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Contribution à l'étude de l'existence de solutions de quelques équations différentielles d'ordre fractionnaire

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Abstract

The main purpose of this thesis is to study the existence of solutions for some classes of initial and boundary value problems involving the Caputo fractional derivative in Banach Spaces. Our results are based on some standard fixed point theorems combined with the technique of measures of noncompactness. Furthermore, examples are presented to illustrate the application of our main results.

Key words and phrases : Caputo's fractional derivative, boundary value problems, initial value problem, Banach space, fixed-point, measure of noncompactness.
AMS Subject Classification : 26A33, 34A08, 34B15, 34G20.

List of symbols

We use the following notations throughout this thesis

Acronyms

- ✓ FC : Fractional calculus.
- ✓ FD : Fractional derivative.
- ✓ FDE : Fractional differential equation.
- ✓ FI : Fractional integral.
- ✓ IVP : Initial value problem.
- ✓ BVP : Boundary value problem.
- ✓ FHDE : Fractional hybrid differential equation.
- ✓ MNC : Measure of noncompactness.
- ✓ DND Degree of nondensifiability.

Notation

- ✓ $\mathbb{N}$: Set of natural numbers.
- ✓ $\mathbb{R}$: Set of real numbers.
- ✓ $\mathbb{R}^n$: Space of n-dimensional real vectors.
- ✓ J : be a finite interval on the half-axis $\mathbb{R}^+$.
- ✓ $\in$: belongs to.
- ✓ sup : Supremum.
- ✓ max : Maximum.
- ✓ $n!$: Factorial $(n), n \in \mathbb{N}$: The product of all the integers from1 to n.
- ✓ $\Gamma(\cdot)$: Gamma function.
- ✓ $\mathrm{B}(\cdot,\cdot)$: Beta function.
- ✓ $I_{0^+}^{\alpha}$: The Riemann-Liouville fractional integral of order $\alpha > 0$.

- ✓ ${}^{RL}D^{\alpha}_{0^+}$: The Riemann-Liouville fractional derivative of orde $\alpha > 0$.
- ✓ ${}^{c}D^{\alpha}_{0^+}$: The Caputo fractional derivative of orde $\alpha > 0$.
- ✓ $\phi_p(u)$: The p-Laplacian operator.
- ✓ $C(J,\mathbb{R})$: Space of continuous functions on J.
- ✓ $C^n(J,\mathbb{R})$: Space of n time continuously. differentiable functions on J
- ✓ $AC(J,\mathbb{R})$: Space of absolutely continuous functions on J.
- ✓ $L^1(J,\mathbb{R})$: space of Lebesgue integrable functions on J.
- ✓ $L^p(J,\mathbb{R})$: space of measurable functions u with $|u|^p$ belongs to $L^1(J,\mathbb{R})$.
- ✓ $L^\infty(J,\mathbb{R})$: space of functions u that are essentially bounded on J .

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Introduction

Fractional calculus is a recent field of mathematical analysis, and it is a generalization of integer differential calculus, involving derivatives and integrals of real or complex order [93, 116]. The first note about this idea of differentiation, for non-integer numbers, dates back to 1695, with a famous correspondence between Leibniz and L'Hôpital. In a letter, L'Hôpital asked Leibniz about the possibility of the order n in the notation $\frac{d^n y}{dx^n}$, for the n-th derivative of the function y, to be a non-integer, $n = \frac{1}{2}$. Since then, several mathematicians investigated this approach, like Lacroix, Fourier, Liouville, Riemann, Letnikov, Grünwald, Caputo, and contributed to the grown development of this field. Currently, this is one of the most intensively developing areas of mathematical analysis as a result of its numerous applications. The first book devoted to the fractional calculus was published by Oldham and Spanier in 1974, where the authors systematized the main ideas, methods, and applications about this field [111]. There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas, and Marichev which was published in Russian in 1987 and in English in 1993 [124]. The works devoted substantially to fractional differential equations are : the book of Miller and Ross (1993) [106], Podlubny (1999) [117], Kilbas et al. (2006) [93], Diethelm (2010) [61], Ortigueira (2011) [114], Abbas et al. (2012) [1], and Baleanu et al. (2012) [33].

In the last two decades, it was established that a series of phenomena can be studied in terms of fractional calculus. Moreover, fractional diferential equations represents a powerful tool in applied mathematics to study many problems from different fields of science and engineering, since their nonlocal property is suitable to describe memory phenomena such as nonlocal elasticity, polymers, propagation in complex medium, biological, electrochemistry, porous media, viscoelasticity, electromagnetics, see for instance [83, 90, 94, 112, 120, 131]. Recent developments of fractional differential and integral equations are given in [2, 4, 22, 136, 138, 139, 140, 141, 142].

On the other hand, fixed point theory is an important tool in nonlinear analysis, in particular, in obtaining existence results for a variety of mathematical problems. In addition, in most of the existed articles, Banach contraction principle, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem, etc. have been employed to obtain the existence and uniqueness of solution of various problems with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions, and periodic boundary conditions for fractional differential equations, under some restrictive conditions for more details see for instance [9, 11, 17, 18, 19, 25, 23, 28, 29, 43, 46, 53, 66, 75, 77, 78, 79, 80, 84, 99, 102, 109, 130]. But, in the absence of compacity and the Lipschitz condition, the previously mentioned theorems are not applicable. In such cases, the measure of noncompactness argument appears as the most convenient and useful in applications. The notion of a measure of noncompactness (briefly, MNC) was first introduced by Kuratowski [97] in 1930 which was further extended to general Banach spaces by Banás and Goebel (see [34]). Later Darbo formulated his celebrated fixed point theorem in 1955 for the case of the Kuratowski measure of non-compactness (cf. [49]) which guarantees the existence of fixed point for condensing operators. It extends both the classical Schauder's fixed point principle and (a special variant of) Banach's contraction mapping principle. After that, the Darbo's fixed point theorem has been generalized in many different directions we suggest some works [13, 88]. Moreover, we can also highlight the papers [3, 7, 12, 24, 26, 41, 42, 44, 54, 76, 118, 135] in which many researchers turned to the existence of solutions for differential equations involving different kinds of fractional derivatives under various boundary conditions. To this end, the celebrated Darbo fixed point theorem and Mönch fixed point theorem have been employed. The reader may also consult the recent book [40], where several applications of the measure of noncompactness can be found.

This thesis is arranged as follows :

In **Chapter** 1, we collect definitions, auxiliary results, lemmas and notions of measures of noncompactness, fixed point theorems that are used throughout this thesis.

Chapter 2, is devoted to the existence results of weak solutions for certain classes of nonlinear differential equation involving the Caputo fractional derivative in Banach Spaces. The arguments are based on Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. More specifically, in Section 2.2 we are interested in the existence of weak solutions for the following fractional boundary value problem

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} x(t) = f\big(t, x(t)\big), \quad t \in J := [0,T], \\ a_1 x(0) + b_1 x(T) = \lambda_1 I^{\sigma_1} x(\eta), \\ a_2\, {}^{c}D_{0^+}^{\sigma_2} x(\xi) + b_2\, {}^{c}D_{0^+}^{\sigma_3} x(\eta) = \lambda_2, \end{cases}$$

where ${}^{c}D_{0^+}^{\mu}$ is the Caputo fractional derivative of order $\mu \in \{\alpha, \sigma_2, \sigma_3\}$ such that $1 < \alpha \leq 2, 0 < \sigma_2, \sigma_3 \leq 1, I^{\sigma_1}$ the Riemann-Liouville fractional integral of order $\sigma_1 > 0$ and $f : [0,T] \times X \longrightarrow X$ is a given function satisfying some assumptions that will be specified later, X is a Banach space with norm $\|.\|, \lambda_1, a_i, b_i, (i = 1,2)$ are suitably chosen real constants, and $\lambda_2 \in X$. The main results of this problem are published in [54]

In Section 2.3, we give similar result to the following fractional Langevin equations involving two fractional orders with initial value problems

$$\begin{cases} {}^{c}D^{\beta}_{0^+}\left({}^{c}D^{\alpha}_{0^+}+\gamma\right)x(t)=f(t,x(t)), & t\in J=[0,1],\\ x^{(k)}(0)=\mu_k, & 0\le k<l,\\ x^{(\alpha+k)}(0)=\nu_k, & 0\le k<n. \end{cases}$$

Where ${}^{c}D^{\alpha}_{0^+}, {}^{c}D^{\beta}_{0^+}$ are the Caputo fractional derivatives $m-1<\alpha\le m,\ n-1<\beta\le n, l=\max\{m,n\}, m,n\in\mathbb{N}, \gamma\in\mathbb{R}$, $f:[0,1]\times X\longrightarrow X$ is a given function satisfying some assumptions that will be specified later, X is a Banach space with norm $\|\cdot\|$, $\mu_k,\nu_k\in X$. This problem has been considered in the paper [55].

Finally, an example is given at the end of each section to illustrate the theoretical results.

