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On the Existence of Solution for some Differential Equations of Fractional order with Nonlocal Conditions and Infinite Delay

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Dédicace

I dedicate this Modest work to those who supported me throughout my studies, who mount

always pushed towards the path of knowledge, to my source of love and affection, both the most expensive beings in the world

My dear parents :

Mohammed , Salha

My Grand Mother :

Bouhafsi Mordia

To my sister any brothers :

Mariem,Mohammed Alward,Haidar.

To my big brother and his wife and daughter

Abd-elkader, Saiida,Hanane

To my big **Baghaffar** family.

To my fiance **Abd-Elkader Boukarziya.**

To my friends :

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To all math , info students.

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Mordia Baghaffar
Ghardaia 2019

Résumé,Abstract

Résumé

Dans ce travail, nous étudions l'existence d'une solution faible d'une équation différentielle d'ordre fractionnaire avec conditions non locales et un retard infini impliquant une condition de Lipschitz sur le terme I_k. Nous nous appuierons sur un théorème de points fixes en tout que la somme de deux opérateurs l'un une contraction l'auter completement continu du à Burton et Krik. **Mots clés :(Dérivé fractionnaire; impulsif; Dérivé fractionnaire de Caputo; existence; retard dépendant de l'état; point fixe.)**

Abstract

In this work ,we study the existence of mild solution of some differential equation of fractional order with nonlocal condition and an infinite delay involving a Lipschitz condition on term I_k .We shall rely on a fixed point theorem for the sum of completly continuous and contraction operators due to Burton and Krik. **Key words(Fractional derivative ; impulsive; Caputo fractional derivative ;existence ;state-dependent delay;fixed -point .)**

Contents

1 Preliminaries **3**

1.1 **Fractionnal Calculus** 4

1.1.1 Riemann-Liouville fractional Integrel and derivatives : 4

1.1.2 Fractional Integral: 4

1.1.3 Fractional derivatives: 4

1.1.4 Caputo farctional derivative : 4

1.2 α **-resolvant familly** 5

1.3 Burton and Kirk fixed point 6

2 Study Of Existence Solution **7**

2.1 Problems 7

2.2 Existece study 9

3 Application **25**

Bibliography **27**

General Introduction

Our goal interest in this work is to study the existence of mild solution defined on a compact interval for some differential equation of fractional order with nonlocal condition and infinite delay in separable Banach space E of the form :

$$
\begin{aligned}
{}^cD^{\alpha}_{t_k}y(t) &= Ay(t) + f(t, y_{\rho(t,y_t)}); t \in J := [0, b], k = 0, 1, \ldots, m-1 \qquad (1)\\
\Delta y|_{y=y_k} &= I_k(y(t_k^-)), k = 1, \ldots, m-1 \qquad (2)\\
y(t) + h_t(y) &= \phi(t), t \in]-\infty, 0] \qquad (3)
\end{aligned}
$$

where $0 < \alpha \leqslant 1, f : J \times \mathcal{D} \longrightarrow E$ is a given function , $\mathcal{D}$ is the phase space defined axiomatically which contains the mapping from $]-\infty, 0]$into$E, \phi \in \mathcal{D}, \Delta|_{y=y_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h \longrightarrow 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \longrightarrow 0^+} y(t_k + h)$ represent the right and left limits of
$y(t)$ at $t = t_k, A : D(A) \subset E \longrightarrow E$ is generator of analytic $\alpha-$resolvant operator family (α-ROF for short) S_α
$0 = t_0 < t_1 < \ldots < t_m = b, I_k : \mathcal{D} \longrightarrow E(k = 1, 2, \ldots, m)$,
$\rho : J \times \mathcal{D} \longrightarrow]-\infty, b], A : D(A) \subset E \longrightarrow E$ is a densely defined closed linear operator on E , and E a real separable Banach space with norm $|.|$.For any function y defined on $(-\infty, b] \setminus \{t_1, t_2, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in]-\infty, 0]$$

$h_t : PC(]-\infty, T], E) \longrightarrow E$ is given function ,where $PC(]-\infty, T], E) = \{y :]-\infty, T] \longrightarrow E :$ $y(t)$ is continuous every where except for some t_k at which $y(t_k^+)$
and $y(t_k^-), k = 1, ..., m$ exist and $y(t_k^-) = y(t_k^+)$ which is a Banach space equipped with the norm

$$\|y\| = \sup\{|y(t)| : t \in]-\infty, T]\}.$$

We assume as usual in the theory of impulsive differential equation that the solution of (1) - (3) is such that at the point of discontinuity , t_k satisfies $y(t_k) = y(t_k^-)$.
Note that the concept of non-local conditions was initiated by Byzewski, proved in [8] that the non-local condition may be more helpful in describing certain physical phenomenon . Since then a series of studies of problems with non-local condition started to apear .Deng in [21] used the non-local condition to describe the phenomenon of diffusion of a small amount of gas in tansparent tube where

$$h_t(y) = \sum_{i=1}^{p} c_i y(t_i)$$

The study of fractional differential equation is linked to the wide applications of fractional calculus in physics , quantum mechanics ,signal processing , and electromagnitics .

The theory of fractional differential equation has seen considerable devlopment using different technics .Some existence results were given in the book by Abbas at all [18] and the papers by Hammouche at all . [5], Wang at all . [14], Balachandran at all [15], Shu at all. [21] and the references therien . The literature related to ordinary and partial functional differential equations with delay for which $\rho(t, \psi) = t$ is very extensive , see for instance the books by Hale [9] , Hale and Verduyn Lunel [10] , Kolmanovskii , Myshkis [20] and Wu [11] and the references therien .
This work is organized as follows : in chapter one , we will recall briefly some basic definition and preliminary facts which will be used thoughout the following sectins .In second chaper we prove the existence of integral solution for the problem (1)-(3) .Our approach will be based , for the existence of integral solutions, on a fixed point theorem of Burton and Kirk for the sum of a contraction map and a completely continuous map .Finally in third chapter we give an example of an application of our study.

Chapter 1

Preliminaries

In this chapter , we introduce notations , definitions ,lemmas and fixed point theorem which are used throughout this memory .
Let$\mathcal{D}$ the linear space of function mapping $]-\infty,0]$ into E endowed with a semi-norm $\|.\|_{\mathcal{D}}$ see [22] For $\psi \in \mathcal{D}$, the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup|\psi(\theta)|; \theta \in]-\infty,0].$$

$PC[J,E]$ is the banach space of all continuous function from J into E with the norm

$$\|u\| = \sup|u(t)| : t \in J.$$

(A1) there exist a positive constante H and function $K(.), M(.) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with K continuous and M locally bounded , such that for any $b > 0$, if $y : (-\infty,b] \longrightarrow E, y_0 \in \mathcal{D}$, and $y|_J \in PC(J,E)$,then for every $t \in [0,b]$ the following conditions hold :
(i) y_t is in$\mathcal{D}$;
(ii)$|y(t)| \leqslant H\|y_t\|_{\mathcal{D}}$;
(iii) $\|y_t\|_{\mathcal{D}} \leqslant K(t)\sup|y(s)| : 0 \leqslant s \leqslant t + M(t)\|y_0\|_{\mathcal{D}}$, and H, K and M are independent of $y(.)$
Denote

$$K_b = \sup\{K(t) : t \in J\} \text{ and } M_b = \sup\{M(t) : t \in J\}.$$

(A2) The space $\mathcal{D}$ is complete .
$L^1[J,E]$ is the Banach space of mesurable function $u : J \longrightarrow E$ which are Bochner integrable normed by

$$\|u\|_{L^1} = \int_0^b |u(t)|dt.$$

To consider the impulsive condition (3) ,it is convenient to introduce some additional concepts and notations.Consider the following spaces

$$PC(J,E) = \{y : [0,b] \to E : y \text{is continuous at } t \neq t_k, y(t_k^-) = y(t_k) \text{ and } y(t_k^+) \text{ exists} \, \forall k = 1,...,m\}$$

and

$$\mathcal{B}_b = \{y :]-\infty,b] \to E : y|_{]-\infty,0]} \in \mathcal{D}\text{and}y|_J \in PC(J,E)\}.$$

Let $\|.\|_b$be the semi-norm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y_0\|_{\mathcal{D}} + \sup\{|y(s)| : 0 \leqslant s \leqslant b\}, y \in \mathcal{B}_b.$$

In this work .we use an axiomatic definition for the phase space $\mathcal{D}$ which is similar to those introduced in [22] Sepecifically , $\mathcal{D}$ will be a linear space of function mapping $(-\infty,0]$ into

E endowed with a semi - norm $\|.\|_{\mathcal{D}}$, and satisfies the following axioms introduced at first by Hale and Kato in [7] Also $B(E)$denote the Banach space of bounded linear operators from E to E , with norm

$$\|N\|_{B(E)} = sup\{|N(y)| : |y| = 1\}.$$

Let $K \subset E$, denote by $\overline{Conv(K)}$ the closure of the convex hull of the set K.

1.1 Fractionnal Calculus

1.1.1 Riemann-Liouville fractional Integrel and derivatives :

1.1.2 Fractional Integral:

***Definition* 1.1.** The α- Riemann - Liouville fractional -order derivative of f ,is defined by

$$(I_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, (x > a; Re(\alpha)$$

1.1.3 Fractional derivatives:

***Definition* 1.2.** The α- Riemann - Liouville fractional -order derivative of f , is defined by

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)ds,$$

where $n = [\alpha] + 1 and [\alpha]$ denotes the integer part of α.

1.1.4 Caputo farctional derivative :

***Definition* 1.3.** : For a fonction f define on the interval $[a, b]$, the Caputo fractional-order derivative of order α of f , is defined by

$$({}_{a^+}{}^c D_t^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{(n-\alpha-1)} f^n(s)ds.$$

where $n = [\alpha] + 1]$.

