{2.30}$$

Similarly, we obtain

$$\|((\mathrm{G}_2v_n)(t)-(\mathrm{G}_2v)(t))\|_{C_{1-\gamma}}\le\Lambda_5\|\mathscr{G}(s,u_n(s),v_n(s))-\mathscr{G}(s,u(s),v(s))\|_{C_{1-\gamma}}+\Lambda_6\|v_n-v\|_{C_{1-\gamma}}. \tag{2.31}$$

Adding the inequalities (2.30) and (2.31), we obtain

$$\begin{aligned}
\|((\mathrm{G}(u_n,v_n))(t)-(\mathrm{G}(u,v))(t))\|_{C_{1-\gamma}}&\le\Lambda_3\|\mathscr{F}(s,u_n(s),v_n(s))-\mathscr{F}(s,u(s),v(s))\|_{C_{1-\gamma}}+\Lambda_4\|u_n-u\|_{C_{1-\gamma}}\\
&+\Lambda_5\|\mathscr{G}(s,u_n(s),v_n(s))-\mathscr{G}(s,u(s),v(s))\|_{C_{1-\gamma}}+\Lambda_6\|v_n-v\|_{C_{1-\gamma}}.
\end{aligned}\tag{2.32}$$

Since $\mathscr{F}$ and $\mathscr{G}$ are continuous, so $\|((\mathrm{G}(u_n,v_n))(t)-(\mathrm{G}(u,v))(t))\|_{C_{1-\gamma}}\to 0$ as $n\to\infty$.

Step 2 : G maps bounded sets into bounded sets in $\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}$.

Due to the first claim of G in Theorem 2.9, then G is a maps bounded sets into bounded sets in $\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}$ too.

Step 3 : G maps bounded sets into equicontinuous sets of $\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}$.

Let $t_1, t_2 \in \mathrm{J}, t_1 < t_2$, B_r be a bounded set of $\mathscr{C}_{1-\gamma} \times \mathscr{C}_{1-\gamma}$ as in Step 2, and let $(u,v) \in B_r$. Then

$$
\begin{aligned}
&\|((t_2-a)^{1-\gamma_1}\mathrm{G}u(t_2) - (t_1-a)^{1-\gamma_1}\mathrm{G}u(t_1))\| \\
&\leq \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{1}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}\left|(t_2-a)^{1-\gamma_1}\mathscr{F}(s,u(s),v(s))(t_2) - (t_1-a)^{1-\gamma_1}\mathscr{F}(s,u(s)_1,v_1(s))(t_1)\right|\right) \\
&- \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{q_1^*}{p_1^*}|(t_2-a)^{1-\gamma_1}u(t_2) - (t_1-a)^{1-\gamma_1}u(t_1)|\right) \\
&- \frac{\gamma_1(\gamma_1)\left((t_2-a)^{1-\gamma_1} - (t_1-a)^{1-\gamma_1}\right)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\} \\
&\leq \frac{L_1}{p_1^*\gamma_1(\alpha)}\left|(t_2-a)^{1-\gamma_1}\int_1^{t_1}(t_2-s)^{\alpha_1+\alpha_2-1}(1)ds - (t_1-a)^{1-\gamma_1}\int_1^{t_1}(t_1-s)^{\alpha_1+\alpha_2-1}(1)ds\right| \\
&+ \frac{L_1}{p_1^*\gamma_1(\alpha)}\left|(t_2-a)^{1-\gamma_1}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1+\alpha_2-1}(1)ds\right| + \frac{rq_1^*}{p_1^*\gamma_1(\alpha_1+\alpha_2)}\left|(t_2-a)^{1-\gamma_1}\int_1^{t_1}(t_2-s)^{\alpha_2-1}(1)ds\right. \\
&\left.- (t_1-a)^{1-\gamma_1}\int_1^{t_1}(t_1-s)^{\alpha_2-1}(1)ds\right| + \frac{rq_1^*}{p_1^*\gamma_1(\alpha_2)}\left|(t_2-a)^{1-\gamma_1}\int_{t_1}^{t_2}(t_2-s)^{\alpha_2-1}(1)ds\right| \\
&- \frac{\gamma_1(\gamma_1)\left((t_2-a)^{1-\gamma_1} - (t_1-a)^{1-\gamma_1}\right)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\} \\
&\leq \frac{p_1^*L_1}{\gamma_1(\alpha_1+\alpha_2+1)}\left((t_2-a)^{\alpha_1+\alpha_2-\gamma_1+1} - (t_1-a)^{\alpha_1+\alpha_2-\gamma_1+1}\right) \\
&+ \frac{p_1^*r}{q_1^*\gamma_1(\alpha_2+1)}\left((t_2-a)^{\alpha_2-\gamma_1+1} - (t_1-a)^{\alpha_2-\gamma_1+1}\right) \\
&- \frac{\gamma_1(\gamma_1)\left((t_2-a)^{1-\gamma_1} - (t_1-a)^{1-\gamma_1}\right)}{\gamma_1(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\}.
\end{aligned}
$$