In **Chapter** 3, we study the existence of solutions for certain classes of fractional hybrid differential equations. Our results are based on fixed point theorem for three operators in a Banach algebra due to Dhage. In Section 3.2 we look into the existence of solutions for the following hybrid Caputo fractional differential equation :

$$\begin{cases} {}^{c}D^{\alpha}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]=h(t,x(t)), & 1<\alpha\le 2, t\in J:=[0,T],\\ a_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0}+b_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T}=\lambda_1, \\ a_2\,{}^{c}D^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta}+b_2\,{}^{c}D^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T}=\lambda_2, 0<\eta<T, \end{cases}$$

where ${}^{c}D^{\alpha}_{0^+}, {}^{c}D^{\beta}_{0^+}$ denote the Caputo fractional derivatives of order α and β, respectively, $0<\beta\le 1$, $a_i,b_i,c_i, i=1,2$ are real constants such that $a_1+b_1\neq 0, a_2\eta^{1-\beta}+b_2T^{1-\beta}\neq 0$ $g\in C\big(J\times\mathbb{R},\mathbb{R}\setminus\{0\}\big), f,h\in C\big(J\times\mathbb{R},\mathbb{R}\big)$.The main results of this problem are published in [56]

As a second problem we discuss in Section 3.3 the existence of solutions for the following boundary value problem of hybrid fractional differential equations with p-Laplacian operator

$$\begin{cases} {}^{c}D^{\beta}_{0^+}\left(\phi_p\left[{}^{c}D^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)=h(t,x(t)),\ t\in I:=[0,1],\\ \left(\phi_p\left[{}^{c}D^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)^{(i)}\Bigg|_{t=0}=0, \quad i=0,2,3\dots,n-1\\ \left(\phi_p\left[{}^{c}D^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)\Bigg|_{t=1}=0,\\ \left(\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)^{(j)}\Bigg|_{t=0}=0, \quad \text{for } j=2,3\dots,n-1\\ {}^{c}D^{\mu}_{0^+}\left[\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right]_{t=1}=0, \quad x(0)=0, \end{cases}$$

where ${}^cD^{\nu}_{0^+}$ is the Caputo fractional derivative of order $\nu \in \{\alpha,\beta,\mu\}$ such that $n-1 < \alpha,\beta \leq n$, $0<\mu\leq 1, I^{\theta}_{0^+}$ is the Riemann–Liouville fractional integral of order $\theta>0, \theta\in\{\sigma_1,\sigma_2,\ldots,\sigma_m\}$, $\phi_p(u)$ is a p-Laplacian operator, i.e., $\phi_p(u)=|u|^{p-2}u$ for $p>1, \phi_p^{-1}=\phi_q$ where $\frac{1}{p}+\frac{1}{q}=1$ and $g\in C\big(I\times\mathbb{R},\mathbb{R}\setminus\{0\}\big), h\in C\big(I\times\mathbb{R},\mathbb{R}\big), f_k\in C\big(I\times\mathbb{R},\mathbb{R}\big), 0<\sigma_k, k=1,2,\ldots,m$. This problem has been considered in the paper [57].

In Section 3.4, we present an existence results for fractional hybrid differential equations with hybrid conditions. More precisely, we will consider the following problem :

$$\begin{cases} {}^cD^{\alpha}_{0^+}\left[\dfrac{x(t)}{f\big(t,x(t),x(\varphi(t))\big)}\right]=g\big(t,x(t),x(\rho(t))\big), \forall t\in I=[0,1], \\ \\ \left[\dfrac{x(t)}{f\big(t,x(t),x(\varphi(t))\big)}\right]_{t=1}=0,\ {}^cD^{\beta}_{0^+}\left[\dfrac{x(t)}{f\big(t,x(t),x(\varphi(t))\big)}\right]_{t=\eta}=0,\ x^{(2)}(0)=0. \end{cases}$$

Where $2<\alpha\leq 3$, $0<\beta\leq 1$ are a real number, ${}^cD^{\alpha}_{0^+}, {}^cD^{\beta}_{0^+}$ are the Caputo fractional derivative, $f\in C\big(I\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\}\big), g\in C\big(I\times\mathbb{R}\times\mathbb{R},\mathbb{R}\big)$, φ and ρ are functions from $[0,1]$ into itself. The results are obtained using the technique of measures of noncompactness in the Banach algebras and a fixed point theorem for the product of two operators verifying a Darbo type condition. This problem has been considered in the paper [58].

Finally, to illustrate the theoretical results, an example is given at the end of each section.

Chapitre 1

Preliminaries and Background Materials

In this chapter, we discuss the necessary mathematical tools we need in the succeeding chapters. We look at some essential properties of fractional differential operators, limiting our scope to the Riemann-Liouville and Caputo versions. We also review some of the basic properties of measures of noncompactness and fixed point theorems which are crucial in our results regarding fractional differential equations.

1.1 Functional spaces

Let $\mathbb{R} = (-\infty, +\infty)$ and let $J := [0, T]$ the compact interval of $\mathbb{R}$. we present the following functional spaces :

Definition 1.1. Let $C(J, \mathbb{R})$ is the Banach space of continuous functions $u : J \longrightarrow \mathbb{R}$ have the valued in $\mathbb{R}$, equipped with the norm

$$\|u\|_\infty = \sup_{t \in J} |u(t)|.$$

Analogoustly, $C^n(J, \mathbb{R})$ the Banach space of functions $u : J \longrightarrow \mathbb{R}$ where u is n time continuously differentiable on J.

Denote by $L^1(J, \mathbb{R})$ the Banach space of functions u Lebesgues integrable with the norm

$$\|u\|_{L^1} = \int_0^T |u(t)| \mathrm{dt}.$$

and we denote $L^p(J, \mathbb{R})$ the space of Lebesgue integrable functions on J where $|u|^p$ belongs to $L^1(J, \mathbb{R})$, endowed with the norm

$$\|u\|_{L^p} = \left[\int_0^T |u(t)|^p \mathrm{dt} \right]^{\frac{1}{p}}.$$

In particular, if $p = \infty$, $L^\infty(J, \mathbb{R})$ is the space of all functions u that are essentially bounded on J with essential supremum

$$\|u\|_{L^\infty} = \operatorname{ess\,sup}_{t \in J} |u(t)| = \inf\{c \geq 0 : |u(t)| \leq c \text{ for a.e.} t\}.$$

Definition 1.2. A function $u\colon J \to \mathbb{R}$ is said absolutly continuous on J if for all $\varepsilon > 0$ there exists a number $\delta > 0$ such that ; for all fnite partition $[a_i, b_i]_{i=1}^n$ in J then $\sum_{k=1}^n (b_k - a_k) < \delta$ implies that $\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$

We denote by $AC(J, \mathbb{R})$ (or $AC^1(J, \mathbb{R})$) the space of all absolutely continuous functions defined on J. It is known that $AC(J, \mathbb{R})$ coincides with the space of primitives of Lebesgue summable functions :

$$u \in AC(J, \mathbb{R}) \Leftrightarrow u(t) = c + \int_0^t \psi(s)\mathrm{ds}, \quad \psi \in L^1(J, \mathbb{R}), \tag{1.1}$$

and therefore an absolutely continuous function u has a summable derivative $u'(t) = \psi(t)$ almost everywhere on J. Thus (1.1) yields

$$u'(t) = \psi(t) \text{ and } \ c = u(0).$$

Definition 1.3. For $n \in \mathbb{N}^*$ we denote by $AC^n(J, \mathbb{R})$ the space of functions $u : J \longrightarrow \mathbb{R}$ which have continuous derivatives up to order $n-1$ on J such that $u^{(n-1)}$ belongs to $AC(J, \mathbb{R})$:

$$\begin{aligned} AC^n(J, \mathbb{R}) &= \left\{ u \in C^{n-1}(J, \mathbb{R}) : u^{(n-1)} \in AC(J, \mathbb{R}) \right\} \\ &= \left\{ u \in C^{n-1}(J, \mathbb{R}) : u^{(n)} \in L^1(J, \mathbb{R}) \right\}. \end{aligned}$$

The space $AC^n(J, \mathbb{R})$ consists of those and only those functions u which can be represented in the form

$$u(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} \psi(s)\mathrm{ds} + \sum_{k=0}^{n-1} c_k t^k, \tag{1.2}$$

where $\psi \in L^1(J, \mathbb{R}), c_j \ (k = 1, \ldots, n-1) \in \mathbb{R}$.
It follows from (1.2) that

$$\psi(t) = u^{(n)}(t) \text{ and } c_k = \frac{u^{(k)}(0)}{k!}, \ (k = 1, \ldots, n-1).$$

Throughout this thesis, X denotes a real Banach space with a norm $\|\cdot\|$ and X^* its dual. By X_w we denotes the space X when endowed with the weak topology (i.e., generated by the continuous linear functionals on X). We will let $C(J, X_w)$ denotes the Banach space of continuous functions from J to X, with its weak topology (see also [45, 107]).

Denote by $C(J, X)$ the Banach space of continuous functions $J \to X$, with the usual supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\|, t \in J\}.$$

A measurable function $x : J \to X$ is said to be Bochner integrable if and only if $\|x\|$ is Lebesgue integrable.

Let $L^1(J, X)$ denote the Banach space of measurable functions $x : J \to X$ which are Bochner integrable with the norm

$$\|x\|_{L^1} = \int_0^T \|x(t)\|\mathrm{dt}.$$

Definition 1.4 ([115])**.** The function $x : J \to X$ is said to be Pettis integrable on J if and only if there is an element $x_I \in X$ corresponding to each $I \subset J$ such that $\varphi(x_I) = \int_I \varphi(x(s))ds$ for all $\varphi \in X^*$, where the integral on the right is supposed to exist in the sense of Lebesgue. By definition, $x_I = \int_I x(s)ds$.

Let $P(J,X)$ be the space of all X-valued Pettis integrable functions in the interval J.

Proposition 1.5 ([115]). *If $x(\cdot)$ is Pettis integrable and $h(\cdot)$ is a measurable and essentially bounded real-valued function, then $x(\cdot)h(\cdot)$is Pettis integrable.*

For the Pettis integral properties, see the monograph of Schwabik [126].