Therefore ,for $0 < \alpha < 1, n = [\alpha]+1 = 1$ and for $a = 0$,the Caputo's fractional derivative for $t \in [0, b]$ is given by

$$({}_0^c D_t^{\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds,$$

In order to defined the mild solution to problem (1)-(3), we consider the following space :

$$PC = PC([-\infty, 0], E) = \{y : [-\infty, 0] \longrightarrow E; y \in C([0, b], E); K = 0, 1, 2, \dots, m \\ \text{such that } y(t_k^-), y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), k = 1, 2, \dots, m\}$$

PC is a Banach space equipped with the following norm :

$$\|y\|_{PC} := max\|y_k\|_{\infty} : k = 1, 2, \dots, m$$

Let us introduce the definition of Caputo's derivative in each interval $[0, b]$,

$$({}^c D_{t_k}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_k}^t (t-s)^{-\alpha} f'(s)ds$$

1.2 α -resolvant familly

Let A be a densely defined operator; $A : D(A) : E \longrightarrow E$

***Definition* 1.4.** :A family $(S_\alpha(t))_{t\geqslant 0} \subset B(E)$ of bounded linear operators in E is called an α-resolvent operator function generatet by A if the following conditions hold:
a)$(S_\alpha(t))_{t\geqslant 0}$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$.
b)$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$ and $t \geqslant 0$.
c)For all $x \in E$,$I_t^\alpha S_\alpha(t)x \in D(A)$ and

$$S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x, t \geqslant 0.$$

d)$x \in D(A)$and$Ax = y$ if only if

$$S_\alpha(t)x = x + AI_t^\alpha S_\alpha(t)x, t \geqslant 0.$$

e)A is closed and densely defined.The generator A of $(S_\alpha(t))_t \geqslant 0$ is defined by

$$D(A) := \left\{ x \in X : \lim_{t \longrightarrow 0^+} \frac{S_\alpha(t)x - x}{\psi_{\alpha+1}(t)} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \longrightarrow 0^+} \frac{S_\alpha(t)x - x}{\psi_{\alpha+1}(t)}, x \in D(A),$$

where $\psi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$ and $\psi_\alpha(t) = 0$ for $t \leqslant 0$ and $\psi_\alpha(t) \longrightarrow \delta(t) as\ \alpha \longrightarrow 0$, where the function delta is defined by

$$\delta_a : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}; \phi \longrightarrow \phi(a)$$

and

$$\mathcal{D}(\Omega) = \{\phi \in C^\infty(\Omega); supp\phi \subset \Omega \text{ is compact}\}$$

***Definition* 1.5.** : An α-ROF $(S_\alpha(t))_{t\geq 0}$ is called analitic, if the function $S_\alpha(t) : \mathbb{R}_+ \longrightarrow B(E)$ admity analytic extension to a sector $\sum(0, \theta_0)$for some $0 < \theta_0 < \dfrac{\pi}{2}$.
An analytic α-ROF$(S_\alpha)$is said to be of analyticity type (ω_0, θ_0)if for each $\theta < \theta_0$and $\omega > \omega_0$ there exists $M_1 = M_1(\omega, \theta)$ such that

$$\|S_\alpha(z)\| \leq M_1 e^\omega Rez$$

for $z \in \sum(0, \theta)$, where Rez denotes the real part of z and

$$\sum(\omega, \theta) := \{\lambda \in \mathbb{C} \mid arg(\lambda - \omega \mid < \theta, \omega\theta \in \mathbb{R}\}$$

***Definition* 1.6.** : An α-ROF $(S_\alpha(t))_{t\geq 0}$ is called compact $t > 0$ if for every $t > 0$,$S_\alpha(t)$ is a compact operator .

***Theorem* 1.7.** *:([23]) Let A generate a compact analytic semigroup $(T(t))_{t\geq 0}$, then for any $\alpha \in (0, 1)$,it also generates a compact analytic resolvent family $(S_{\alpha(t)})_{t\geq 0}$*

***Lemma* 1.1.** *: ([23, 24]) Assum that the α-ROF $(S_\alpha(t))_t \geq 0$ is compact for $t > 0$ and analytic of type (ω_0, θ_0).Then the following assertions hold :*
i) $\lim_{h \longrightarrow 0} \| S_\alpha(t + h) - S_\alpha(t) \| = 0$, *for $t > 0$;*
ii) $\lim_{h \longrightarrow 0^+} \| S_\alpha(t + h) - S_\alpha(h)S_\alpha(t) \| = 0$ *, for $t > 0$;*

***Definition* 1.8.** : An α-ROF $(S_\alpha(t))_t \geq$ is said to be exponentially bounded if there exist constants $M \geq 1$, $\omega \geq 0$ such that

$$\| S_\alpha(t) \| \leq Me^{\omega}t, fort \geq 0$$

in this case we write $A \in C_\alpha(M, \omega)$.

***Proposition* 1.1.** *:Let $\alpha > 0. A \in C_\alpha(M, \omega)$ if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there exists a strongly continuous function $S_\alpha : \mathbb{R}_+ \longrightarrow B(X)$ such that*

$$\| S_\alpha(t) \| \leq Me^{\omega t}, \quad for\ t \geq 0$$

and

$$\int_0^\infty Me^{-\lambda t} S_\alpha(t) x dt = \lambda^{\alpha-1} R(\lambda^\alpha, A)x, \lambda > \omega and x \in E$$

Furthermore , $(S_\alpha(t))_{t\geq 0}$ is the α-ROF generated by the operator A.

***Definition* 1.9.** A map $f : [0, b] \times D \longrightarrow E$ is said to be carathéodory if it satisfies the following condition :
(1) the function $t \longmapsto f(t, x)$ is measurable for each $x \in D$;
(2) the function $t \longmapsto f(t, x)$ is continuous for almost all $t \in J := [0, b], k = 0, 1, \ldots, m.$

1.3 Burton and Kirk fixed point

***Theorem* 1.10.** *Let E be a banach space and $A, B : E \longrightarrow E$ be two operators satisfying :*
i) A is contraction
ii) B is completely continuous
Then either:
a) the operator equation $y = A(y) + B(y)$ has a solution , or
b) the set $\Upsilon = \{u \in E : \lambda A\Big(\dfrac{u}{\lambda}\Big) + \lambda B(u) = u, \lambda \in (0, 1)\}$ is unbounded.

***Note* 1.1.** *the operator B is completly continuous if it is continuous and maps any bounded subset of D into a relatively copmact subset of E.*

Chapter 2

Study Of Existence Solution

2.1 Problems

We consider the problem of fractional differential equation in separable Banach space E :

$$ {}^cD^{\alpha}_{t_k}y(t) = Ay(t) + f(t, y_{\rho(t,y_t)}); t \in J := [0, b] \quad (2.1)$$
$$ \Delta y|_{y=y_k} = I_k(y(t_k^-)), k = 1, \ldots, m-1 \quad (2.2)$$
$$ y(t) + h_t(y) = \phi(t), t \in]-\infty, 0] \quad (2.3)$$

where $0 < \alpha \leqslant 1, f : J \times \mathcal{D} \longrightarrow E$ is a given function , $\mathcal{D}$ is the phase space defined axiomatically which contains the mapping from $]-\infty, 0]$into$E, \phi \in \mathcal{D}$,
$0 = t_0 < t_1 < \ldots < t_m = b, I_k : \mathcal{D} \longrightarrow E(k = 1, 2, \ldots, m-1)$,
$\rho : J \times \mathcal{D} \longrightarrow]-\infty, b], A : D(A) \subset E \longrightarrow E$ is a densely defined closed linear operator on E , and E a real separable Banach space with norm $|.|$.For any function y defined on $(-\infty, b] \setminus \{t_1, t_2, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in]-\infty, 0]$$

$h_t : PC(]-\infty, T], E) \longrightarrow E$ is given function , where $PC(]-\infty, T], E) = \{y :]-\infty, T] \longrightarrow E : y(t)$ is continuous everywhere except for some $\{ t_k$ at which $y(t_k^-)$ and ,
$k = 1, ..., m-1$ exist and $y(t_k^-) = y(t_k^+)\}$ which is a Banach space equipped with the norm

$$\|y\| = sup\{|y(t)| : t \in]-, T]\}.$$

Lemma 2.1. *: A function $y \in PC([-\infty, 0]$ is a mild solution of problem (2.1)-(2.3) if $y(t) = \phi(t) - h_t(y), t \in [-\infty, 0], \Delta y|_y = y_k = I_{k(y(t_k^-))}, k = 1...m-1$ and such that y satisfies the flowing integral equation :*

$$y(t) = \begin{cases} S_\alpha(t)(\phi(0) - h_0(y)) + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds, & \text{if } t \in [0, t_1]; \\ S_\alpha(t - t_k) \prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(y) + \sum_{i=1}^{k} \int_{i-1}^{t_i} S_\alpha(t - t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) S_\alpha(t_i - s) \\ f(s, y_{\rho(s,y_s)}) ds + \int_{t_k}^{t} S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + \sum_{i=1}^{k} S_\alpha(t - t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), \\ & \text{if } t \in (t_k, t_{k+1}). \end{cases}$$

Proof. For $t \in [0, t_1]$ the previous problem is given:

$$ {}^cD^{\alpha}_{t_1}y(t) = Ay(t) + f(t, y_{\rho(t,y_t)}); t \in [0, t_1]$$
$$ y(t) + h_t(y) = \phi(t), t \in]-\infty, 0]$$

the solution of previous problems is :

$$y(t) = s_\alpha(t-0)(\phi(0) - h_0(y) + \int_0^t s_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds, t \in [0, t_1]$$

For $k = 2$ we have that $I_k(y(t_k^-)) = y(t_k^+) - y(t_k^-) \Leftrightarrow y(t_k^+) = I_k(y(t_k^-)) + y(t_k^-)$
Then ,$t \in [t_1, t_2]$
$y(t) = S_\alpha(t-t_1) y(t_1^+) + \int_{t_1}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds$
$= S_\alpha(t-t_1)(I_1(y(t_1^-)) + y(t_1^-)) + \int_{t_1}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds$
$= S_\alpha(t-t_1) y(t_1^-) + \int_{t_1}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + S_\alpha(t-t_1)(I_1(y(t_1^-))$
$= S_\alpha(t-t_1)\Big(S_\alpha(t_1)(\phi(0) - h_0(y)) + \int_0^{t_1} S_\alpha(t_1 - s) f(s, y_{\rho(s,y_s)}) ds$
$+ \int_{t_1}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + S_\alpha(t-t_1) I_1(y(t_1^-))\Big)$
$= S_\alpha(t-t_1) s_\alpha(t_1)(\phi(0) - h_0(y)) + S_\alpha(t-t_1) \int_0^{t_1} S_\alpha(t_1 - s) f(s, y_{\rho(s,y_s)}) ds$
$+ \int_{t_1}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + S_\alpha(t-t_1) I_1(y(t_1^-))$.
And for $t \in [t_2, t_3]$
$y(t) = S_\alpha(t-t_2) y(t_2^+) + \int_{t_2}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds$
$= S_\alpha(t-t_2)(I_2(y(t_2^-)) + y(t_1^-)) + \int_{t_2}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds$
$= S_\alpha(t-t_2) y(t_2^-) + \int_{t_2}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + S_\alpha(t-t_2)(I_2(y(t_2^-))$
$= S_\alpha(t-t_2)\Big(S_\alpha(t_2 - t_1) S_\alpha(t_1)(\phi(0) - h_0(y)) + S_\alpha(t_2 - t_1) \int_0^{t_1} S_\alpha(t_1 - s) f(s, y_{\rho(s,y_s)} ds$
$+ \int_{t_1}^{t_2} S_\alpha(t_2 - s) f(s, y_{\rho(s,y_s)}) ds + S_\alpha(t_2 - t_1) I_1(y(t_1^-))\Big) + \int_{t_2}^t S_\alpha(t-s) f(s, y(s)) + S_\alpha(t-t_2)(I_2(y(t_2^-))$
$= S_\alpha(t-t_2) \prod_{i=1}^{2} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(y)) + \sum_{i=1}^{2} \int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) S_\alpha(t_i - s) f(s, y_{\rho(s,y_s)}) ds$
$+ \int_{t_2}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + S_\alpha(t-t_2) I_2(y(t_2^-))$
We can deduce that for $t \in [t_k, t_{k+1}]$. the solution of our problems is :

$$y(t) = S_\alpha(t-t_k) \prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(y)) + \sum_{i=1}^{k} \int_{i-1}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) S_\alpha(t_i - s) f(s, y_{\rho(s,y_s)}) ds$$

$$+ \int_{t_k}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), \text{ if } t \in (t_k, t_{k+1})$$