which gives

$$
\begin{aligned}
\|\mathrm{G}_1u(t_2) - \mathrm{G}_1u(t_1)\|_{\mathscr{C}_{1-\gamma}} \leq& \frac{p_1^*L_1}{\Gamma(\alpha_1+\alpha_2+1)}\left((t_2-a)^{\alpha_1+\alpha_2-\gamma_1+1} - (t_1-a)^{\alpha_1+\alpha_2-\gamma_1+1}\right) \\
&+ \frac{p_1^*r}{q_1^*\Gamma(\alpha_2+1)}\left((t_2-a)^{\alpha_2-\gamma_1+1} - (t_1-a)^{\alpha_2-\gamma_1+1}\right) \\
&- \frac{\Gamma(\gamma_1)\left((t_2-a)^{1-\gamma_1} - (t_1-a)^{1-\gamma_1}\right)}{\Gamma(\alpha_2+\gamma_1)}\frac{(b-a)^{\alpha_2}}{p_1^*} \\
&\times \mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\}.
\end{aligned}
\tag{2.33}
$$

In a similar way, we get

$$\begin{aligned}\|\mathrm{G}_2 v(t_2) - \mathrm{G}_2 v(t_1)\|_{\mathscr{C}_{1-\gamma_2}} \le & \frac{p_2^* L_2}{\Gamma(\alpha_3+\alpha_4+1)}\left((t_2-a)^{\alpha_3+\alpha_4-\gamma_2+1} - (t_1-a)^{\alpha_2+\alpha_4-\gamma_2+1}\right) \\ & + \frac{p_2^* r}{q_2^*\Gamma(\alpha_4+1)}\left((t_2-a)^{\alpha_4-\gamma_2+1} - (t_1-a)^{\alpha_4-\gamma_2+1}\right) \\ & - \frac{\Gamma(\gamma_2)\Big((t_2-a)^{1-\gamma_2} - (t_1-a)^{1-\gamma_2}\Big)}{\Gamma(\alpha_4+\gamma_2)}\frac{(b-a)^{\alpha_4}}{p_2^*} \\ & \times \mathscr{I}_{a^+}^{\alpha_3}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\}. \end{aligned} \tag{2.34}$$

From (2.33) and (2.34), we obtain

$$\begin{aligned}\|\mathrm{G}(u,v)(t_2) - \mathrm{G}(u,v)(t_1)\|_{\mathscr{C}_{1-\gamma_1}\times\mathscr{C}_{1-\gamma_1}} \le & \frac{p_1^*(L_1)}{\Gamma(\alpha_1+\alpha_2+1)}\left((t_2-a)^{\alpha_1+\alpha_2-\gamma_1+1} - (t_1-a)^{\alpha_1+\alpha_2-\gamma_1+1}\right) \\ & + \frac{p_1^* r}{q_1^*\Gamma(\alpha_2+1)}\left((t_2-a)^{\alpha_2-\gamma+1} - (t_1-a)^{\alpha_2-\gamma+1}\right) \\ & - \frac{\Gamma(\gamma_1)\Big((t_2-a)^{1-\gamma_1} - (t_1-a)^{1-\gamma_1}\Big)}{\Gamma(\alpha_2+\gamma_1)} \\ & \times \frac{(b-a)^{\alpha_2}}{p_1^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\} \\ & + \frac{p_2^*(L_2)}{\Gamma(\alpha_3+\alpha_4+1)}\left((t_2-a)^{\alpha_3+\alpha_4-\gamma_2+1} - (t_1-a)^{\alpha_3+\alpha_4-\gamma_2+1}\right) \\ & + \frac{p_2^* r}{q_2^*\Gamma(\alpha_4+1)}\left((t_2-a)^{\alpha_2-\gamma_2+1} - (t_1-a)^{\alpha_4-\gamma_2+1}\right) \\ & - \frac{\Gamma(\gamma_2)\Big((t_2-a)^{1-\gamma_2} - (t_1-a)^{1-\gamma_2}\Big)}{\Gamma(\alpha_4+\gamma_2)} \\ & \times \frac{(b-a)^{\alpha_4}}{p_2^*}\mathscr{I}_{a^+}^{\alpha_3}|\mathscr{F}(s,u(s),v(s)) - \mathscr{F}(s,u(s)_1,v_1(s))|(b)\Big\} \end{aligned} \tag{2.35}$$

which implies $\|\mathrm{G}u(t_2) - \mathrm{G}u(t_1)\|_{\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}} \to 0$ as $t_1 \to t_2$, As consequence of Step1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that G is continuous and completely continuous.