Definition 1.6 ([122]). A function $h : X \to X$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in X to weakly convergent sequence in X (i.e. for any $(x_n)_n$ in X with $x_n \to x$ in (X,w) then $h(x_n) \to h(x)$ in (X,w) for each $t \in J$).

Definition 1.7 ([122]). If the function $\varphi(x(\cdot))$ is differentiable for every $\varphi \in X^*$ and if there is a function $y\colon [0,T] \to X$ such that $\frac{d}{dt}(\varphi(x(t))) = \varphi(y(t))$ for every $\varphi \in X^*$ and every $t \in [0,T]$, then x is weakly differentiable and we write $x'(t) = y(t)$, where $x'(t)$ denotes the weak derivative of the function x.

1.2 Special Functions

Before introducing the basic facts on fractional operators, we recall two types of functions that are important in Fractional Calculus : the Gamma and Beta functions. Some properties of these functions are also recalled. More details about these functions can be found in [5, 63, 64, 117].

1.2.1 Gamma function

The Euler Gamma function is an extension of the factorial function to real numbers and is considered the most important Eulerian function used in fractional calculus because it appears in almost every fractional integral and derivative definitions.

Definition 1.8 ([117]). The Gamma function, or second order Euler integral, denoted $\Gamma(\cdot)$ is defined as :

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} \mathrm{d}t, \quad \alpha > 0. \tag{1.3}$$

For positive integer values n, the Gamma function becomes $\Gamma(n) = (n-1)!$ and thus can be seen as an extension of the factorial function to real values.
An important property of the gamma function $\Gamma(\alpha)$ is that it satisfies :

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0.$$

1.2.2 Beta Function

Definition 1.9 ([117]). The Beta function, or the first order Euler function, can be defined as

$$\mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} \mathrm{d}t, \quad p,q > 0.$$

We use the following formula which expresses the beta function in terms of the gamma function :

$$\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q > 0.$$

1.3 Elements From Fractional Calculus Theory

In this section, we recall some definitions of fractional integral and fractional differential operators that include all we use throughout this thesis.

1.3.1 Fractional Integrals

Definition 1.10 ([93])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f \in L^1([0,T])$ is defined by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \quad (t > 0,\ \alpha > 0).$$

Moreover, for $\alpha = 0$, we set $I_{0^+}^{0} f := f$.

Example 1.11. Let $\alpha > 0$, and $\beta > -1$. Then

$$I_{0^+}^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}.$$

Lemma 1.12 ([93])**.** *The following basic properties of the Riemann-Liouville integrals hold :*

1. *The integral operator $I_{0^+}^{\alpha}$ is linear ;*
2. *The semigroup property of the fractional integration operator $I_{0^+}^{\alpha}$ is given by the following result*
$$I_{0^+}^{\alpha}(I_{0^+}^{\beta} f(t)) = I_{0^+}^{\alpha+\beta} f(t), \quad \alpha, \beta > 0,$$
holds at every point if $f \in C([0,T])$ and holds almost everywhere if $f \in L^1([0,T])$,
3. *Commutativity*
$$I_{0^+}^{\alpha}(I_{0^+}^{\beta} f(t)) = I_{0^+}^{\beta}(I_{0^+}^{\alpha} f(t)), \quad \alpha, \beta > 0;$$
4. *The fractional integration operator $I_{0^+}^{\alpha}$ is bounded in $L^p[0,T]$ $(1 \le p \le \infty)$;*
$$\|I_{0^+}^{\alpha} f\|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|f\|_{L^p}.$$

1.3.2 Fractional Derivatives

From the definition of the Riemann-Liouville fractional integral, the fractional derivative is obtained not by replacing α with $-\alpha$ because the integral $\int_0^t (t-s)^{-\alpha-1} f(s)\,\mathrm{d}s$ is, in general, divergent. Instead, differentiation of arbitrary order is defined as the composition of ordinary differentiation D^n and fractional integration, i.e.,we can define fractional derivative of order $n-1 < \alpha \le n$ by two ways :

(1) Riemann-Liouville fractional derivative : Take fractional integral of order $(n-\alpha)$ and then take a n derivative, i.e.,

$${}^{RL}D_{0^+}^{\alpha} f(t) = D^n I_{0^+}^{n-\alpha} f(t), \quad n = [\alpha]+1.$$

(2) Caputo fractional derivative : Take n-order derivative and then take a fractional integral of order $(n-\alpha)$

$${}^{c}D_{0^+}^{\alpha} f(t) = I_{0^+}^{n-\alpha} D^n f(t), \quad n = [\alpha]+1.$$

The Riemann-Liouville fractional derivative

Definition 1.13 ([93, 117])**.** The Riemann-Liouville fractional derivative of order α of a function $f \in L^1([0,T])$ is defined by

$$^{RL}D^{\alpha}_{0^+}f(t) = D^n I^{n-\alpha}_{0^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{\mathrm{d}t^n}\int_0^t (t-s)^{n-\alpha-1}f(s)\mathrm{d}s,\ 0<t<T,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Example 1.14. Let $\alpha > 0$, and $\beta > -1$. Then

$$^{RL}D^{\alpha}_{0^+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

Remark 1.15. If we let $\beta = 0$ in the previous example, we see that the Riemann-Liouville fractional derivative of a constant is not 0. In fact,

$$^{RL}D^{\alpha}_{0^+}1(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Remark 1.16. On the other hand, for $j = 1,2,\cdots,[\alpha]+1$,

$$^{RL}D^{\alpha}_{0^+}t^{\alpha-j}(t) = 0.$$

We could say that $t^{\alpha-j}$ plays the same role in Riemann-Liouville fractional differentiation as a constant does in classical integer-ordered differentiation.

As a result, we have the following fact :

Lemma 1.17 ([93, 117])**.** *$\alpha > 0$, and $n = [\alpha]+1$ then*

$$^{RL}D^{\alpha}_{0^+}f(t) = 0 \Leftrightarrow f(t) = \sum_{j=1}^{n} c_j t^{\alpha-j},$$

where c_j $(j = 1,\ldots,n)$ are arbitrary constants.

The next result describes $^{RL}D^{\alpha}_{0^+}$ in the space $AC^n([0,T])$.

Lemma 1.18 ([117, 93])**.** *Let $\alpha > 0$, and $n = [\alpha]+1$. If $f \in AC^n([0,T])$, then the fractional derivative $^{RL}D^{\alpha}_{0^+}$ exists almost everywhere on $[0,T]$ and can be represented in the form :*

$$^{RL}D^{\alpha}_{0^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^n(s)ds + \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)}t^{k-\alpha}.$$

The following lemma shows that the fractional differentiation is an operation inverse to the fractional integration from the left.

Lemma 1.19 ([93, 117])**.** *If $\alpha > 0$ and $f \in L^p([a,b])$ $(1 \le p \le \infty)$, then the following equalities*

$$^{RL}D^{\alpha}_{0^+}I^{\alpha}_{0^+}f(t) = f(t), \tag{1.4}$$

hold almost everywhere on $[a,b]$.

The Caputo Fractional Derivative

Definition 1.20 ([93, 117])**.** For a function $f \in AC^n([0,T])$, the Caputo fractional derivative of order α defined by

$$\begin{aligned} {}^cD^{\alpha}_{0^+} f(t) &= I^{n-\alpha}_{0^+} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \end{aligned}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Example 1.21. Let $\alpha, \beta > 0$ and $n = [\alpha]+1$ Then the following relation hold

$$ {}^cD^{\alpha}_{0^+} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & (\beta \in \mathbb{N} \text{ and } \beta \geq n \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > n-1), \\ 0, & \beta \in \{0, \ldots, n-1\}. \end{cases} \tag{1.5}$$

Remark 1.22. We see that consistent with classical integer-ordered derivatives, for any constant C

$$ {}^cD^{\alpha}_{0^+} C = 0.$$

We also recognize from (1.5) that :

Lemma 1.23 ([93, 117])**.** *Let $\alpha > 0$ and $n = [\alpha]+1$ then the differential equation*

$$ {}^cD^{\alpha}_{0^+} f(t) = 0$$

has solutions

$$ f(t) = \sum_{j=0}^{n-1} c_j t^j, \quad c_j \in \mathbb{R}, j = 0 \cdots n-1.$$

Lemma 1.24 ([93, 117])**.** *Let $\alpha > \beta > 0$, and $f \in L^1([0,T])$. Then we have :*

(1) *The Caputo fractional derivative is linear ;*

(2) $I^{\alpha}_{0^+}\left({}^cD^{\alpha}_{0^+} f(t)\right) = f(t) + \sum_{j=0}^{n-1} c_j t^j$, *for some* $c_j \in \mathbb{R}, j = 0,1,2,\cdots,n-1$, *where* $n = [\alpha]+1$ *;*

(3) ${}^cD^{\alpha}_{0^+} I^{\alpha}_{0^+} f(t) = f(t)$ *;*

(4) ${}^cD^{\beta}_{0^+} I^{\alpha}_{0^+} f(t) = I^{\alpha-\beta}_{0^+} f(t)$.

Remark 1.25. We note that if $f \in AC^n([0,T])$, then Lemma 1.18 is equivalent to saying that

$$ {}^{RL}D^{\alpha}_{0^+} f(t) = {}^cD^{\alpha}_{0^+} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(1+k-\alpha)} t^{k-\alpha}.$$

Clearly, we see that if $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n-1$ then we have

$$ {}^cD^{\alpha}_{0^+} f(t) = {}^{RL}D^{\alpha}_{0^+} f(t).$$

Remark 1.26. Note that for an abstract function $u : [0,T] \longrightarrow X$, the integrals which appear in the previous definitions are taken in Bochner's sense for background and details about fractional calculus in Banach spaces see for instance [10, 60, 122, 123].