Now assume that the result is realized for $n \in \mathbb{N}$

$$y(t) = S_\alpha(t-t_n) \prod_{i=1}^{n} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(y)) + \sum_{i=1}^{n} \int_{i-1}^{t_i} S_\alpha(t-t_n) \prod_{j=i}^{n-1} S_\alpha(t_{j+1} - t_j) S_\alpha(t_i - s) f(s, y_{\rho(s,y_s)}) ds$$

$$+ \int_{t_n}^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds + \sum_{i=1}^{n} S_\alpha(t-t_n) \prod_{j=i}^{n-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), \text{ if } t \in (t_k, t_{n+1})$$

And proved its validity for $n+1 \in \mathbb{N}$

Let $y(t) = S_\alpha(t-t_{n+1})y(t_{n+1}^{+}) + \int_{t_{n+1}}^{t} S_\alpha(t-s)f((s, y_{\rho(s,y_s)})ds.$

$= S_\alpha(t-t_{n+1})[I_{n+1}y(t_{n+1}^{-}) + y(t_{n+1}^{-})] + \int_{t_{n+1}}^{t} S_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds.$

$= S_\alpha(t-t_{n+1})y(t_{n+1}^{-}) + \int_{n+1}^{t} S_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds + S_\alpha(t-t_{n+1})I_{n+1}(y(t_{n+1}^{-})).$

$= S_\alpha(t-t_{n+1})\Big(S_\alpha(t_{n+1}-t_n)\prod_{i=1}^{n} S_\alpha(t_i-t_{i-1})(\phi(0)-h_0(y))+\sum_{i=1}^{n}\int_{i-1}^{t_i} S_\alpha(t_{n+1}-t_n)\prod_{j=i}^{n-1} S_\alpha(t_{j+1}-t_j)$

$S_\alpha(t_i-s)f(s, y_{\rho(s,y_s)})ds + \int_{t_n}^{t_{n+1}} S_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds + \sum_{i=1}^{n} S_\alpha(t_{n+1}-t_n)\prod_{j=i}^{n-1} S_\alpha(t_{j+1}-t_j)I_i(y(t_i^{-}))\Big)$

$+ \int_{t_{n+1}}^{t} S_\alpha(t-s)f(s, y_{\rho_{(s,y_s)}})ds + S_\alpha(t-t_{n+1})I_{n+1}(y(t_{n+1}^{-})).$

$= S_\alpha(t-t_{n+1})S_\alpha(t_{n+1}-t_n)\prod_{i=1}^{n} S_\alpha(t_i-t_{i-1})(\phi(0)-h_0(y))+S_\alpha(t-t_{n+1})\sum_{i=1}^{n}\int_{i-1}^{t_i} S_\alpha(t_{n+1}-t_n)$

$\prod_{j=i}^{n-1} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)f(s, y_{\rho_{(s,y_s)}})ds + S_\alpha(t-t_{n+1})\int_{t_n}^{t_{n+1}} S_\alpha(t-s)f(s, y_{\rho_{(s,y_s)}})ds$

$+ S_\alpha(t-t_{n+1})\sum_{i=1}^{n} S_\alpha(t_{n+1}-t_n)\prod_{j=i}^{n-1} S_\alpha(t_{j+1}-t_j)I_i(y(t_i^{-})) + \int_{t_{n+1}}^{t} S_\alpha(t-s)f(s, y_{\rho_{(s,y_s)}}ds$

$+ S_\alpha(t-t_{n+1})I_{n+1}(y(t_{n+1}^{-})).$

Thus

$$y(t) = S_\alpha(t-t_{n+1})\prod_{i=1}^{n+1} S_\alpha(t_i-t_{i-1})(\phi(0)-h_0(y))+\sum_{i=1}^{n+1}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_{n+1})\prod_{j=i}^{n} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)$$

$$f(s, y_{\rho(s,y_s)})ds\int_{t_{n+1}}^{t} S_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds+\sum_{i=1}^{n+1} S_\alpha(t-t_{n+1})\prod_{j=i}^{n} S_\alpha(t_{j+1}-t_j)I_{n+1}(y(t_{n+1}^{-})).$$

So the solution is verified for each natural number n □

2.2 Existece study

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{D}, \rho(s,\phi) \leqslant 0\}.$$

We always assume that $\rho : I \times \mathcal{D} \longrightarrow (-\infty, b]$ is continuous .Additionally , we introduce the following hypotheses :

(H_ϕ) The function $t \longrightarrow \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{D}$ and there existe a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \longrightarrow (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{D}} \leqslant L^\phi(t)\|\phi\|_{\mathcal{D}} \text{ for every } t \in \mathcal{R}(\rho^-).$$

(H_1) A generate a compact and analytic α-ROF $(S_\alpha(t))_{t\geq 0}$ which is exponentially bounded i.e .there exist constants $M \geq 1, \Omega \geq 0$ such that

$$\|S_\alpha(t)\| \leqslant Me^{wt}, t \geq 0$$

(H_2) The function $f : J \times \mathcal{D} \longrightarrow E$ is continuous and there existes a constant $N > 0$ such that

$$\|f(t,y_1) - f(t,y_2)\| \leqslant N\|y_1 - y_2\|, \forall y_1, y_2 \in \mathcal{D}$$

(H_3) The function $I_k : E \longrightarrow E$are Lipschitz .Let M_k,for $k = 1, 2, ..., m$ be such that

$$\|I_k(y) - I_k(x)\| \leqslant M_k\|y - x\| \text{ for each } y, x \in E$$

and

$$1 - \sum_{i=1}^{k} M^{k-i+1}e^{w(b-t_i)}M_i > 0$$

(H_4) The function $f : J \times \mathcal{D} \longrightarrow E$ is Carathéodory ;
(H_5) The function $h_t : PC(]0, -\infty], E) \longrightarrow E$ is continuous with respect to t , and there exists a constant $G > 0$ such that

$$\|h_t(y_1) - h_t(y_2)\| \leqslant L\|y_1 - y_2\|, \text{for all } y_1, y_2 \in PC(] - \infty, 0], E);$$

with

$$L + GN\Big(\frac{1}{w}e^{wt_1} - \frac{1}{w}\Big) < 1$$

and
(H_6) There existe a function $p \in L^1(J, \mathbb{R}_+)$and a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that $|f(t, y)| \leqslant p(t)\psi(\|y\|_{\mathcal{D}})$,a.e $t \in J$,for all $y \in \mathcal{D}$ with $\displaystyle\int_{C_0}^{\infty} \frac{du}{\psi(u)} = \infty$and $\displaystyle\int_{C_3}^{\infty} \frac{du}{\psi(u)} = \infty$, where

$$C_0 = Me^{wb}\|\phi(0) - h_0(x(t) + \tilde{\phi}(t))\|, C_3 = min(C_1, C_2),$$

$$\bullet\, C_1 = \Bigg(M^{k+1}e^{wb}\|\phi(0) - h_0(x(t) + \tilde{\phi}(t))\| + \sum_{i=1}^{k} M^{k-i+1}e^{w(b-t_i)}|I_i(0)| + \sum_{i=1}^{k} M^{k-i+2}e^{w(b-t_{k-1})}$$
$$\times \int_{t_{i-1}}^{t_i} e^{-ws}p(s)\psi(K_b|x(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}})ds\Bigg) \Bigg/ \Bigg(1 - \sum_{i=1}^{k} M^{k-i+1}e^{w(b-t_i)}K_bM_i\Bigg).$$

$$\bullet C_2 = \frac{Me^{wb}}{1 - \sum_{i=1}^{k} M^{k-i+1}e^{w(b-t_i)}K_bM_i}.$$

(H_7)

$$[LM^{k+1}e^{wt} + \sum_{i=1}^{k} M_iM^{k-i+1}e^{w(t-t_i)}]K_b < 1.$$

Lemma 2.2. *: [12] If $y : (-\infty, b] \longrightarrow E$ is a function such that $y_0 = \phi$ and $y|_J \in PC(J : D(A))$, then $\|y_s\|_{\mathcal{D}} \leqslant (M_b + L^{\phi})\|\phi\|_{\mathcal{D}} + K_b sup\{\|y(\theta)\|; \theta \in [0, max\{0, s\}]\}, s \in \mathcal{R}(\rho^-) \cup J$, where $L^{\phi} = \sup_{t \in \mathcal{R}(\rho^-)} L^{\phi}(t), M_b = \sup_{t \in J} M(t)$ and $K_b = \sup_{t \in J} K(t)$.*

Theorem 2.1. *Assume that (H_{ϕ}) and $(H_1) - (H_7)$ hold .Then the problem (2.1)-(2.3) has at least one mild solution on $] - \infty, 0]$.*

Proof. Transform the problem (2.1)-(2.3) into a fixed point problem .Consider the operator

$N : PC(]-\infty,0],E) \longrightarrow PC(]-\infty,0],E)$ defined by :

$$N(y)(t) = \begin{cases} \phi(t) - h_t(y), & \text{if } t \in]-\infty, 0] \\ S_\alpha(t)(\phi(0) - h_0(y) + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds \ \text{ if } \ t \in [0, t_1]; \\ S_\alpha(t-t_k) \prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(y) + \sum_{i=1}^{k} \int_{i-1}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) \\ S_\alpha(t_i - s) f(s, y_{\rho(s,y_s)}) ds + \int_{t_k}^{t} S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds \\ + \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), & \text{if } t \in (t_k, t_{k+1}). \end{cases}$$

Let $\tilde{\phi}(.) :]-\infty, b] \longrightarrow E$ be the function defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) - h_t(y), & t \in (-\infty, 0] \\ S_\alpha(t)(\phi(0) - h_0(y)) + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds & \text{if } \ t \in [0, t_1] \end{cases}$$

Then $\tilde{\phi}_0 = \phi(t) - h_t(y)$. For each $x \in \mathcal{B}_b$ with $x(0) = 0$, we denote by $\overline{x}$ the function defined by

$$\overline{x}(t) = \begin{cases} 0, t \in (-\infty, 0] \\ x(t), t \in J. \end{cases}$$

If $y(.)$ satisfies (lemma 2.2) , we can decompose it as $y(t) = \tilde{\phi}(t) + x(t), 0 \leqslant t \leqslant b$ which implies $y_t = x_t + \tilde{\phi}_t$, for every $0 \leqslant t \leqslant b$ and the function $x(.)$ satisfies
For $t \in [0, t_1]$

$$x(t) = S_\alpha(t)(\phi(0) - h_0(y)) + \int_0^t S_\alpha(t-s) f(s, y_{\rho(s,y_s)}) ds \ \text{ if } \ t \in [0, t_1]$$