Step 4 : A priori bounds.
Now it remains to show that the set $\bar{\Omega} = \{(u,v) \in \mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma} : (u,v) = \mu\mathrm{G}(u,v)$ for some $0 < \mu < 1\}$ is bounded.

For such $(u,v) \in \bar{\Omega}$. Thus, for each $t \in \mathrm{J}$, we have

$$u(t) \le \mu \Bigg\{ \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{1}{p_1}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}\right)(t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{q_1}{p_1}u\right)(t) \\ - \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\frac{(b-a)^{1-\gamma_1}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b),v(b))(t-a)^{\gamma_1+\alpha_2-1}\Bigg\}.$$

and

$$v(t) \le \mu \Bigg\{ \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{1}{p_2}\mathscr{I}_{a^+}^{\alpha_3}\mathscr{F}\right)(t,u(t),v(t)) - \mathscr{I}_{a^+}^{\alpha_4}\left(\frac{q_2}{p_2}v\right)(t) \\ - \frac{\Gamma(\gamma_2)}{\Gamma(\alpha_4+\gamma_2)}\frac{(b-a)^{1-\gamma_2}}{p_2(t)}\mathscr{I}_{a^+}^{\alpha_3}\mathscr{F}(b,u(b),v(b))(t-a)^{\gamma_2+\alpha_4-1}\Bigg\}.$$

For $\mu \in [0,1]$, let u be such that for each $t \in \mathrm{J}$

$$\begin{aligned}\|(\mathrm{G}_1u(t))(t-a)^{1-\gamma_1}\| \le L_1 \Bigg\{ & \mathscr{I}_{a^+}^{\alpha_2}\left(\frac{1}{p_1}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}\right)(t,u(t),v(t)) \\ & - \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}\frac{(b-a)^{1-\gamma_1}}{p_1(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b),v(b))(t-a)^{\gamma_1+\alpha_2-1}\Bigg\} \\ & - r\left\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q_1}{p_1}u\right)(t)\right\} \\ & := L_1\Lambda_3 + r\Lambda_4.\end{aligned}$$

and

$$\begin{aligned}\|(\mathrm{G}_2v(t))(t-a)^{1-\gamma}\| \le L_1 \Bigg\{ & \mathscr{I}_{a^+}^{\alpha_4,\beta}\left(\frac{1}{p_2}\mathscr{I}_{a^+}^{\alpha_3}\mathscr{F}\right)(t,u(t),v(t)) \\ & - \frac{\Gamma(\gamma)}{\Gamma(\alpha_4+\gamma)}\frac{(b-a)^{1-\gamma}}{p_2(t)}\mathscr{I}_{a^+}^{\alpha_3}\mathscr{F}(b,u(b),v(b))(t-a)^{\gamma+\alpha_4-1}\Bigg\} \\ & - r\left\{\mathscr{I}_{a^+}^{\alpha_4}\left(\frac{q_2}{p_2}v\right)(t)\right\} \\ & := L_2\Lambda_5 + r\Lambda_6.\end{aligned}$$

Thus

$$\|(\mathrm{G}(u,v)(t))\|_{\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}} \le \infty$$

This implies that the set $\bar{\Omega}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that G has a fixed point which is a solution on J of the problem (2.14)-(2.15).

□

Existence result via the Leray-Schauder nonlinear alternative

Theorem 2.11. *Assume the hypotheses (As4)-(As5) hold. Then the boundary value problem* (2.14)-(2.15) *has at least one solution on* J.