We will also need the following well-known lemmas :

Lemma 1.27 ([50])**.** *Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the function defined by $f(t) = t^{\alpha}$.*

(1) *If $\alpha \geq 1$ and $t_1, t_2 \in [0,1]$ with $t_2 > t_1$, then $t_2^{\alpha} - t_1^{\alpha} \leq \alpha(t_2 - t_1)$;*

(2) *If $0 < \alpha < 1$ and $t_1, t_2 \in [0,1]$ with $t_2 > t_1$, then $t_2^{\alpha} - t_1^{\alpha} \leq (t_2 - t_1)^{\alpha}$.*

Lemma 1.28 ([89])**.** *Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the function defined by $\varphi(t) = (t+1)^k - 1$ for $t \in [0, \infty)$, where $k \in (0,1)$. then, we have.*

(a) *φ is nondecreasing ;*

(b) *$|\varphi(t) - \varphi(t')| \leq \varphi(|t - t'|)$ for any $t, t' \in [0, \infty)$.*

Lemma 1.29 ([27, 96])**.** *Let $a, b \in \mathbb{R}, p > 0$*

(i) *If $0 < p \leq 1$, then*

$$(|a| + |b|)^p \leq |a|^p + |b|^p.$$

(ii) *If $p > 1$, then*

$$(|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p).$$

Observe that from Lemma 1.29 if $p > 0$ and $a, b \in \mathbb{R}$ then

$$(|a| + |b|)^p \leq \max\{1, 2^{p-1}\}(|a|^p + |b|^p).$$

1.4 Background about measures of non-compactness

In this section, we present the main definitions and we give several examples measures of noncompactness in certain specific spaces that will be used in the sequel

1.4.1 The general notion of a measure of noncompactness

Firstly, we need to fix the notation. In what follows, (E, d) will be a metric space, and $(X, \|\cdot\|)$ a Banach space. By $B(x, r)$ we denote the closed ball centered at x with radius r. By B_r we denote the ball $\mathrm{B}(0, r)$. If Q is non-empty subset of X , then $\overline{Q}$ and $\mathrm{Conv}Q$ denote the closure and the closed convex closure of Q, respectively. When Q is a bounded subset, $\mathrm{Diam}(Q)$ denotes the diameter of Q. Also, we denote by $\mathscr{B}_E$ (resp. $\mathscr{B}_X$) the class of non-empty and bounded subsets of E (resp. of X),

We begin with the following general definition.

Definition 1.30 ([30, 35])**.** A mapping $\mu : \mathscr{B}_E \longrightarrow \mathbb{R}_+ = [0, \infty)$ will be called a measure of noncompactness in E if it satisfies the following conditions :

(1) Regularity : $\mu(Q) = 0$ if, and only if, Q is a precompact set.

(2) Invariant under closure : $\mu(Q) = \mu(\overline{Q})$, for all $Q \in \mathscr{B}_E$.

(3) Semi-additivity : $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$, for all $Q_1, Q_2 \in \mathscr{B}_E$.
To have a MNC in a Banach space X we need to add the two following additional properties :

(4) Semi-homogeneity : $\mu(\lambda Q) = |\lambda|\mu(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathscr{B}_X$.

(5) Invariant under translations : $\mu(x+Q) = \mu(Q)$, for all $x \in X$ and $Q \in \mathscr{B}_X$.

Three main and most frequently used MNCs : the Kuratowski MNC κ , the Hausdorff MNC χ, and the De Blasi Measure of Weak Noncompactness β.

1.4.2 The Kuratowski and Hausdorff measure of noncompactness

we present a list of three important examples of measures of noncompactness which arise over and over in applications. The first example is the Kuratowski measure of noncompactness (or set measure of noncompactness).

Definition 1.31 ([97, 98])**.** Let (E,d) be a metric space and Q be a bounded subset of E. Then the Kuratowski measure of noncompactness (the set-measure of noncompactness, κ-measure) of Q, denoted by $\kappa(Q)$, is the infimum of the set of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of sets with diameters$< \varepsilon$, i.e.,

$$\kappa(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i,\ S_i \subset E, \operatorname{diam}(S_i) < \varepsilon, i = 1,2,\ldots,n, n \in \mathbb{N}\right\}.$$

In general, the computation of the exact value of $\kappa(Q)$ is difficult. Another measure of noncompactness, which seems to be more applicable, is so-called Hausdorff measure of noncompactness (or ball measure of noncompactness). It is defined as follows.

Definition 1.32. Let (E,d) be a complete metric space. The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of E, denoted by $\chi(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of balls with radi $< \varepsilon$, i.e.,

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in E, r_i < \varepsilon, i = 1,2,\ldots,n, n \in \mathbb{N}\right\}.$$

If $(X, \|\cdot\|)$ is a Banach space, we have the following equivalent definition.

Definition 1.33. Let $(X, \|\cdot\|)$ be a Banach space. The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of X, denoted by $\chi(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q has a finite ε-net in X, i.e.,

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \subset S + \varepsilon B(0,1), S \subset X, S \text{ is finite}\}.$$

We list below some properties are common to κ and χ and so we are going to use ϕ to denote either of them. These properties follow immediately from the definitions and show that both mappings are measures of noncompactness in the sense of Definition 1.30.

Properties 1. ([30, 97]) Let ϕ denote κ or χ. Then the following properties are satisfied in any complete metric space E :

(a) Regularity : $\phi(Q) = 0$ if, and only if, Q is a precompact set.

(b) Invariant under passage to the closure : $\phi(Q) = \phi(\overline{Q})$, for all $Q \in \mathscr{B}_E$.

(c) Semi-additivity : $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$, for all $Q_1, Q_2 \in \mathscr{B}_E$.

(d) Monotonicity :$Q_1 \subset Q_2 \Rightarrow \phi(Q_1) \leq \phi(Q_2)$.

(e) $\phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\}$, for all $Q_1, Q_2 \in \mathscr{B}_E$.

(f) Non-singularity : If Q is a finite set, then $\phi(Q) = 0$.

(g) Generalized Cantor's intersection. If $\{Q_n\}_{n=1}^{\infty}$ is a decreasing sequence of bounded and closed nonempty subsets of E and $\lim_{n\to\infty} \phi(Q_n) = 0$ then $\bigcap_{n=1}^{\infty} Q_n$ is nonempty and compact in E. If X is a Banach space, then we also have :

(h) Semi-homogeneity : $\phi(\lambda Q) = |\lambda|\phi(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathscr{B}_X$.

(i) Algebraic semi-additivity : $\phi(Q_1 + Q_2) \leq \phi(Q_1) + \phi(Q_2)$, for all $Q_1, Q_2 \in \mathscr{B}_X$.

(j) Invariant under translations : $\phi(x + Q) = \phi(Q)$, for all $x \in X$ and $Q \in \mathscr{B}_X$.

(k) invariant under passage to the convex hull : $\phi(A) = \phi(conv(A))$,

(l) Lipschitzianity : $|\phi(Q_1) - \phi(Q_2)| \leq L_\phi d_H(Q_1, Q_2)$, where $L_\chi = 1$, $L_\kappa = 2$ and d_H denotes the Hausdorff semimetric.

(m) Continuity : For every $Q \in \mathscr{B}_X$ and for all $\varepsilon > 0$, there is $\delta > 0$ such that $|\phi(Q) - \phi(Q_1)| \leq \varepsilon$ for all Q_1satisfying $d_H(Q, Q_1) < \delta$.

Theorem 1.34 ([67, 110]). *Let $B(0,1)$ be the unit ball in a Banach space X. Then $\kappa(B(0,1)) = \chi(B(0,1)) = 0$ if X is finite dimensional, and $\kappa(B(0,1)) = 2, \chi(B(0,1)) = 1$ otherwise.*

The next result shows the equivalence between the Kuratowski's measure of noncompactness and the Hausdorff measure of noncompactness.

Theorem 1.35 ([39]). *Let (E,d) be a complete metric space and B be a nonempty and bounded subset of E. Then*

$$\chi(Q) \leq \kappa(Q) \leq 2\chi(Q).$$

1.4.3 The De Blasi Measure of Weak Noncompactness

The measure of weak noncompactness is an MNC in the sense of the general definition provided X is endowed with the weak topology. The important example of a measure of weak noncompactness was defined by De Blasi [52] in 1977 and it is the map $\beta : \mathscr{B}(X) \longrightarrow [0,\infty)$ defined by

$$\beta(Q) = \inf\big\{\varepsilon > 0 : \text{there exists } W \in \mathscr{K}^w(X) \text{ with } \ Q \subset W + \varepsilon B_1\big\}.$$

for every $Q \in \mathscr{B}(X)$. Here, $\mathscr{B}(X)$ means the collection of all nonempty bounded subsets of X and $\mathscr{K}^w(X)$ is the subset of $\mathscr{B}(X)$ consisting of all weakly compact subsets of X. Now, we are going to recall some basic properties of $\beta(\cdot)$.