For $t \in [t_k, t_{k+1}]$

$$\begin{aligned} x(t) = S_\alpha(t-t_k) \prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) \\ + \sum_{i=1}^{k} \int_{i-1}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) S_\alpha(t_i - s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds \\ + \int_{t_k}^{t} S_\alpha(t-s) f(s, x_{\rho(s,x_s^n+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds + \sum_{i=1}^{k} S_\alpha(t-t_k) \\ \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(x_{t_i^-} + \tilde{\phi}_{t_i^-}), if t \in J. \end{aligned}$$

Let

$$\mathcal{B}_b^0 = \{x \in \mathcal{B}_b : x_0 = 0 \in \mathcal{D}\}.$$

For any $x \in \mathcal{B}_b^0$ we have

$$\|x\|_b = \|x_0\|_{\mathcal{D}} + \sup\{|x(s)| : 0 \leqslant s \leqslant b\} \sup\{|x(s)| : 0 \leqslant s \leqslant b\}.$$

Thus $(\mathcal{B}_b^0, \|.\|_b)$ is a Banach space . We define the operators $\mathcal{A}, \mathcal{B} : \mathcal{B}_b^0 \longrightarrow \mathcal{B}_b^0$ by :

$$\mathcal{A}(y)(t) = \begin{cases} \phi(0) - h_0(y), \quad \text{if } t \in]-\infty, 0] \\ S_\alpha(t)(\phi(0) - h_0(x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})) + \int_0^t S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi})})ds, \\ \quad \text{if } \ t \in [0, t_1] \\ S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})(\phi(0) - h_0(y)) \sum_{i=1}^{k} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) I_i(y(t_i^-)), \\ \hfill \text{if } t \in (t_k, t_{k+1}). \end{cases}$$

and

$$\mathcal{B}(y)(t) = \begin{cases} 0, if t \in]-\infty, 0] \\ \int_0^t S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds, \hfill \text{if } t \in [0, t_1]; \\ \sum_{i=1}^{k} \int_{i-1}^{t_i} S_\alpha(t-t_k) \prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j) S_\alpha(t_i - s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \\ + \int_{t_k}^{t} S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds, \hfill \text{if } t \in (t_k, t_{k+1}). \end{cases}$$

Then the solution of the problem (2.1)-(2.3) is reduced to finding the solution of operator equation $\mathcal{A}(y)(t) + \mathcal{B}(y)(t) = y(t), t \in [-\infty, 0]$,We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$satisfy all conditions of Theorem (1.10) we give the proof into a sequence of steps .
step 1. $\mathcal{B}$ is continous
Let $(x_n)_{n\geq 0}$ be a sequence such that $x_n \longrightarrow x$ in $\mathcal{B}_b^0$ then for $w > 0 (if w < 0 one has e^{wt} < 1)$.
At first , we study the convergence of the sequence $(x^n_{\rho(s,x^n_s)})_{n\in\mathbb{N}}, s \in J$. We distinguish two cases . Let $s \in J$ be such that $\rho(s, x^n_s) > 0$ for every $n > N$. In this case , for $n > N$ we see that

$$\begin{aligned} \|x^n_{\rho(s,x^n_s)} - x_{\rho(s,x_s)}\|_{\mathcal{D}} &\leqslant \|x^n_{\rho(s,x^n_s)} - x_{\rho(s,x^n_s)}\|_{\mathcal{D}} + \|x_{\rho(s,x^n_s)} - x_{\rho(s,x_s)}\|_{\mathcal{D}} \\ &\leqslant K_b\|x_n - x\|_{\mathcal{D}} + \|x_{\rho(s,x^n_s)} - x_{\rho(s,x_s)}\|_{\mathcal{D}}, \end{aligned}$$

which prove that $x^n_{\rho(s,x^n_s)} \longrightarrow x_{\rho(s,x_s)}$ in $\mathcal{D}$ as $n \longrightarrow \infty$ for every $s \in J$ such that $\rho(s, x_s) > 0$.
Similarly , if $\rho(s, x_s) < 0$ and $n \ in \mathbb{N}$ is such that $\rho(s, x^n_s) < 0$ for every $n > N$ we get

$$\|x^n_{\rho(s,x^n_s)} - x_{\rho(s,x_s)}\|_{\mathcal{D}} = \|\phi_{\rho(s,x^n_s)} - \phi_{\rho(s,x_s)}|_{\mathcal{D}} = 0$$

which also shows that $x_{\rho(s,x^n_s)} \longrightarrow x_{\rho(s,x_s)}$ in $\mathcal{D}$ as $n \longrightarrow \infty$ for every $s \in J$ such that $\rho(s, x_s) < 0$. Combining the previous argument , we can prove that $x^n_{\rho(s,x^n_s)} \longrightarrow \phi$
for every $s \in J$ such that $\rho(s, x_s) = 0$.
Finally
(i) For $t \in [0, t_1]$,we have

$$\begin{aligned} &|\mathcal{B}(x_n)(t) - \mathcal{B}(x)(t)| \\ &= \left| \int_0^t S_\alpha(t-s) f(s, x^n_{\rho(s,x^n_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})ds - \int_0^t S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \right| \\ &\quad \leqslant \int_0^t \|S_\alpha(t-s)\| |f(s, x^n_{\rho(s,x^n_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)}) - f(s, x^n_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})|ds \\ &\leqslant Me^{wt} \int_0^t e^{-ws} |f(s, x^n_{\rho(s,x^n_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)}) - f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds \longrightarrow 0 \end{aligned}$$

since f is continuous , and $f(s,x^n_{\rho(s,x^n_s)}) \longrightarrow f(s,x_{\rho(s,x_s)})$.
(ii) For $t \in [t_k,t_{k+1}]$

$$\begin{aligned}
&|\mathcal{B}(y_n)(t)-\mathcal{B}(y)(t)|\\
&\quad = |\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)[f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})\\
&\quad -f(s,x_{\rho(s,x_s+\tilde{\phi}_s)})+\tilde{\phi}_{\rho(s,x+\tilde{\phi}_s)})]ds+\int_{t_k}^{t} S_\alpha(t-s)[f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})\\
&\qquad -f(s,x_{\rho(s,x_s+\tilde{\phi}_s}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})]ds|\\
&\leqslant \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\|S_\alpha(t-t_k)\|\prod_{j=i}^{k-1}\|S_\alpha(t_{j+1}-t_j)\|\|S_\alpha(t_i-s)\||f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})\\
&\quad -f(s,x_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})|ds+\int_{t_k}^{t}\|S_\alpha(t-s)\||f(s,x_{\rho(s,x_s+\tilde{\phi}_s}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\quad \leqslant \sum_{i=1}^{k}\int_{t_{i-1}}^{t_k} Me^{w(t-t_k)}\prod_{j=i}^{k-1} Me^{w(t_{j+1}-t_j)}Me^{w(t_i-s)}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})\\
&\quad -f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds+\int_{t_k}^{t} Me^{w(t-s)}\times|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})\\
&\qquad -f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\quad \leqslant \sum_{i=1}^{k}\int_{i-1}^{t_i} Me^{w(t-k_k)}[e^{w(t_{i+1}-t_i)}Me^{w(t_{i+2}-t_{i+1})}Me^{w(t_{i+3}-t_{i+2})}\times\ldots\times Me^{w(t_k-t_{k-1})}]\\
&\qquad Me^{w(t_i-s)}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})-f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\quad +\int_{t_k}^{t} Me^{w(t-s)}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})-f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\leqslant \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} Me^{wt}[M^{k-1-i+1}]Me^{-ws}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})-f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\quad +Me^{wt}\int_{t_k}^{t} e^{-ws}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})-f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\leqslant \sum_{i=1}^{k} M^{k-i+2}e^{wt}\int_{t_k}^{t} e^{-ws}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})-f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\\
&\quad +Me^{wt}\int_{t_k}^{t} e^{-ws}|f(s,x^n_{\rho(s,x^n_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)})-f(s,x_{\rho(s,x_s+\tilde{\phi}_s}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds\longrightarrow 0.
\end{aligned}$$

Since f is continuous then $f(s,x^n_{\rho(s,x^n_s)}) \longrightarrow f(s,x_{\rho(s,x_s)})$, we have $\|\mathcal{B}(y_n)-\mathcal{B}(y)\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.
Thus B is continuous.
Step2. $\mathcal{B}$ maps bounded sets into bounded sets in $PC(]-\infty,0],E)$.
it is enough to show that for any $q>0$,there exists a positive constant $l_k; k=1...m$ such that for each $y \in \mathbf{B}_q=\{y \in PC(]-\infty,0],E) : \|y\| \leqslant q\}$, we have $\|\mathcal{B}(y)\| \leqslant l_k$.
Let $y \in \mathbf{B}_q$. Then from Lemma 3.1 it fllows that
$\|x^n_{\rho(s,x^n_s+\tilde{\phi}_s}+\tilde{\phi}_{\rho(s,x^n_s+\tilde{\phi}_s)}\|_{\mathcal{D}} \leq K_b q+(M_b+L^{\phi})\|\phi(t)-h_t(y)\|_{\mathcal{D}}+K_b M|\phi(0)h_0(y)|=q_*$.