Proof. We shall use the Leray-Schauder theorem to prove that G defined by (2.21) has a fixed point. As shown in Theorem 2.10, we see that the operator G is continuous, uniformly bounded, and maps bounded sets into equicontinuous sets. So by the Arzela-Ascoli theorem G is completely continuous.
Let (u,v) be such that for each $t\in$ J, we take the equation $(u,v)=\lambda \mathrm{G}(u,v)$ for $\lambda\in(0,1)$ and let (u,v) be a solution. After that, the following is obtained.

$$\begin{aligned}
|u(t)(t-a)^{1-\gamma}| &\leq \lambda\Big\{(t-a)^{1-\gamma}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p}\mathscr{I}_{a^+}^{\alpha_1,\beta}\mathscr{F}\right)(t,u(t))-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q}{p}u\right)(t)\\
&\quad-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p(t)}\mathscr{I}_{a^+}^{\alpha_1}\mathscr{F}(b,u(b))(t-a)^{\gamma+\alpha_2-1}\Big\}\Big\}\\
&\leq (t-a)^{1-\gamma}\Big\{\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}|\mathscr{F}(t,u(t))|\right)-\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}|u(t)|\right)\Big\}\\
&\quad-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{1-\gamma}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}|\mathscr{F}(b,u(b))|(t-a)^{\alpha_2}\Big\}\\
&\leq \Phi_1(\|(u,v)\|)\|\omega_1\|\Big\{(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{1}{p^*}\mathscr{I}_{a^+}^{\alpha_1,\beta}(1)(t)\right)\\
&\quad-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_2-\gamma+1}}{p^*}\mathscr{I}_{a^+}^{\alpha_1}(1)(b)|\Big\}-r(b-a)^{1-\gamma}\mathscr{I}_{a^+}^{\alpha_2,\beta}\left(\frac{q^*}{p^*}(1)(t)\right)\\
&\leq \Phi_1(\|(u,v)\|)\|\omega_1\|\left\{\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+\alpha_2+1)}-\frac{\Gamma(\gamma)}{\Gamma(\alpha_2+\gamma)}\frac{(b-a)^{\alpha_1+\alpha_2}}{p^*\Gamma(\alpha_1+1)}\right\}-\|u\|\left(\frac{q^*(b-a)^{\alpha_2}}{p^*\Gamma(\alpha_2+1)}\right)\\
&:=\Phi_1(\|(u,v)\|)\|\omega_1\|\Lambda_3+\|u\|\Lambda_4.
\end{aligned}$$

Thus

$$\|u\|\leq\frac{\|\omega_1\|\Phi_1(\|(u,v)\|)\Lambda_3}{(1-\Lambda_4)}. \tag{2.36}$$

In a similar way, we get

$$\|v\|\leq\frac{\|\omega_2\|\Phi_2(\|(u,v)\|)\Lambda_5}{(1-\Lambda_6)}. \tag{2.37}$$

From (2.36) and (2.37), we obtain

$$\|(u,v)\|\leq\frac{\|\omega_1\|\Phi_1(\|(u,v)\|)\Lambda_3}{(1-\Lambda_4)}+\frac{\|\omega_2\|\Phi_2(\|(u,v)\|)\Lambda_5}{(1-\Lambda_6)}. \tag{2.38}$$

which leads to

$$\frac{\|(u,v)\|\big(1-\Lambda_4\big)\big(1-\Lambda_6\big)}{\big(1-\Lambda_6\big)\|\omega_1\|\Phi_1(\|(u,v)\|)\Lambda_3+\big(1-\Lambda_4\big)\|\omega_2\|\Phi_2(\|(u,v)\|)\Lambda_5}\leq 1. \tag{2.39}$$

In view of (As5), there exists ε such that $\|(u,v)\|\neq\bar{\varepsilon}$. Let us set $\bar{U}=\big\{(u,v)\in\mathscr{C}_{1-\gamma}\times\mathscr{C}_{1-\gamma}:\|(u,v)\|_{1-\gamma}<\bar{\varepsilon}\big\}$.

Obviously, the operator $\mathrm{G} : \overline{U} \to \mathscr{C}_{1-\gamma} \times \mathscr{C}_{1-\gamma}$ is completely continuous. From the choice of $\bar{U}$, there is no $(u,v) \in \partial \bar{U}$ such that $(u,v) = \lambda \mathrm{G}(u,v)$ for some $\lambda \in (0,1)$. As a result, by the Leray-Schauder's nonlinear alternative theorem, G has a fixed point $(u,v) \in \bar{U}$ which is a solution of the (2.1)-(2.2). The proof is completed.