Properties 2. Let Q_1, Q_2 be two elements of $\mathscr{B}(X)$. Then De Blasi measure of noncompactness has the following properties. For more details and the proof of these properties see [52]

(a) $Q_1 \subset Q_2 \Rightarrow \beta(Q_1) \leq \beta(Q_2)$,

(b) $\beta(Q) = 0 \Leftrightarrow Q$ is relatively weakly compact,

(c) $\beta(Q_1 \cup Q_2) = \max\{\beta(Q_1), \beta(Q_2)\}$,

(d) $\beta(\overline{Q}^w) = \beta(Q)$, where $\overline{Q}^w$ denotes the weak closure of Q,

(e) $\beta(Q_1 + Q_2) \leq \beta(Q_1) + \beta(Q_2)$,

(f) $\beta(\lambda Q) \leq |\lambda|\beta(Q), \lambda \in \mathbb{R}$,

(g) $\beta(\mathrm{conv}(Q)) = \beta(Q)$,

(h) $\beta(\cup_{|\lambda| \leq h} \lambda Q) = h\beta(Q)$.

Lemma 1.36 ([76]). *Let $V \subset C(J,X)$ be a bounded and equicontinuous subset. Then the function $t \to \beta(V(t))$ is continuous on J,*

$$\beta_C(V) = \max_{t \in J} \beta(V(t)),$$

and

$$\beta\left(\int_J u(s)\mathrm{ds} : u \in V\right) \leq \int_J \beta(V(s))\mathrm{ds},$$

where $V(s) = \{u(s) : u \in V\}, s \in J$ and β_C is the De Blasi measure of weak noncompactness defined on the bounded sets of $C(J,X)$.

On the other hand, the use of the Hausdorff measure χ in practice requires expressing of χ with the help of a handy formula associated with the structure of the underlying Banach space X in which the measure χ is considered. Unfortunately, it turns out that such formulas are known only in a few Banach spaces such as the classical space $C([0,T])$ of real functions defined and continuous on the interval $[0,T]$ or Banach sequence spaces c_0 and ℓ^p. We illustrate this assertion by a few examples.

1.4.4 Measures of Noncompactness in Some Spaces

The Hausdorff MNC in the Spaces $C[0,T]$

Let $C[0,T]$ denote the classical Banach space consisting of all real functions defined and continuous on the interval $[0,T]$. We consider $C[0,T]$ furnished with the standard maximum norm, i.e.,

$$\|x\| = \max_{t \in [0,T]} |x(t)|.$$

Keeping in mind the Arzelà-Ascoli criterion for compactness in $C[0,T]$ we can express the Hausdorff measure of noncompactness in the below described manner.

Namely, for $x \in C[0,T]$ denote by $\omega(x,\varepsilon)$ the modulus of continuity of the function x :

$$\omega(x,\varepsilon) = \sup\{|x(t) - x(s)| : t,s \in [0,T], |t-s| \leq \varepsilon\}.$$

Next, for an arbitrary set $Q \in \mathscr{B}_{C[a,b]}$ let us put :

$$\omega(Q,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in Q\},$$

and

$$\omega_0(Q) = \lim_{\varepsilon \to 0} \omega(Q, \varepsilon). \tag{1.6}$$

It can be shown [36] that for $Q \in \mathscr{B}_{C[a,b]}$ the following equality holds :

$$\chi(Q) = \frac{1}{2}\omega_0(Q).$$

This equality is very useful in applications.

The Hausdorff MNC in the Space c_0

Let c_0 denote the space of all real sequences $x = \{x_n\}$ converging to zero and endowed with the maximum norm, i.e.,

$$\|x\| = \|\{x_n\}\| = \max\{|x_n| : n = 1, 2, 3, \ldots\}.$$

To describe the formula expressing the Hausdorff measure χ in the space c_0 fix arbitrarily a set $Q \in \mathscr{B}_{c_0}$. Then, it can be shown that the following equality holds (cf. [36]) :

$$\chi(Q) = \lim_{n \to \infty} \left\{ \sup_{x \in Q} \left(\max_{i \geq n} |x_i| \right) \right\}.$$

The formula expressing the Hausdorff measure of noncompactness is also known in the space ℓ^p for $1 \leq p < \infty$ [36]. On the other hand in the classical Banach spaces $L^p(a, b)$ and ℓ^∞ we only know some estimates of the Hausdorff measure of noncompactness with the help of formulas that define measures of noncompactness in those spaces. Refer to [36] for details.

1.5 Fixed-Point Theorem

In this section, first we recall Schauder's and Darbo's fixed-point theorem and we review some important generalizations of Darbo's theorem, which has been proved recently.

Definition 1.37 ([30])**.** If E_1 and E_2 are metric spaces, μ_1 and μ_2 measures of noncompactness defined on E_1 and E_2 respectively, and $\mathscr{T} : D(\mathscr{T}) \subset E_1 \longrightarrow E_2$ a mapping, then

(a) $\mathscr{T}$ is a (μ_1, μ_2)-contractive operator with constant $k > 0$ (or simply k-(μ_1, μ_2)- contractive) if $\mathscr{T}$ is continuous and verifies that for every bounded subset Q of $D(\mathscr{T})$ we have

$$\mu_2(\mathscr{T}(Q)) \leq k\mu_1(Q).$$

In the particular case when $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$ we simply say that $\mathscr{T}$ is a k-μ-contractive operator.

(b) $\mathscr{T}$ is a (μ_1, μ_2)-condensing operator with constant $k > 0$ (or simply k-(μ_1, μ_2)-condensing) if $\mathscr{T}$ is continuous and verifies that for every bounded and nonprecompact subset Q of $D(\mathscr{T})$ we have

$$\mu_2(\mathscr{T}(Q)) < k\mu_1(Q).$$

In the particular case when $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$ we simply say that $\mathscr{T}$ is a k-μ-condensing operator. Moreover, if $k = 1$ we say that $\mathscr{T}$ is a μ-condensing operator.

Remark 1.38. (1) If $\mu = \kappa$, the k-κ-contractive (or k-κ-condensing) operators are usually called k-set-contractive (or k-set-condensing) operators.

(2) If $\mu = \chi$, the k-χ-contractive (or k-χ-condensing) operators are usually called k-ball-contractive (or k-ball-condensing) operators.

We are now going to give some easy properties of these operators (see [30]).

Proposition 1.39. *If E_1, E_2 and E_3 are metric spaces, μ_1, μ_2 and μ_3 measures of non-compactness defined on E_1, E_2 and E_3 respectively, and $\mathscr{T} : D(\mathscr{T}) \subset E_1 \longrightarrow E_2$ and $\mathscr{S} : E_2 \longrightarrow E_3$ mappings, then*

(i) If $\mathscr{T}$ is k-(μ_1, μ_2)-contractive, then $\mathscr{T}$ is k'-(μ_1, μ_2)-condensing for every $k' > k$.

(ii) If $\mathscr{T}$ is k_1-(μ_1, μ_2)-contractive (condensing) and $\mathscr{S}$ is k_2-(μ_1, μ_2) -contractive (condensing), then $\mathscr{S} \circ \mathscr{T}$ is $k_1 k_2$-(μ_1, μ_2)-contractive (condensing).

(iii) If X and Y are Banach spaces, μ_2 algebraically semi-additive, $\mathscr{T}_1 : D(\mathscr{T}_1) \subset X \longrightarrow Y$ k_1-(μ_1, μ_2)-contractive (condensing) and $\mathscr{T}_2 : D(\mathscr{T}_2) \subset X \longrightarrow Y$ k_2-(μ_1, μ_2)-contractive (condensing), then $\mathscr{T}_1 + \mathscr{T}_2$ $(k_1 + k_2)$-(μ_1, μ_2)-contractive (condensing).

At first, let us recall the well-known Schauder fixed point theorem that will be used later.

Theorem 1.40. (*Schauder's fixed point theorem* [*125*]) *. Let C be a nonempty, convex and compact subset of a Banach space X. Then, every continuous mapping $\mathscr{T} : C \longrightarrow C$ has at least one fixed point.*

Darbo formulated his celebrated fixed point theorem in 1955 for the case of the Kuratowski measure of non-compactness (cf. [49, 121]). This was the first theorem that involves the notion of measure of non-compactness. Quite recently in [13, 88], given an extension of Darbo's fixed point theorem and used it to study the problem of existence of solutions for a general system of nonlinear integral equations. Here we present Darbo's theorem.

Theorem 1.41. (*Darbo and Sadovskii* [*36*, *49*]) *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $\mathscr{T} : \Omega \to \Omega$ be a continuous operator. If $\mathscr{T}$ is a μ-condensing operator. Then $\mathscr{T}$ has at least one fixed point.*

Remark 1.42. Schauder's theorem extends Brouwer's theorem (from $\mathbb{R}^n$ to infinite dimensional Banach spaces) and Darbo's theorem extends Schauder's theorem (from compact to condensing operators).

1.5.1 A Fixed Point Theorem for θ-μ-Contractive Mappings

The following generalization of Darbo's fixed point theorem appears in [88] and it is the version in the context of measures of non-compactness of a recent result about fixed point theorem which appears in [87], we present this result. Previously, we need to introduce the class Θ of functions. By Θ we denote the class of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following condition : For any sequence $(t_n) \subset (0, \infty)$

$$\lim_{n\to\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\to\infty} t_n = 0.$$

Theorem 1.43. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $\mathscr{T} : \Omega \to \Omega$ be a continuous mapping. Suppose that there exist $\theta \in \Theta$ and $k \in [0,1)$ such that, for any nonempty subset X of Ω with $\mu(\mathscr{T}B) > 0$,*

$$\theta(\mu(\mathscr{T}B)) \leq (\theta(\mu(B)))^k,$$

for any non-empty subset B of Ω, where μ is a measure of non-compactness in X. Then $\mathscr{T}$ has a fixed point in Ω.

Examples of functions belonging to the class of Θ are $\theta(t) = e^{\sqrt{t}}, \theta(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{t^\alpha}\right)$ with $0 < \alpha < 1$ and $\theta(t) = (1+t^2)^\beta$ with $\beta > 0$, [88].