Then we have

$$|\mathcal{B}(y)(t)| \leqslant \begin{cases} \int_0^t \|S_\alpha(t-s)\| |f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})| ds, & if t \in [0,t_1]; \\ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|S_\alpha(t-t_k)\| \times \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1}-t_j)\| \|S_\alpha(t_i-s)\| \\ |f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})| ds \\ + \int_{t_k}^t \|S_\alpha(t-s)\| |f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})| ds, & \text{if } t \in (t_k, t_{k+1}). \end{cases}$$

Since $\|S_\alpha(t)\| \leqslant Me^{wt}$ and $|f(t,y) \leqslant p(t)\psi(\|y\|_D)$, we have

$$|\mathcal{B}(y)(t) \leqslant \begin{cases} \int_0^t Me^{w(t_1-s)} p(s)\psi(q_*) ds, & \text{if } t \in [0,t_1]; \\ \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} Me^{w(t-t_k)} \times \prod_{j=i}^{k-1} Me^{w(t_{j+1}-t_j)} Me^{w(t_i-s)} p(s)\psi(q_*) ds \\ + \int_{t_k}^t Me^{w(t-s)} p(s)\psi(q_*) ds, & \text{if} t \in (t_k, t_{k+1}). \end{cases} .$$

Which gives

$$|\mathcal{B}(y)(t)| \leqslant \begin{cases} Me^{wt_1}\psi(q_*) \int_0^t e^{-ws} p(s) ds = l_1, & \text{if } t \in [0,t_1]; \\ \sum_{i=1}^{k} [Me^{w(t-t_k)} [Me^{w(t_{i+1}-t_i} Me^{w(t_{i+2}-t_{i+1})} \ldots Me^{w(t_{k-1}-t_{k-2})} \\ \times Me^{w(t_k-t_{k-1})}] Me^{w(t_i)} \times \psi(q_*) \int_{t_{i-1}}^{t_i} p(s) e^{-w(s)} ds \\ + Me^{w(t)}\psi(q_*) \int_{t_k}^t p(s) e^{-w(s)} ds, & \text{if} t \in (t_k, t_{k+1}). \end{cases}$$

Then

$$|\mathcal{B}(y)(t)| \leqslant \begin{cases} Me^{w(t_1)}\psi(q_*) \int_0^t e^{-ws} p(s) ds = l_1, & \text{if} t \in [0,t_1]; \\ \sum_{i=1}^{k} M^{k-i+2} e^{w(t-t_k+t_{i+1}-t_i+t_{i+2}-t_{i+1}+\ldots+t_{k-1}-t_{k-2}+t_k-t_{k-1}+t_i)} \\ \times\psi(q_*) \int_{t_{i-1}}^{t_i} p(s) e^{-w(s)} ds + Me^{w(t)}\psi(q_*) \int_{t_k}^t p(s) e^{-w(s)} ds, & \text{if } t \in (t_k, t_{k+1}). \end{cases}$$

Using characteristic of the exponential function ,we get

$$|\mathcal{B}(y)(t)| \leqslant \begin{cases} Me^{-ws} p(s) ds = l_1, & if t \in [0,t_1]; \\ \sum_{i=1}^{k} M^{k-i+2} e^{w(t-t_{k-1})} \times \psi(q_*) \int_{t_{i-1}}^{t_i} p(s) e^{-w(s)} ds \\ + Me^{w(t)}\psi(q_*) \int_{t_k}^t p(s) e^{-w(s)} ds, if t \in (t_k, t_{k+1}). \end{cases}$$

Finally ,we obtain

$$|\mathcal{B}(t)| \leqslant \begin{cases} Me^{w(t_1)}\psi(q_*)\int_0^t e^{-ws}p(s)ds = l_1, & ift \in [0,t_1];\\ \sum_{t_{i=1}}^{k} M^{k-i+2}e^{w(t_{k+1}-t_{k-1})} \times \psi(q_*)\int_{t_{i-1}}^{t_i} p(s)e^{-w(s))}ds + Me^{w(t_{k+1}}\psi(q_*) & \\ \times \int_{t_k}^{t} p(s)e^{-w(s)}ds = l_k, \quad k = 2...m, & ift \in (t_k, t_{k+1}). \end{cases}$$

Step3. $\mathcal{B}$ maps bounded sets into equicontinuous sets of $PC(]-\infty,0],E)$.
Let $\tau_1,\tau_2 \in J \setminus t_1,t_2,....t_m$ with $\tau_1 < \tau_2$, let $\mathbf{B}_q$ be a bounded set in $PC(]-\infty,0],E)$ as in Step 2,and let $y \in \mathbf{B}_q$.
•If $\tau_1,\tau_2 \in [0,t_1]$, we have

$$|\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| = |\int_0^{\tau 2} S_\alpha(\tau_2 - s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds - \int_0^{\tau_1} S_\alpha(\tau - s) \\ f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds|$$

Using the linearity of the integrale operator and hypotheses (H_4) , we get

$$\begin{aligned} |\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| &= \\ &|\int_0^{\tau_1} S_\alpha(\tau_2 - s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds + \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s) \\ &f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds - \int_0^{\tau_1}(S_\alpha(\tau_1 - s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds| \\ &= |\int_0^{\tau_1}(S_\alpha(\tau_2 - s) - (S_\alpha(\tau_1 - s))f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \\ &+ \int_{\tau_1}^{\tau_2} S_\alpha(\tau_2 - s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds| \\ &\leqslant \int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\||f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds \\ &+ \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\||f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})|ds \\ &\leqslant \psi(q_*)\int_0^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds + Me^{w\tau_2}\psi(q_*)\int_{\tau_1}^{\tau_2} e^{-ws}p(s)ds. \end{aligned}$$

If $\tau_1 = 0$, the right-hand side of the previous inequality tends to zero as $\tau_2 \longrightarrow 0$ uniformly for $y \in PC$
if $0 < \tau_1 < \tau_2$ for $\epsilon < \tau_1 < \tau_2$, we have

$$\begin{aligned} |\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| &\leqslant \int_0^{\tau_1-\epsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\||f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)|})ds \\ &+ \int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\||f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}|ds \\ &+ \int_{\tau_1}^{\tau_2} \||f(s,x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}|ds \\ \leqslant \psi(q_*)\int_0^{\tau_1-\epsilon} &\|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds + \psi(q_*)\int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ &+ Me^{w\tau_2}\psi(q_*)\int_{\tau_1}^{\tau_2} e^{-ws}p(s)ds. \end{aligned}$$

From Lemma 1.1 , the operator $S_\alpha(t)$ is a uniformly continuous operator for $t \in [\epsilon, t_1]$.
Combining this and the arbitrariness of ϵ with the above estimation on $|\mathcal{B}(y(\tau_2))-\mathcal{B}(y(\tau_1))|$, we can conclude that

$$\lim_{[\tau_1,\tau_2]\longrightarrow 0} |\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| = 0.$$

Thus the operator $\mathcal{B}$ is equicontinuous on $[0, t_1]$.

• if $\tau_1, \tau_2 \in [t_k, t_k + 1]$

$$\begin{aligned}|\mathcal{B}(y(\tau_2))-\mathcal{B}(y(\tau_1))| = \Big\| \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(\tau_2-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}ds \\ -\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(\tau_1 - t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1} - t_j)S_\alpha(t_i - s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}ds \\ + \int_{t_k}^{\tau_2} S_\alpha(\tau_2 - s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \\ - \int_{t_k}^{\tau_1} S_\alpha(\tau_1 - s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds\Big\|.\end{aligned}$$

Then

$$\begin{aligned}&|\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| \\ &\quad\leqslant \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_i)\| \\ &\times\|S_\alpha(t_i-s)\||f(s, x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}ds+\Big\|\int_{t_k}^{\tau_1} S_\alpha(\tau_2-s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \\ &+\int_{\tau_1}^{\tau_2} S_\alpha(\tau_2-s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds-\int_{t_k}^{\tau_1} S_\alpha(\tau_1-s)f(s, x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds\Big\|.\end{aligned}$$

Which gives

$$\begin{aligned}|\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| \leqslant \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\| \\ \times \|S_\alpha(t_i - s)\||f(s, y(s, y_{\rho(s,x_s)})|ds \\ + \int_{t_k}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\||f(s, y_{\rho(s,y_s)})|ds \\ + \int_{\tau_1}^{\tau_2} \|S_\alpha(\tau_2 - s)\||f(s, y_{\rho(s,y_s)})|ds.\end{aligned}$$

Under the hypothesis (H_6),we obtain

$$\begin{aligned}|\mathcal{B}(y(\tau_2)) - \mathcal{B}(y(\tau_1))| \leqslant \sum_{i=1}^{k}\psi(q_*)\int_{t_{i-1}}^{t_i} \|S_\alpha(\tau_2 - t_k) - S_\alpha(\tau_1 - t_k)\| \\ \times\prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\|\|S_\alpha(t_i - s)\|p(s)ds + \psi(q_*)\int_{t_k}^{\tau_1-\epsilon} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ + \psi(q_*)\int_{\tau_1-\epsilon}^{\tau_1} \|S_\alpha(\tau_2 - s) - S_\alpha(\tau_1 - s)\|p(s)ds \\ + M\psi(q_*)e^{w\tau_2}\int_{\tau_1}^{\tau_2} e^{-ws}p(s)ds.\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ and ϵ becomes sufficiently small, the right -hand side of the a bove inequality tends to zero , since S_α is an analytic operator and the compactness of $S_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology [23, 24] .
This proves the equicontinuity for the case where $t \neq t_i, i = 1...., m+1$.
Now ,it remains to examine equicontinuity at $t = t_l$.We have for $y \in \mathbf{B}_q$,
for each $t \in J$.
First, we prove equicontinuity at $t = t_l^-$.
Fix $\delta_1 > 0$ such that $\{t_k, k \neq l\} \cap [t_l - \delta_1, t_l + \delta_1] = \{\}$.
For $0 < h < \delta_1$ we have
• if $l = 1$ i.e .$t_1 - h, t_1 \in [0, t_1]$

$$\begin{aligned}|\mathcal{B}(y)(t_1 - h) - \mathcal{B}(y)(t_1)| \\ &\leqslant \psi(q_*) \int_0^{t_1-h} \|S_\alpha(t_1 - s) - S_\alpha(t_1 - h - s)\| p(s) ds \\ &\qquad + M e^{wt_1} \psi(q_*) \int_{t_1-h}^{t_1} e^{-ws} p(s) ds,\end{aligned}$$

which tends to zero as $h \longrightarrow 0$ since $S_\alpha(t)$ is a uniformly continuous operator for $t \in [0, t_1]$ thus the operator $\mathcal{B}$ is equicontinuous at $t = t_1^-$
• if $t_l - h, t_l \in [t_k, t_{k+1}]$.
Then:

$$\begin{aligned}|\mathcal{B}(y(t_l - h)) - \mathcal{B}(y(t_l))| &\leqslant \sum_{i=1}^{k} \psi(q_*) \int_{t_{i-1}}^{t_i} \|S_\alpha(t_l - t_k) - S_\alpha(t_{l-h} - t_k)\| \\ &\quad \times \prod_{j=i}^{k-1} \psi \|S_\alpha(t_{j+1} - t_j)\| \|S_\alpha(t_i - s)\| p(s) ds \\ &+ \psi(q_*) \int_{t_k}^{t_l-h} \|S_\alpha(t_l - s) - S_\alpha(t_l - h - s)\| p(s) ds \\ &\qquad + M \psi(q_*) e^{wt_l} \int_{t_l-h}^{t_l} e^{-ws} p(s) ds\end{aligned}$$

The right -hand side of the privious enequality tends to zero as $h \longrightarrow 0$.
So the operator $\mathcal{B}$ is equicontinuous at t_l^-.
Now , define

$$\widehat{B_0}(y)(t) = \mathcal{B}(y)(t), \text{ift} \in [0, t_1]$$

and

$$\widehat{B_i}(y)(t) = \begin{cases} \mathcal{B}(y)(t), & \text{if } t \in [t_i, t_{i+1}] \\ \mathcal{B}(y)(t_i^+), & t = t_i \end{cases} .$$

Next ,we prove equicontinuity at $t = t_i^+$.
Fix $\delta_2 > 0$ such that $\{t_k, k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$.
First , we study the equicontinuity at $t = 0^+$.
If $t \in [0, t_1]$,we have

$$\widehat{B_0}(y)(h) = \begin{cases} \mathcal{B}(y)(t), & \text{if } t \in [0, t_1] \\ 0, ift = 0 \end{cases}$$

For $0 < h < \delta_2$, we have

$$\begin{aligned}|\widehat{B_0}(y)(t) - \widehat{B_0}(y)(t)| &= |\mathcal{B}(y)(h)| \\ &= \left\| \int_0^h S_\alpha(h-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds \right| \\ &\leqslant \int_0^h \|S_\alpha(h-s)\| f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds \\ &\leqslant \psi(q_*) e^{wh} \int_0^h e^{-ws} p(s) ds.\end{aligned}$$

The right -hand side to tends to zero as $h \longrightarrow 0$.