□

2.3.3 Example

Example 2.12. Let us consider coupled system (2.14)-(2.15) with specific data :

$$p_1(t) \equiv 1, \quad p_2(t) \equiv 1, \quad q_1(t) = \lambda_1, \quad q_2(t) = \lambda_2, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

and

$$\begin{array}{llll} a = 0, & b = 1, & \alpha_1 = \dfrac{5}{8}, & \alpha_2 = \dfrac{1}{2}, \\ \alpha_3 = \dfrac{4}{5}, & \alpha_4 = \dfrac{3}{4} & ,\beta_1 = \dfrac{3}{7}, & \beta_2 = \dfrac{1}{5}. \end{array}$$

$$\gamma_1 = (\alpha_1 + \alpha_2)(1 - \beta_1) + \beta_1 = \left(\frac{5}{8} + \frac{1}{2}\right)\left(1 - \frac{3}{7}\right) + \frac{3}{7} \simeq 1.0714$$

$$\gamma_2 = (\alpha_3 + \alpha_4)(1 - \beta_2) + \beta_2 = \left(\frac{4}{5} + \frac{3}{4}\right)\left(1 - \frac{1}{5}\right) + \frac{1}{5} \simeq 1.4400$$

With

$$\mathscr{F}(t,u,v) = \frac{t}{\frac{e}{12}} \sin|u(t) + v(t)|, \quad t \in [0,1].$$

$$\mathscr{G}(t,u,v) = \frac{t\sqrt{|u(t)+v(t)|}}{23\sqrt{|u(t)+v(t)|+1}}, \quad t \in [0,1].$$

Clearly, the functions $\mathscr{F}$ and $\mathscr{G}$ are continuous.
For each $u \in \mathbb{R}^+$ and $t \in [0,1]$, we have

$$|\mathscr{F}(t,u(t),v(t))| \leq \left|\frac{t}{12}\sin\right| |u(t)+v(t)| \leq \frac{t}{12} \leq \frac{1}{12},$$

$$|\mathscr{G}(t,u(t),v(t))| \leq \frac{t}{23} \frac{\sqrt{|u(t)+v(t)|}}{\sqrt{|u(t)+v(t)|+1}} \leq \frac{t}{23} \leq \frac{1}{23},$$

and

$$|\mathscr{F}(t,u(t),v(t)) - \mathscr{F}(t,u_1(t),v_1(t))| \leq \frac{1}{12}(|u-u_1| + |v-v_1|),$$

$$|\mathscr{G}(t,u(t),v(t)) - \mathscr{G}(t,u_1(t),v_1(t))| \leq \frac{1}{23}(|u-u_1| + |v-v_1|).$$

Hence, the hypothesis (As 2) is satisfied with $L_{\mathscr{F}} = \frac{1}{12}$ and $L_{\mathscr{G}} = \frac{1}{23}$. Further,

$$\Lambda_1 :\simeq 0.2311, \quad \Lambda_2 :\simeq 0.1705 \tag{2.40}$$

and

$$\Lambda_1 :\simeq 0.0977, \quad \Lambda_2 :\simeq 0.1890 \tag{2.41}$$

From (2.40) and (2.41), we get

$$\Delta_1 + \Delta_2 \simeq 0.4033 < 1.$$

Therefore, by the conclusion of Theorem (2.9), It follows that the problem (2.14)-(2.15) has a unique solution defined on $[0, 1]$.

2.4 Conclusion

In this chapter, we have given some results of the existence and uniqueness of solutions for BVPs of nonlinear Sturm-Liouville-Langevin FDEs involving the Hilfer fractional derivative. As a first step, the BVP is turned to a fixed point problem. Based on this, the existence results are established via the generalized theorem of Schauder, Schaefer's and Banach's fixed point theorems. We give an example for each section to justify the theoretical results. We confirm that the results of this work are novel and generalize some previous works. Especially, problem (2.1) is formed as an overarching structure comprising both fractional SLE and LE, subjected to boundary conditions involving Hilfer FDs. In fact, choosing $q(t) \equiv 0$ on the one hand, and $p(t) \equiv 1$, $q(t) = \lambda$, $\lambda \in \mathbb{R}$, on the other hand, reduces the problem (2.1) into the fractional Sturm-Liouville problem and the fractional Langevin problem, respectively. our studied problem covers many problems that contain classical operators, which are incorporated into the operators used in our study.

Chapitre 3

Conclusion and Perspective

Our main scientific contributions in this Master project focused on the existence and uniqueness for different classes of fractional differential equations. We have shown the interest of the fractional derivative of Hilfer, for different classes of differential systems in the modeling of several complex phenomena, for which, it can be considered as an interpolant between the derivatives of Riemann - Liouville and Caputo.

In order to describe the various real-world problems in physical sciences and engineering, subject to abrupt changes for certain instants during the process of evolution, fractional differential equations have become important in recent years as mathematical models for many physical phenomena and social. This observation motivated us to use this condition, where we have devoted an important part of this thesis.

Another recent and active field is the notion of fractional derivative of variable order, where the fractional derivative can also be considered as a function. This notion has been successfully exploited in certain practical cases.

As perspectives, we project the study of dynamic and biomathematical models using fractional differential equations of variable orders.

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