Remark 1.44. Taking $\theta(t) = e^t$ in Theorem 1.43, we obtain Darbo's fixed point result (see Theorem 1.41).

1.5.2 A Fixed Point Result in a Banach Algebra

In this section, we recall some definitions and we give some results that we will need in the sequel.

Definition 1.45. An algebra $\mathscr{X}$ is a vector space endowed with an internal composition law noted by $(\cdot)$ that is,

$$\begin{cases} \mathscr{X} \times \mathscr{X} & \longrightarrow \mathscr{X} \\ (x,y) & \longrightarrow x \cdot y, \end{cases}$$

which is associative and bilinear.
A normed algebra is an algebra endowed with a norm satisfying the following property

$$\text{for all } x,y \in \mathscr{X} \ \|x \cdot y\| \leq \|x\|\|y\|.$$

A complete normed algebra is called a Banach algebra.

Now, we recall a useful concept (see [38]).

Definition 1.46. Let $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ be a Banach algebra. A measure of non-compactness μ in $\mathscr{X}$ said to satisfy condition (m) if it satisfies the following condition :

$$\mu(AB) \leq \|A\|\mu(B) + \|B\|\mu(A),$$

for any $A, B \in \mathscr{B}_{\mathscr{X}}$.

It is known that the family of all real-valued and continuous functions defined on the interval $[0,T]$ is denoted by $C([0,T],\mathbb{R})$. Also, $C([0,T],\mathbb{R})$ is a Banach space with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0,T]\}.$$

Obviously, the space $C([0,T],\mathbb{R})$ has also the structure of Banach algebra.

In [36], it is proved that ω_0 is a measure of non-compactness in $C[0,T]$ where ω_0 is given in (1.6) .

Proposition 1.47 ([**?**, 51]). *The measure of noncompactness ω_0 on $C[0,T]$ satisfies condition (m).*

The following hybrid fixed point theorem for three operators in a Banach algebra $\mathscr{X}$ due to Dhage [59] will be used to prove the existence result for the nonlocal boundary value problem.

Lemma 1.48. *Let S be a closed convex , bounded and nonempty subset of a Banach algebra $\mathscr{X}$, and let $\mathscr{A},\mathscr{C} : \mathscr{X} \longrightarrow \mathscr{X}$ and $\mathscr{B} : S \longrightarrow \mathscr{X}$ be three operators such that*

(a) *$\mathscr{A}$ and $\mathscr{C}$ are Lipschitzian with Lipschitz constants δ and ξ , respectively ;*

(b) *$\mathscr{B}$ is compact and continuous ;*

(c) *$x = \mathscr{A}x\mathscr{B}y + \mathscr{C}x \Rightarrow x \in S$ for all $y \in S$,*

(d) *$\delta M + \xi < 1$ where $M = \left\|\mathscr{B}(S)\right\|$.*

Then the operator equation $\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x$ has a solution in S.

Theorem 1.49 ([113]). *Let D be a closed convex and equicontinuous subset of a Banach space X such that $0 \in D$. Assume that $\mathscr{N} : D \to D$ is weakly sequentially continuous. If the implication*

$$\overline{V} = \overline{conv}(\{0\} \cup \mathscr{N}(V)) \Rightarrow V \text{ is relatively weakly compact}, \tag{1.7}$$

holds for every subset $V \subset D$, then $\mathscr{N}$ has a fixed point.

Chapitre 2

The existence of solutions for a nonlinear differential equation involving the Caputo fractional-order in Banach Spaces

2.1 Introduction

In this chapter, we are concerned with the existence of weak solutions for certain classes of nonlinear fractional differential equations in Banach Spaces via Caputo's fractional derivative. First, we investigate the problem of existence for a boundary value problem for fractional differential equations with fractional integral boundary conditions in Banach spaces. Next, we study the existence of weak solutions for an initial value problem for fractional Langevin equations involving two fractional orders posed on an arbitrary Banach space. The used approach is based on Mönch's fixed point theorem combined with the technique of measures of weak noncompactness. We also provide some illustrative examples in support of our existence theorems.

2.2 Weak Solutions For some Nonlinear Fractional Differential Equations with fractional integral boundary conditions in Banach Spaces

In this section, we give conditions for the existence of solution for a class of fractional differential equations with fractional integral boundary conditions of the type :

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = f(t,x(t)), \quad t \in J := [0,T], \\ a_1x(0) + b_1x(T) = \lambda_1 I^{\sigma_1}_{0^+}x(\eta), \\ a_2\, {}^cD^{\sigma_2}_{0^+}x(\xi) + b_2\, {}^cD^{\sigma_3}_{0^+}x(\eta) = \lambda_2. \end{cases} \tag{2.1}$$

where ${}^cD^{\mu}_{0^+}$ is the Caputo fractional derivative of order $\mu \in \{\alpha, \sigma_2, \sigma_3\}$ such that $1 < \alpha \leq 2, 0 < \sigma_2, \sigma_3 \leq 1, I^{\sigma_1}_{0^+}$ the Riemann-Liouville fractional integral of order $\sigma_1 > 0$ and $f : [0,T] \times$

$X \longrightarrow X$ is a given function satisfying some assumptions that will be specified later, X is a Banach space with norm $\|.\|, \lambda_1, a_i, b_i, (i = 1,2)$ are suitably chosen real constants, and $\lambda_2 \in E$

We remark that when $X = \mathbb{R}, b_1 = T = \sigma_2 = 1, a_1 = \lambda_2 = 0$, the problem (2.1) reduces to the one considered in [77] in the scalar case using Banach contraction principal, Schaefer's fixed point theorem and Krasnoselskii's fixed point theorem. Here we extend the results of [77] to cover the abstract case.

2.2.1 Existence of solutions [1]

First of all, we define what we mean by a weak solution for the boundary value problem (2.1).

Definition 2.1. By a weak solution of (2.1), we mean a function $x : J \longrightarrow X$ such that the weak fractional derivative ${}^cD^{\alpha}_{0^+}$ exists and is weakly continuous and satisfies problem (2.1).

For the existence of weak solutions for the boundary value problem (2.1), we need the following auxiliary lemma.

Lemma 2.2. *Let $1 < \alpha \leq 2$ and h be continuous function on $J := [0,T]$. Then the linear problem*

$$ {}^cD^{\alpha}_{0^+}x(t) = h(t), \tag{2.2}$$

with boundary conditions

$$a_1x(0) + b_1x(T) = \lambda_1 I^{\sigma_1}_{0^+}x(\eta), \;\; a_2\, {}^cD^{\sigma_2}_{0^+}x(\xi) + b_2\, {}^cD^{\sigma_3}_{0^+}x(\eta) = \lambda_2. \tag{2.3}$$

is equivalent to the fractional integral equation

$$\begin{aligned} x(t) = {} & I^{\alpha}_{0^+}h(t) + \frac{1}{v_0}\{\lambda_1 I^{\alpha+\sigma_1}_{0^+}h(\eta) - b_1 I^{\alpha}_{0^+}h(T)\} \\ & + \frac{v_1}{v_0v_2}\{\lambda_2 - (a_2 I^{\alpha-\sigma_2}_{0^+}h(\xi) + b_2 I^{\alpha-\sigma_3}_{0^+}h(\eta))\} \\ & + \frac{t}{v_2}\{a_2 I^{\alpha-\sigma_2}_{0^+}h(\xi) + b_2 I^{\alpha-\sigma_3}_{0^+}h(\eta) - \lambda_2\}. \end{aligned} \tag{2.4}$$

where

$$v_0 = a_1 + b_1 - \frac{\lambda_1\eta^{\sigma_1}}{\Gamma(\sigma_1+1)}, \; v_1 = b_1T - \frac{\lambda_1\eta^{\sigma_1+1}}{\Gamma(\sigma_1+2)}, \; v_2 = \frac{a_2\xi^{1-\sigma_2}}{\Gamma(2-\sigma_2)} + \frac{b_2\eta^{1-\sigma_3}}{\Gamma(2-\sigma_3)}, \tag{2.5}$$

and $v_0v_2 \neq 0$.

Proof. By applying Lemmas 1.12, 1.24, we may reduce (2.2) to an equivalent integral equation

$$x(t) = I^{\alpha}_{0^+}h(t) - c_0 - c_1t, \quad c_0, c_1 \in \mathbb{R}. \tag{2.6}$$

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Applying the boundary conditions (2.3) in (2.6) we may obtain

$$\begin{aligned}
I_{0^+}^{\sigma_1}x(\eta) &= I_{0^+}^{\sigma_1+\alpha}h(\eta) - c_0\frac{\eta^{\sigma_1}}{\Gamma(\sigma_1+1)} - c_1\frac{\eta^{\sigma_1+1}}{\Gamma(\sigma_1+2)},\\
{}^cD_{0^+}^{\sigma_i}x(t) &= I_{0^+}^{\alpha-\sigma_i}h(t) - c_1\frac{\Gamma(2)}{\Gamma(2-\sigma_i)}t^{1-\sigma_i},\ i=2,3.
\end{aligned}$$

After collecting the similar terms in one part, we have the following equation :

$$\left(a_1+b_1-\frac{\lambda_1\eta^{\sigma_1}}{\Gamma(\sigma_1+1)}\right)c_0+\left(b_1T-\frac{\lambda_1\eta^{\sigma_1+1}}{\Gamma(\sigma_1+2)}\right)c_1 = b_1I_{0^+}^{\alpha}h(T)-\lambda_1I_{0^+}^{\sigma_1+\alpha}h(\eta) \tag{2.7}$$

$$\left(\frac{a_2\xi^{1-\sigma_2}}{\Gamma(2-\sigma_2)}+\frac{b_2\eta^{1-\sigma_3}}{\Gamma(2-\sigma_3)}\right)c_1 = \lambda_2-\left(a_2I_{0^+}^{\alpha-\sigma_2}h(\xi)+b_2I_{0^+}^{\alpha-\sigma_3}h(\eta)\right). \tag{2.8}$$