Now,we study the equicontinuity at $t_1^+, t_2^+, ..., t_m^+$(i.e. at $t_l^+, 1 \leqslant l \leqslant m$).
For $0 < h < \delta_2$, we have

$$\begin{aligned}&\|\mathcal{B}(y)(t_l + h) - \mathcal{B}(y)(t_l)\| \\ &\leqslant \sum_{i=1}^{k} \psi(q_*) \int_{t_{i-1}}^{t_i} \|S_\alpha(h) - S_\alpha(0)\| \prod_{j=i}^{k-1} \|S_\alpha(t_{j+1} - t_j)\| \|S_\alpha(t_i - s)\| p(s) ds \\ &\qquad + M\psi(q_*) e^{w(t_l+h)} \int_{t_l}^{t_l+h} e^{-ws} p(s) ds.\end{aligned}$$

It is clear that the right hand-side tends to zero as $h \longrightarrow 0$.
Then $\mathcal{B}$ is equicontinuous at $t_l^+, (1 \leqslant l \leqslant m)$.
The equicontiuity for the cases $\tau_1 < \tau_2 \leqslant 0$ and $\tau_1 \leqslant 0 \leqslant \tau_2$ follows from the uniform continuity of ϕ on the interval $]-\infty, 0]$,
as a consequence of Setps1 and 3 together with Arzel-Ascoli Theorem it suffices to show that $\mathcal{B}$ maps $\mathcal{B}_q$ into a precompact set in E i.e: we show that the set $\mathcal{B}(y)(t); y \in \mathbf{B}_q$ is percompact in E for every $t \in [0, b]$.

Now,let$x \in \mathcal{B}_q$ and let ϵ be apositive real number satisfiying $0 < \epsilon < t \leqslant b$.
For $y \in \mathcal{B}_q$ and $t \in [0, t_1]$.
We have if $t = 0$ the set $\{\mathcal{B}(y)(0); y \in \mathcal{B}_q\} = \{0\{$ which is precompact as a finite set.
For $0 < \epsilon < t \leqslant t_1$, we have

$$\begin{aligned}\mathcal{B}(y)(t) &= \int_0^t S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds \\ &= \int_0^{t-\epsilon} S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds \\ &\qquad + \int_{t-\epsilon}^{t} S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds.\end{aligned}$$

Set $F_0 = \{S_\alpha(t-\theta) f(\theta, y(\theta)); \theta \in [0, t-\epsilon], y \in \mathcal{B}_q\}$, from the mean value Theorem for the Bochner integrable we have

$$\int_0^{t-\epsilon} S_\alpha(t-s) f(s, x_{s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)}) ds \in (t-\epsilon)\overline{Conv(F_0)} \tag{2.4}$$

On the other hands , using hypotheses (H_1) and (H_6), we obtain

$$\begin{aligned}\int_{t-\epsilon}^{t} |S_\alpha(t-s) f(s, x_{\rho(s,x_s+\tilde{\phi}_s)} + \tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)|}) ds &\leqslant M e^{wt} \psi(q_*) \int_{t-\epsilon}^{t} e^{-ws} p(s) ds \\ &\leqslant M e^{wt} \psi(q_*) \int_{t-\epsilon}^{t} p(s) ds.\end{aligned}$$

Let C_ϵ^0 the circle who's diameter

$$d_\epsilon^0 \leqslant Me^{wt}\psi(q_*)\int_{t-\epsilon}^{t} e^{-ws)}p(s)ds. \tag{2.5}$$

As a consequence of (2.4) and (2.5), we conclude that

$$\mathcal{B}(y)(t) \in (t-\epsilon)\overline{Conv(F_0)} - C_\epsilon^0, \forall 0 < \epsilon < t \leqslant t_1. \tag{2.6}$$

For $t_k < \epsilon < t \leqslant t_{k+1}$ and $y \in \mathcal{B}$, we have

$$\mathcal{B}(y)(t) = \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\times\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds$$
$$+\int_{t_k}^{t-\epsilon} S_\alpha(t-s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds+\int_{t-\epsilon}^{t} S_\alpha(t-s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \tag{2.7}$$

Set $F_k = \{S_\alpha(t-\theta)f(\theta,y(\theta)); \theta \in [t_k, t-\epsilon], y \in \mathcal{B}\}$,from the mean value Theorem for Bochner integral , we have

$$\int_{t_k}^{t-\epsilon} S_\alpha(t-s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds \in (t-t_k-\epsilon)\overline{Conv(F_k)}. \tag{2.8}$$

From $(H_1), (H_6)$, we obtain

$$\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k)\times\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)ds+\int_{t-\epsilon}^{t} S_\alpha(t-s)f(s,x_{\rho(s,x_s+\tilde{\phi}_s)}+\tilde{\phi}_{\rho(s,x_s+\tilde{\phi}_s)})ds$$
$$\leqslant \psi(q_*)\sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})}\int_{t_{i-1}}^{t_i} e^{-ws}p(s)ds + M\psi(q_*)e^{wt}\int_{t-\epsilon}^{t} e^{-ws}p(s)ds.$$

Let C_ϵ^k the circle who's diameter d_ϵ^k is such that

$$d_\epsilon^k \leqslant \psi(q_*)\sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})}\int_{t_{i-1}}^{t_i} e^{-ws}p(s)ds + M\psi(q_*)e^{wt}\int_{t-\epsilon}^{t} e^{-ws}p(s)ds. \tag{2.9}$$

From (2.8) and (2.9),it follows that

$$\mathcal{B}(y)(t) \in (t-t_k-\epsilon)\overline{Conv(F_k)} + C_\epsilon^k, \forall t_k < \epsilon < t < t_{k+1}. \tag{2.10}$$

From (2.6) and(2.10) , we conclude that $\mathcal{B}(y)(t)$ is precompact in E.
From **Step1-Step3**,we deduce that $\mathcal{B} : PC(]-\infty,0],E) \longrightarrow PC(]-\infty,0],E)$ is completely continuous.
Step4. $\mathcal{A}$ is a contraction
For $t \in [-\infty, 0]$, we have

$$\begin{aligned} |\mathcal{A}(y_1)(t) - \mathcal{A}(y_2)(t)| &= \|h_0(y_1) - h_0(y_2)\| \\ &\leqslant L\|y_1 - y_2\|. \end{aligned}$$

from hypothes (H5) , the previous inequality implies tha $\mathcal{A}$ is contraction on $t \in]-\infty, 0]$.
For $t \in [0, t_1]$ we have

$$\begin{aligned} |\mathcal{A}(y_1)(t) - \mathcal{A}(y_2)(t)| &= S_\alpha(t)(h_0(x_t^1+\tilde{\phi}_t) - h_0(x_t^2+\tilde{\phi}_t)) \\ &\leqslant \|S_\alpha(t)\|\|(h_0(x_t^1+\tilde{\phi}_t) - h_0(x_t^2+\tilde{\phi}_t))\| \\ &\leqslant Me^{wt_1}L\|(x_1)-(x_2)\| \end{aligned}$$

From hypotes (H5) we have $MLe^{wt_1} < 1$. Is follows that $\mathcal{A}$ is contraction when $t \in [0, t_1]$. It remains to prove that $\mathcal{A}$ is a contraction operator for $t \in [t_k, t_{k+1}], k \geq 1$

$$\begin{aligned}
&|\mathcal{A}(y_1)(t) - \mathcal{A}(y_2)(t)| \\
&= \|S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1})|h_0(x_t^1 + \tilde{\phi}(t)) - h_0(x_t^2 + \tilde{\phi}(t))| \\
&+ \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)[I_i(x^1_{t_i^-} + \tilde{\phi}_{t_i^-}) - I_i(x^2_{t_i^-} + \tilde{\phi}_{t_i^-})]\| \\
&\leqslant \|S_\alpha(t-t_k)\| \prod_{i=1}^{k} Me^{w(t_i-t_{i-1})}|h_0(x_t^1 + \tilde{\phi}(t)) - h_0(x_t^2 + \tilde{\phi}(t))| \\
&+ \sum_{i=1}^{k} \|S_\alpha(t-t_k)\| \prod_{j=i}^{k} Me^{w(t_{j+1}-t_j)}|I_i(x^1_{t_i^-} + \tilde{\phi}_{t_i^-}) - I_i(x_{t_i^-} + \tilde{\phi}_{t_i^-})| \\
&\leqslant Me^{w(t-t_k)} \prod_{i=1}^{k} Me^{w(t_i-t_{i-1})}|h_0(x_t^1 + \tilde{\phi}(t)) - h_0(x_t^2 + \tilde{\phi}(t))| \\
&+ \sum_{i=1}^{k} Me^{w(t-t_k)} \prod_{j=i}^{k-1} Me^{w(t_{j+1}-t_j}|I_i(x^1_{t_i^-} + \tilde{\phi}_{t_i^-}) - I_i(y_2(I_i(x^2_{t_i^-} + \tilde{\phi}_{t_i^-})| \\
&\leqslant Me^{w(t-t_k)}[e^{w(t_1-t_0)}Me^{w(t_2-t_1)}Me^{w(t_3-t_2)}...Me^{w(t_{k-1}-t_{k-2})}Me^{w(t_k-t_{k-1})}] \\
&\times |h_0(x_t^1 + \tilde{\phi}(t)) - h_0(x_t^2 + \tilde{\phi}(t))| + \sum_{i=1}^{k} Me^{w(t-t_k)}[e^{w(t_{i+1}-t_i)}Me^{w(t_{i+2}-t_{i+1})} \\
&...Me^{w(t_{k-1}-t_{k-2})} \times Me^{w(t_k-t_{k-1})}] \times |I_i(x^1_{t_i^-} + \tilde{\phi}_{t_i^-}) - I_i(x^2_{t_i^-} + \tilde{\phi}_{t_i^-})| \\
&\leqslant Me^{wt}e^{-wt_k}[M^k e^{-w(t_0)}e^{w(t_k)} \times |h_0(x_t^1 + \tilde{\phi}(t)) - h_0(x_t^2 + \tilde{\phi}(t))| \\
&+ \sum_{i=1}^{k} Me^{wt}e^{-wt_k}[M^{k-i}e^{-wt_i}e^{wt_k}]|I_i(x^1_{t_i^-} + \tilde{\phi}_{t_i^-}) - I_i(x^1_{t_i^-} + \tilde{\phi}_{t_i^-})| \\
&\leqslant M^{k+1}e^{wt}|h_0(x_t^1+\tilde{\phi}(t)) - h_0(x_t^2+\tilde{\phi}(t))| + \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}|I_i(x^1_{t_i^-}+\tilde{\phi}_{t_i^-}) - I_i(x^1_{t_i^-}+\tilde{\phi}_{t_i^-})|.
\end{aligned}$$

Since $t \in J := [0, b]$, and the function $I_k; k = 1, 2, ...m.$ are Lipschitz ; then

$$\begin{aligned}
|\mathcal{A}(y_1)(t) - \mathcal{A}(y_2)(t)| &\leqslant LM^{k+1}e^{wt}\|x_t^1 - x_t^2\| + M^{k-i+1}e^{w(t-t_i^-)}\|I_i(x^1_{t_i^-}) - I_i(x^2_{t_i^-})\|_{\mathcal{D}}. \\
&\leqslant LM^{k+1}e^{wt}\|(x_t^1 + \tilde{\phi}(t)) - (x_t^2 + \tilde{\phi}(t))\| + \sum_{i}^{k} M_i M^{k-i+1}e^{w(t-t_i)}\|(x^1_{t_i^-}) - (x^2_{t_i^-})\|_{\mathcal{D}}. \\
&\leqslant [LM^{k+1}e^{wt} + \sum_{i=1}^{k} M_i M^{k-i+1}e^{w(t-t_i)}]K_b\|(x_1) - (x_2)\|_{\mathcal{D}}.
\end{aligned}$$

Thus the operator $\mathcal{A}$ is a contraction , since

$$[LM^{k+1}e^{wt} + \sum_{i=1}^{k} M_i M^{k-i+1}e^{w(t-t_i)}]K_b < 1.$$

Step 5. $\mathcal{A}$ priori bounds.

Now it remains to show that the set $\Upsilon = \{y \in PC(]-\infty, 0], E) : y = \lambda\mathcal{B}(y) + \lambda\mathcal{A}\Big(\frac{y}{\lambda}\Big)\}$,

for some $0 < \lambda < 1\}$ is bounded .
Let $y \in \Upsilon$ be any element , then $y = \{\lambda\mathcal{B}(y) + \lambda\mathcal{A}\Big(\dfrac{y}{\lambda}\Big)\}$,for some $0 < \lambda < 1$.
First , for each $t \in [0, t_1]$,

$$\begin{aligned}
&|y(t)| = \\
&\qquad \Big|\lambda \int_0^t S_{\alpha)}(t-s) f(s, x_{\rho_{s,x_s+\tilde{\phi})}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi})}}) ds + \lambda(S_\alpha(t)\Big(\phi(0) - h_0\Big(\frac{x(t)}{\lambda} + \tilde{\phi}\Big)\Big)\Big| \\
&\leqslant \lambda\Big(Me^{wt}\|\phi(0) - h_0\Big(\frac{x(t)}{\lambda} + \phi(t)\Big)\Big) + \lambda\Big(Me^{wt}\int_0^t e^{-wt}\|f(s, x_{\rho_{s,x_s+\tilde{\phi})}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi})}})\|ds\Big) \\
&\qquad \leqslant \lambda\Big(Me^{wt}\|\phi(0) - h_0\Big(\frac{x(t)}{\lambda} + \phi(t)\Big)\Big)\| + \lambda M\int_0^t e^{w(t-s)}p(s)\psi(\|y_s\|)ds \\
&\leqslant \lambda\Big(Me^{wt}\|\phi(0) - h_0\Big(\frac{x(t)}{\lambda} + \phi(t)\Big)\Big)\| + M\int_0^{t_1} e^{t_1-s)}p(s)\psi(K_bq + (Mb + L^\phi)\|\phi(0)\|_{\mathcal{D}} + KbM|\phi(0)|ds
\end{aligned}$$

On the other hand, for each $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned}
\|y(t)\| = &\Big\|\lambda\Big(\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} S_\alpha(t-t_k) \times \prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)S_\alpha(t_i-s)f(s, x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}})ds \\
&+ \int_{t_k}^{t} S_\alpha(t-s)f(s, x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}})ds\Big) + \lambda\Big(S_\alpha(t-t_k)\prod_{i=1}^{k} S_\alpha(t_i - t_{i-1}) \\
&\qquad (\phi(0) - h_0(\frac{x(t)}{\lambda} + \tilde{\phi}(t))) + \sum_{i=1}^{k} S_\alpha(t-t_k)\prod_{j=i}^{k-1} S_\alpha(t_{j+1}-t_j)I_i\Big(\frac{x_{t_i^-}}{\lambda} + \phi_{t_i^-}\Big)\Big\|
\end{aligned}$$

From$(H_1), (H_3)$ and since $\lambda < 1$, we obtain

$$\begin{aligned}
\|y(t)\| \leqslant{}& \lambda\Big(M^{k+1}e^{wt}\|\phi(0) - h_0\Big(\frac{x(t)}{\lambda} + \tilde{\phi}(t)\Big)\|\sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})}\int_{t_{i-1}}^{t_i} e^{-ws}p(s) \\
&\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds\Big) + Me^{wt}\int_{t_k}^{t} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds \\
&+ \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}|I_i\Big(\frac{x_{t_i^-}}{\lambda} + \phi_{t_i^-}\Big)\Big)| \\
\leqslant{}& M^{k+1}e^{wt}\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\| + \sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})}\int_{t_{i+1}}^{t_i} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds \\
&+ Me^{wt}\int_{t_k}^{t} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds + \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}\Big|I_i\Big(\frac{x_{t_i^-}}{\lambda} + \phi_{t_i^-}\Big) - I_i(0)\Big| \\
&+ \sum_{i=1}^{k} M^{k-i-1}e^{w(t-t_i)}|I_i(0)| \\
\leqslant{}& M^{k+1}e^{wt}\|\phi(0) - h_0\Big(\frac{x(t)}{\lambda} + \tilde{\phi}(t)\Big)\| + \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}|I_i(0)| + \sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})} \\
&\int_{t_{i-1}}^{t_i} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)})}\|)ds + Me^{wt}\int_{t_k}^{t} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}} + \tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds \\
&+ \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}\Big|I_i\Big(\frac{x_{t_i^-}}{\lambda} + \phi_{t_i^-})(t_i^-)\Big) - I_i(0)\Big|.
\end{aligned}$$

SinceI_i are Lipschitz, then

$$\|y(t)\| \leqslant M^{k+1}e^{wt}\|\phi(0)-h_0\Big(\frac{x(t)}{\lambda}+\tilde{\phi}(t)\Big)\| + \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}|I_i(0)|$$
$$+\sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})}\int_{t_{i-1}}^{t_i} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}}+\tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds$$
$$+Me^{wt}\int_{t_k}^{t} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}}+\tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds+\sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}\Big|I_i\Big((\frac{x_{t_i^-}}{\lambda}+\phi_{t_i^-})\Big)-I_i(0)\Big|$$
$$\leqslant M^{k+1}e^{wt}\|\phi(0)-h_0\Big(\frac{x(t)}{\lambda}+\tilde{\phi}(t)\Big)\| + \sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}|I_i(0)|$$
$$+\sum_{i=1}^{k} M^{k-i+2}e^{w(t-t_{k-1})}\int_{t_{i-1}}^{t_i} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}}+\tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds$$
$$+Me^{wt}\int_{t_k}^{t} e^{-ws}p(s)\psi(\|x_{\rho_{(s,x_s+\tilde{\phi}_s)}}+\tilde{\phi}_{\rho_{(s,x_s+\tilde{\phi}_s)}}\|)ds+\sum_{i=1}^{k} M^{k-i+1}e^{w(t-t_i)}K_bM_i|y(t_i^-)|$$

Now , we consider the function $\mu(t)$ defined by

$$\mu(t) = sup\Big\{K_b|x(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}} : 0 \leqslant s \leqslant t\Big\}, 0 \leqslant t \leqslant b.$$

Then , we have $\|y_s\| \leqslant \mu(t)$ for all $t \in J$, and there is a point $\xi \in [0,t]$ such that $\mu(t) = |y(\xi)|$.if $\xi \in [0,b]$, by the previous inequality, we have for $t \in [0,b]$ (note $\xi < t$).

- if $t \in [0,t_1]$

$$\mu(t) \leqslant Me^{wb}\|\phi(0)-h_0(x(t)+\tilde{\phi(t)})\| + Me^{wb}\int_0^t e^{-ws}p(s)\psi(\mu(s))ds,$$
$$\leqslant C_0 + Me^{wb}\int_0^t e^{-ws}p(s)\psi(\mu(s))ds,$$

where

$$C_0 = Me^{wb}\|\phi(0)-h_0(x(t)+\tilde{\phi(t)})\|.$$

- $t \in [t_k, t_{k+1}]$

$$\mu(t) \leqslant M^{k+1}e^{wb}\|\phi(0)-h_0(x(t)+\tilde{\phi}(t)\|+\sum_{i=1}^{k} M^{k-i+1}e^{w(b-t_i)}|I_i(0)|+\sum_{i=1}^{k} M^{k-i+2}e^{w(b-t_{k-1})}$$
$$\times\int_{t_{i-1}}^{t_i} e^{-ws}p(s)\psi(K_b|y(s)| + (M_b+L^{\phi}+MK_b)\|\phi\|_{\mathcal{D}})ds+$$
$$Me^{wb}\int_{t_k}^{t} e^{-ws}p(s)\psi(K_b|x(s)|+(M_b+L^{\phi}+MK_b)\|\phi\|_{\mathcal{D}})ds+\sum_{i=1}^{k} M^{k-i+1}e^{w(b-t_i)}K_bM_i\mu(t).$$

Then

$$\left[1-\sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} K_b M_i\right] \mu(t)$$
$$\leqslant M^{k+1} e^{wb} \|\phi(0)-h_0(x(t)+\tilde{\phi}(t))\| + \sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} |I_i(0)| + \sum_{i=1}^{k} M^{k-i+2} e^{w(b-t_{k-1}}$$
$$\times \int_{t_{i-1}}^{t_i} e^{-ws} p(s)\psi(K_b|x(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}})ds$$
$$+ Me^{wb} \int_{t_k}^{t} p(s)\psi(K_b|y(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}})ds.$$

Thus, we have

$$\mu(t) \leqslant \left(M^{k+1} e^{wb} \|\phi(0)-h_0(x(t)+\tilde{\phi}(t))\| + \sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} |I_i(0)| + \sum_{i=1}^{k} M^{k-i+2} e^{w(b-t_{k-1})}\right.$$
$$\left.\times \int_{ti-1}^{t_i} e^{-ws} p(s)\psi(\mu(s))ds\right) \Big/ \left(1-\sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} K_b M_i\right)$$
$$+ \frac{Me^{wb}}{[1-\sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i) M_i}]} \int_{t_k}^{t} e^{-ws} p(s)\psi(\mu(s))ds$$
$$\leqslant C_1 + C_2 \int_{t_k}^{t} e^{-ws} p(s)\psi(\mu(s))ds.$$

where

- $$C_1 = \left(M^{k+1} e^{wb} \|\phi(0)-h_0(x(t)+\tilde{\phi}(t))\| + \sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} |I_i(0)| + \sum_{i=1}^{k} M^{k-i+2} e^{w(b-t_{k-1})}\right.$$
$$\left.\times \int_{t_{i-1}}^{t_i} e^{-ws} p(s)\psi(K_b|x(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}})ds\right) \Big/ \left(1-\sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} M_i\right).$$
- $$C_2 = \frac{Me^{wb}}{1-\sum_{i=1}^{k} M^{k-i+1} e^{w(b-t_i)} K_b M_i}.$$

It follows that

$$\begin{cases} \mu(t) \leqslant C_0 + Me^{wb} \int_0^t e^{-ws} p(s)\psi(K_b|x(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}})ds, if t \in [0; t_1], \\ \mu(t) \leqslant C_1 + C_2 \int_{t_k}^{t} e^{-ws} p(s)\psi(K_b|x(s)| + (M_b + L^{\phi} + MK_b)\|\phi\|_{\mathcal{D}})ds, if t \in (t_k, t_{k+1}]. \end{cases}$$

Let us take the right - hand side of the above inequality as $\vartheta(t)$.