Rewriting equations (2.7) and (2.8) by using (2.5) we obtain

$$\begin{aligned}
v_0c_0+v_1c_1 &= b_1I_{0^+}^{\alpha}h(T)-\lambda_1I_{0^+}^{\sigma_1+\alpha}h(\eta)\\
v_2c_1 &= \lambda_2-\left(a_2I_{0^+}^{\alpha-\sigma_2}h(\xi)+b_2I_{0^+}^{\alpha-\sigma_3}h(\eta)\right).
\end{aligned} \tag{2.9}$$

Solving (2.9), we find that

$$\begin{aligned}
c_1 &= \frac{\lambda_2-\left(a_2I_{0^+}^{\alpha-\sigma_2}h(\xi)+b_2I_{0^+}^{\alpha-\sigma_3}h(\eta)\right)}{v_2}\\
c_0 &= \frac{b_1I_{0^+}^{\alpha}h(T)-\lambda_1I_{0^+}^{\sigma_1+\alpha}h(\eta)}{v_0}+\frac{v_1}{v_0v_2}\left\{a_2I_{0^+}^{\alpha-\sigma_2}h(\xi)+b_2I_{0^+}^{\alpha-\sigma_3}h(\eta)-\lambda_2\right\}.
\end{aligned}$$

Substituting the value of c_0, c_1 in (2.6) we get (2.4). The converse follows by direct computation which completes the proof.

□

For simplicity of presentation, we give some notations :

$$\begin{aligned}
\mathscr{M} &= \frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{|b_1|T^\alpha}{|v_0|\Gamma(\alpha+1)}+\frac{|\lambda_1|}{|v_0|\Gamma(\alpha+\sigma_1+1)}\eta^{\alpha+\sigma_1}\\
&\quad+\frac{|a_2|(|v_1|+|v_0|T)\xi^{\alpha-\sigma_2}}{|v_0v_2|\Gamma(\alpha-\sigma_2+1)}+\frac{|b_2|(|v_1|+|v_0|T)\eta^{\alpha-\sigma_3}}{|v_0v_2|\Gamma(\alpha-\sigma_3+1)},\\
\mathscr{L} &= \frac{1}{\Gamma(\alpha)}+\frac{1}{|v_2|}\left[\frac{|a_2|\xi^{\alpha-\sigma_2}}{\Gamma(\alpha-\sigma_2+1)}+\frac{|b_2|\eta^{\alpha-\sigma_3}}{\Gamma(\alpha-\sigma_3+1)}+|\lambda_2|\right].
\end{aligned}$$

We will need to introduce the following hypotheses which are assumed thereafter :

(H1) For each $t\in J$, the function $f(t,\cdot)$ is weakly sequentially continuous ;

(H2) For each $x\in C(J,X)$, the function $f(\cdot,x(\cdot))$ is Pettis integrable on J ;

(H3) There exist $p\in L^\infty(J,\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that

$$\|f(t,x(t))\|\le p(t)\psi(\|x\|);$$

(H4) There exists a constant $R > 0$ such that

$$\frac{R}{\|p\|_{L^\infty}\psi(R)\mathscr{M}+\frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0v_2|}} \geq 1; \tag{2.10}$$

(H5) For each bounded set $D \subset X$, and each $t \in J$, the following inequality holds

$$\beta(f(t,D)) \leq p(t)\beta(D).$$

Now we are in able to establish the main results.

Theorem 2.3. *Assume that assumptions (H1)-(H5) hold. If*

$$\|p\|_{L^\infty}\mathscr{M} < 1, \tag{2.11}$$

then the boundary value problem (2.1) *has at least one solution.*

Proof.

Transform the integral equation (2.4) into a fixed point equation. Consider the operator $\mathscr{N}: C(J,X) \longrightarrow C(J,X)$ defined by :

$$\begin{aligned}
\mathscr{N}x(t) &= I_{0^+}^{\alpha} f(s,x(s))(t) + \frac{1}{v_0}\left\{\lambda_1 I_{0^+}^{\alpha+\sigma_1} f(s,x(s))(\eta) - b_1 I_{0^+}^{\alpha} f(s,x(s))(T)\right\} \\
&+ \frac{v_1}{v_0v_2}\left\{\lambda_2 - (a_2 I_{0^+}^{\alpha-\sigma_2} f(s,x(s))(\xi) + b_2 I_{0^+}^{\alpha-\sigma_3} f(s,x(s))(\eta))\right\} \\
&+ \frac{t}{v_2}\left\{a_2 I_{0^+}^{\alpha-\sigma_2} f(s,x(s))(\xi) + b_2 I_{0^+}^{\alpha-\sigma_3} f(s,x(s))(\eta) - \lambda_2\right\}.
\end{aligned} \tag{2.12}$$

First notice that, for $x \in C(J,X)$, we have $f(\cdot,x(\cdot)) \in P(J,X)$ (assumption (H2)). Since

$$\frac{(t-\cdot)^{\alpha-1}}{\Gamma(\alpha)}, \frac{(T-\cdot)^{\alpha-1}}{\Gamma(\alpha)}, \frac{(\eta-\cdot)^{\alpha+\sigma_1-1}}{\Gamma(\alpha+\sigma_1)}, \frac{(\xi-\cdot)^{\alpha-\sigma_2-1}}{\Gamma(\alpha-\sigma_2)}, \frac{(\eta-\cdot)^{\alpha-\sigma_3-1}}{\Gamma(\alpha-\sigma_3)},$$

are $\in L^\infty(J)$ then

$$\begin{aligned}
&\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(\cdot,x(\cdot)), \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(\cdot,x(\cdot)), \frac{(\eta-s)^{\alpha+\sigma_1-1}}{\Gamma(\alpha+\sigma_1)} f(\cdot,x(\cdot)), \\
&\frac{(\xi-s)^{\alpha-\sigma_2-1}}{\Gamma(\alpha-\sigma_2)} f(\cdot,x(\cdot)), \frac{(\eta-s)^{\alpha-\sigma_3-1}}{\Gamma(\alpha-\sigma_3)} f(\cdot,x(\cdot)),
\end{aligned}$$

for all $t \in J$ are Pettis integrable (Proposition 1.5) and thus, the operator $\mathscr{N}$ is well defined. Let $R > 0$, and consider the set

$$D = \{x \in C(J,X) : \|x\|_\infty \leq R, \|x(t_2)-x(t_1)\| \leq \|p\|_{L^\infty}\psi(R)\mathscr{L}(t_2-t_1)\},$$

where R satisfies inequality (2.10). Notice that D is a closed, convex, bounded, and equicontinuous subset of $C(J,X)$ .We shall show that the operator $\mathscr{N}$ satisfies all the assumptions of Theorem 1.49. The proof will be given in several steps.

Step 1 :We will show that the operator $\mathcal{N}$ maps D into D.

Take $x \in D, t \in J$ and assume that $\mathcal{N}x(t) \neq 0$. Then there exists $\varphi \in X^*$ such that $\|\mathcal{N}x(t)\| = \varphi(\mathcal{N}x(t))$. Thus

$$
\begin{aligned}
\|\mathcal{N}x(t)\| = \varphi\Bigg(& I_{0^+}^{\alpha} f(s,x(s))(t) + \frac{1}{v_0}\{\lambda_1 I_{0^+}^{\alpha+\sigma_1} f(s,x(s))(\eta) - b_1 I_{0^+}^{\alpha} f(s,x(s))(T)\} \\
& + \frac{v_1}{v_0 v_2}\{\lambda_2 - (a_2 I_{0^+}^{\alpha-\sigma_2} f(s,x(s))(\xi) + b_2 I_{0^+}^{\alpha-\sigma_3} f(s,x(s))(\eta))\} \\
& + \frac{t}{v_2}\{a_2 I_{0^+}^{\alpha-\sigma_2} f(s,x(s))(\xi) + b_2 I_{0^+}^{\alpha-\sigma_3} f(s,x(s))(\eta) - \lambda_2\}\Bigg) \\
& \leq I_{0^+}^{\alpha}\varphi(f(s,x(s)))(t) + \frac{|b_1|}{|v_0|} I_{0^+}^{\alpha}\varphi(f(s,x(s)))(T) + \frac{|\lambda_1|}{|v_0|} I_{0^+}^{\alpha+\sigma_1}\varphi(f(s,x(s)))(\eta) \\
& + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} + \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_2}\varphi(f(s,x(s)))(\xi) \\
& + \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_3}\varphi(f(s,x(s)))(\eta).
\end{aligned}
$$