Then, we have for all $t \in J$

$$\mu(t) \leqslant \psi(t)$$

and

$$\begin{cases} \varphi(t) = C_0 + Me^{wb} \int_0^t e^{-ws} p(s)\psi(\mu(s))ds, if t \in [0, t_1], \\ \varphi(t) = C_1 + C_2 \int_{t_k}^{t} e^{-ws} p(s)\psi(s))ds, if t \in (t_k, t_{k+1}]. \end{cases}$$

$$\begin{cases} \varphi(0) = C_0 \\ \varphi(t_k) = C_1, k = 1, ..., m. \end{cases}$$

And differentiating both sides of the above equality , we obtain

$$\begin{cases} \vartheta'(t) = Me^{w(b-t)}p(t)\psi(\mu(t)), if t \in [0, t_1], \\ \vartheta'(t) = C_2 e^{-wt}p(t)\psi(\mu(t)), if t \in (t_k, t_{k+1}]. \end{cases}$$

Using the non decreasing character of the function ψ , i.e. $(\mu(t) \leqslant \vartheta(t) \Rightarrow \psi(\mu(t)) \leqslant \psi(\vartheta(t))$.
We have

$$\begin{cases} \vartheta'(t) \leqslant Me^{w(b-t)}p(t)\psi(\vartheta(t)), if t \in [0, t_1] \\ \vartheta'(t) \leqslant C_2 e^{-wt}p(t)\psi(\vartheta(t)), if t \in (t_k, t_{k+1}). \end{cases}$$

It gives

$$\begin{cases} \dfrac{\vartheta'(t)}{\psi(\vartheta(t))} \leqslant Me^{w(b-t)}p(t), if t \in [0, t_1], \\ \dfrac{\vartheta'(t)}{\psi(\vartheta(t))} \leqslant C_2 e^{-wt}p(t), if t \in (t_k, t_{k+1}]. \end{cases}$$

◄ Integrating from 0 to t , if $t \in [0, t_1]$, we get

$$\int_0^t \frac{\vartheta'(s)}{\psi(u)} \leqslant Me^{wb} \int_0^t e^{-ws}p(s)ds \leqslant \int_{C_0}^{\infty} \frac{du}{\psi(u)}.$$

Hence , there exists a canstant η_1 such that

$$\mu(t) \leqslant \varphi(t) \leqslant \eta_1, t \in [0, t_1].$$

◄ Now, integrating from t_k to t if $t \in [t_k, t_{k+1}]$, we get

$$\int_{t_k}^t \frac{\vartheta'(s)}{\psi(\vartheta(s))} ds \leqslant C_2 \int_{t_k}^t e^{-ws}p(s)ds.$$

By a change of variable $(\vartheta(s) = u)(s : t_k \longrightarrow t; u : \vartheta(t_k) = C_1 \longrightarrow \vartheta(t))$:

$$\int_{C_1}^{\vartheta(t)} \frac{du}{\psi(u)} \leqslant C_2 \int_{t_k}^t e^{-ws}p(s)ds \leqslant \int_{C_3}^{\infty} \frac{du}{\psi(u)},$$

where $C_3 = min(C_1, C_2)$ Hence, there exists a constant η_2 such that

$$\mu(t) \leqslant \vartheta(t) \leqslant \eta_2, t \in [t_k, t_{k+1}].$$

Now from definitin of μ it follows that

$$\|y\| = sup|y(t)| : t \in J \leqslant \mu(b) \leqslant \eta, forall y \in \Upsilon.$$

This shows that the set Υ is bounded .
As a consequence of Theoreme 1.10 , we deduce that $\mathcal{A} + \mathcal{B}$ has a fixed point which is mild solution of the problem (2.1)-(2.3). □

Chapter 3

Application

consider the following impulsive fractional differential equation :

$$\begin{aligned}
{}^cD_t^{\alpha}z(t,x) &= \frac{\partial^2 z(t,x)}{\partial x^2} + Q(t,z(t-r),x)), \quad (3.1)\\
x\in[0,\pi], t\in[0,b], t\neq t_k & \\
z(t_k^+,x)-z(t_k^-,x) &= b_k z(t_k,x), x\in[0,\pi], k=1,2,\ldots,m \quad (3.2)\\
z(t,0) &= z(t,\pi)=0, t\in[0,b]\\
z(t,x)+\sum_{i=1}^{m+1}\int_0^{t_i} h_i(s)z(s,x)ds &= \phi(t,x), t\in]-\infty,0], x\in[0,\pi] \quad (3.3)
\end{aligned}$$

where $r>0, I_k>0, k=1,2,\ldots,m, Q:[0,b]\times\mathbb{R}\to\mathbb{R}$ is a given function ,$\phi\in\mathcal{D}$ where $\mathcal{D}=\{\psi:]-\infty,0]\to\mathcal{R}\}$; such that ψ is continuous everywhere except for a countable number of point at which $\psi(s^-),\psi(s^+)$
existe with $\psi(s^-)=\psi(s), 0=t_0<t_1<\ldots<t_m<t_{m+1}=b, z(t_k^+)=\lim_{h\longrightarrow 0} z(t_k+h,x)$, $z(t_k^-,x)=\lim_{h\longrightarrow 0^-} z(t_k+h,x), h_i\in L^2([0,b];\mathbb{R})$; for al $i=1,\ldots,m+1$
Let

$$\begin{aligned}
y(t)(x) &= z(t,x), t\in[0,b]\neq\{t_1,t_2,\ldots,t_m\}\\
I_k(y(t_k^-))(x) &= b_k z(t_k,x), x\in[0,\pi], k=1,\ldots,m\\
f(t,\phi)(x) &= Q(t,\phi(\theta,x)), x\in[0,\pi], \theta\in]-\infty,0]\\
\phi(\theta)(x) &= \phi(\theta,x), x\in[0,\pi], \theta\in]-\infty,0]\\
g_t(x) &= \sum_{i=1}^{m+1}\int_0^{t_i} h_i(s)y(s)(x)ds.
\end{aligned}$$

Let $E=L^2[0,\pi]$ and define the operator $A:D(A)\subset E\to E$ by :

$$D(A)=\{u\in E, u, u' are absolutely continuous, u''\in E, u(0)=u(\pi)=0\}$$

and

$$Au=u''$$

It is well known that A generates a compact analytic semigroup $(T(t))$ for $t>0$ on E given by

$$T(t)w=\sum_{n=1}^{\infty} e^{-n^2t}\langle w,w_n\rangle w_n, w\in E$$

where $\langle,\rangle$ is the inner product in L^2 and $w_n(s) = \sqrt[2]{\pi}\sin ns, n = 1, 2, ...$ is the orthogonal set of eigenvectors of A .From Theorem 2.1 the operator A also generates an α-resolvent family which is compact for $t > 0$, given by [12]

$$S_\alpha = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{\lambda t}(\lambda^\alpha - A)^{-1}1 d\lambda, t > 0 \\ I, \qquad\qquad\qquad\qquad t = 0 \end{cases}$$

where $\theta \in (\pi, \frac{\pi}{2})$ and Γ_θ is the contour $\{re^{i\theta}; r \geq 0\} \cup \{re^{-i\theta}; r \geq 0\}$.

also there exist constant $M \geq 1 G = \frac{1}{2}\sum_{i=1}^{m+1} \|h_i\|_{L^2([0,t_i])}$ such that $\|S_\alpha(t)\| \leqslant M$ and

$$\|g_t(y_1) - g_t(y_2)\| \leqslant \|y_1 - y_2\|.$$

Assume that there exist an integrable function $\sigma[0, b] \longrightarrow \mathbb{R}^+$ such that

$$|Q(t, g(t - r))| \leqslant \sigma(t)\Gamma(|g|),$$

where $\Gamma : [0, \infty[\rightarrow)$ is continuous and nondecreasing . Using the previous change of variables , we can reformulate the fractional partial differential equation (3.1)-(3.3) as the abstract problem (2.1)-(2.3)

$$\begin{aligned} {}^cD^\alpha_{t_k} z(t) &= Az(t) + Q(t, y_{\rho(t,z_t)}); t \in J := [0, b], \\ \Delta y|_{z=z_k} &= b_k(z(t_k^-)), k = 1, \ldots, m \\ z(t) + g_t(y) &= \phi(t), t \in] - \infty, 0] \end{aligned}$$

where $0 < \alpha \leqslant 1, f : J \times \mathcal{D} \longrightarrow E$ is a given function , $\mathcal{D}$ is the phase space defined axiomatically which contains the mapping from $]-\infty, 0]$into$E, \phi \in \mathcal{D}, \Delta|_{y=y_k} = y(t_k^+)-y(t_k^-)$, where $y(t_k^+) = \lim_{h\longrightarrow 0^+} y(t_k+h)$ and $y(t_k^-) = \lim_{h\longrightarrow 0^+} y(t_k+h)$ represent the right and left limits of$y(t)$ at $t = t_k, A : D(A) \subset E \longrightarrow E$ is generator of analytic $\alpha-$resolvant operator family ($\alpha-$ROF for short) S_α
$0 = t_0 < t_1 < \ldots < t_m = b, I_k : \mathcal{D} \longrightarrow E(k = 1, 2, \ldots, m)$,
$\rho : J \times \mathcal{D} \longrightarrow] - \infty, b], A : D(A) \subset E \longrightarrow E$ is a densely defined closed linear operator on E , and E a real separable Banach space with norm $|.|$.For any function y defined on $(-\infty, b] \setminus \{t_1, t_2, \ldots, t_m\}$ and any $t \in J$, we denote by y_t the element of $\mathcal{D}$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in] - \infty, 0]$$

Therefore under appropriate condition on $\phi(\theta)$, in view of theorem (2.1) the problem (3.1)-(3.3) has a mild solution .

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