From $(H3)$, we have

$$
\begin{aligned}
\|\mathcal{N}x(t)\| \leq \psi(\|x\|) \Bigg\{ & I_{0^+}^{\alpha} p(s)(T) + \frac{|b_1|}{|v_0|} I_{0^+}^{\alpha} p(s)(T) + \frac{|\lambda_1|}{|v_0|} I_{0^+}^{\alpha+\sigma_1} p(s)(\eta) \\
& + \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_2} p(s)(\xi) + \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_3} p(s)(\eta)\Bigg\} \\
& + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} \\
& \leq \|p\|_{L^\infty}\psi(R)\Bigg\{ I_{0^+}^{\alpha}(1)(T) + \frac{|b_1|}{|v_0|} I_{0^+}^{\alpha}(1)(T) + \frac{|\lambda_1|}{|v_0|} I_{0^+}^{\alpha+\sigma_1}(1)(\eta) \\
& + \frac{|a_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_2}(1)(\xi) + \frac{|b_2|(|v_1| + |v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_3}(1)(\eta)\Bigg\} \\
& + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} \\
& \leq \|p\|_{L^\infty}\psi(R)\Bigg\{ \frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{|b_1|T^{\alpha}}{|v_0|\Gamma(\alpha+1)} + \frac{|\lambda_1|\eta^{\alpha+\sigma_1}}{|v_0|\Gamma(\alpha+\sigma_1+1)} \\
& + \frac{|a_2|(|v_1| + |v_0|T)\xi^{\alpha-\sigma_2}}{|v_0 v_2|\Gamma(\alpha-\sigma_2+1)} + \frac{|b_2|(|v_1| + |v_0|T)\eta^{\alpha-\sigma_3}}{|v_0 v_2|\Gamma(\alpha-\sigma_3+1)}\Bigg\} + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} \\
& \leq \|p\|_{L^\infty}\psi(R)\mathcal{M} + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} \leq R.
\end{aligned}
$$

Next, let $t_1, t_2 \in J$ be such that $t_1 < t_2$ and let $x \in D$ be such that

$$\mathcal{N}x(t_2) - \mathcal{N}x(t_1) \neq 0.$$

Then there exists $\varphi \in X^*$ such that

$$\|\mathcal{N}(x)(t_2) - \mathcal{N}(x)(t_1)\| = \varphi(\mathcal{N}(x)(t_2) - \mathcal{N}(x)(t_1)).$$

Then, we have

$$
\begin{aligned}
&\|\mathcal{N}(x)(t_2)-\mathcal{N}(x)(t_1)\| \leq I_{0^+}^{\alpha}\varphi(f(s,x(s))(t_2)-f(s,x(s))(t_1))\\
&\quad+\frac{t_2-t_1}{|v_2|}\{|a_2|I_{0^+}^{\alpha-\sigma_2}\varphi(f(s,x(s)))(\xi)+|b_2|I_{0^+}^{\alpha-\sigma_3}\varphi(f(s,x(s)))(\eta)+|\lambda_2|\},\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\varphi(f(s,x(s)))ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi(f(s,x(s)))ds\\
&\quad+\frac{t_2-t_1}{|v_2|}\{|a_2|I_{0^+}^{\alpha-\sigma_2}\varphi(f(s,x(s)))(\xi)+|b_2|I_{0^+}^{\alpha-\sigma_3}\varphi(f(s,x(s)))(\eta)+|\lambda_2|\},\\
&\quad\leq\frac{\|p\|_{L^\infty}\psi(R)}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\
&\quad+\frac{(t_2-t_1)\|p\|_{L^\infty}\psi(R)}{|v_2|}\{|a_2|I_{0^+}^{\alpha-\sigma_2}(1)(\xi)+|b_2|I_{0^+}^{\alpha-\sigma_3}(1)(\eta)+|\lambda_2|\},\\
&\quad\leq\|p\|_{L^\infty}\psi(R)\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{|v_2|}\left[\frac{|a_2|\xi^{\alpha-\sigma_2}}{\Gamma(\alpha-\sigma_2+1)}+\frac{|b_2|\eta^{\alpha-\sigma_3}}{\Gamma(\alpha-\sigma_3+1)}+|\lambda_2|\right]\right)(t_2-t_1).
\end{aligned}
$$

Hence $\mathcal{N}(D)\subset D$.

Step 2 : We will show that the operator $\mathcal{N}$ has a weakly sequentially continuous.

Let (x_n) be a sequence in D and let $x_n(t)\to x(t)$ in (E,w) for each $t\in J$. Fix $t\in J$. Since f satisfies assumption (H1), we have $f(t,x_n(t))$ converges weakly uniformly to $f(t,x(t))$. Hence the Lebesgue Dominated Convergence theorem for Pettis integral implies $\mathcal{N}x_n(t)$ converges weakly uniformly to $\mathcal{N}x(t)$ in (X,w). We do it for each $t\in J$ so $\mathcal{N}x_n\to\mathcal{N}x$. Then $\mathcal{N}:D\longrightarrow D$ is weakly sequentially continuous.

Step 3 : The implication (1.7) holds.

Now let V be a subset of D such that $\overline{V}=\overline{conv}(\mathcal{N}(V)\cup\{0\})$.
Clearly,

$$V(t)\subset\overline{conv}(\mathcal{N}(V(t))\cup\{0\}),t\in J.$$

Further, as V is bounded and equicontinuous, the function $t\to v(t)=\beta(V(t))$ is continuous on J. By assumption (H5), and the properties of the measure β, for any $t \in J$, we have

$$\begin{aligned}
v(t) &\leq \beta(\overline{conv}(\mathscr{N}(V)(t) \cup \{0\})) \leq \beta(\mathscr{N}(V)(t))\\
&\leq \beta\left(I_{0^+}^{\alpha} f(s,V(s))(t) + \frac{1}{v_0}\{\lambda_1 I_{0^+}^{\alpha+\sigma_1} f(s,V(s))(\eta) - b_1 I_{0^+}^{\alpha} f(s,V(s))(T)\}\right.\\
&+\frac{v_1}{v_0 v_2}\{\lambda_2 - (a_2 I_{0^+}^{\alpha-\sigma_2} f(s,V(s))(\xi) + b_2 I_{0^+}^{\alpha-\sigma_3} f(s,V(s))(\eta))\}\\
&\left.+\frac{t}{v_2}\{a_2 I_{0^+}^{\alpha-\sigma_2} f(s,V(s))(\xi) + b_2 I_{0^+}^{\alpha-\sigma_3} f(s,V(s))(\eta) - \lambda_2\}\right)\\
&\leq I_{0^+}^{\alpha}\beta(f(s,V(s)))(t) + \frac{|b_1|}{|v_0|} I_{0^+}^{\alpha}\beta(f(s,V(s)))(T) + \frac{|\lambda_1|}{|v_0|} I_{0^+}^{\alpha+\sigma_1}\beta(f(s,V(s)))(\eta)\\
&+\frac{|a_2|(|v_1|+|v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_2}\beta(f(s,V(s)))(\xi) + \frac{|b_2|(|v_1|+|v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_3}\beta(f(s,V(s)))(\eta)\\
&\leq I_{0^+}^{\alpha}(p(s)v(s))(t) + \frac{|b_1|}{|v_0|} I_{0^+}^{\alpha}(p(s)v(s))(T) + \frac{|\lambda_1|}{|v_0|} I_{0^+}^{\alpha+\sigma_1}(p(s)v(s))(\eta)\\
&+\frac{|a_2|(|v_1|+|v_0|T)}{v_0 v_2} I_{0^+}^{\alpha-\sigma_2}(p(s)v(s))(\xi) + \frac{|b_2|(|v_1|+|v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_3}(p(s)v(s))(\eta)\\
&\leq \|p\|_{L^\infty}\|v\|_\infty \left\{ I_{0^+}^{\alpha}(1)(T) + \frac{|b_1|}{|v_0|} I_{0^+}^{\alpha}(1)(T) + \frac{|\lambda_1|}{|v_0|} I_{0^+}^{\alpha+\sigma_1}(1)(\eta)\right.\\
&\left.+\frac{|a_2|(|v_1|+|v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_2}(1)(\xi) + \frac{|b_2|(|v_1|+|v_0|T)}{|v_0 v_2|} I_{0^+}^{\alpha-\sigma_3}(1)(\eta)\right\}\\
&\leq \|p\|_{L^\infty}\|v\|_\infty \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{b_1 T^\alpha}{|v_0|\Gamma(\alpha+1)} + \frac{\lambda_1}{|v_0|\Gamma(\alpha+\sigma_1+1)}\eta^{\alpha+\sigma_1}\right.\\
&\left.+\frac{a_2(v_1+|v_0|T)\xi^{\alpha-\sigma_2}}{|v_0|v_2\Gamma(\alpha-\sigma_2+1)} + \frac{b_2(v_1+|v_0|T)\eta^{\alpha-\sigma_3}}{|v_0|v_2\Gamma(\alpha-\sigma_3+1)}\right\}\\
&\leq \|p\|_{L^\infty}\|v\|_\infty \mathscr{M}.
\end{aligned}$$

which gives

$$\|v\|_\infty \leq \|p\|_{L^\infty}\|v\|_\infty \mathscr{M}.$$

This means that

$$\|v\|_\infty(1 - \|p\|_{L^\infty}\mathscr{M}) \leq 0.$$

By (2.11) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively weakly compact in X. Applying Theorem 1.49 we conclude that $\mathscr{N}$ has a fixed point which is a solution of the problem (2.1).

□

2.2.2 An Example

In this section we give an example to illustrate the usefulness of our main result. Let

$$X = \ell^1 = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots), \sum_{n=1}^{\infty} |x_n| < \infty \right\}$$

be the Banach space with the norm $\|x\|_X = \sum_{n=1}^{\infty} |x_n|$

Example 2.4. Let us consider the following fractional boundary value problem :

$$\begin{cases} {}^cD_{0^+}^{\frac{7}{4}} x_n(t) = \frac{1}{e^{t+4}}(\frac{1}{n^2} + |x_n(t)|) \quad \forall t \in J = [0,1] \\ x_n(0) - x_n(1) = 2I_{0^+}^{\frac{3}{4}} x_n(\frac{1}{3}), \\ 3\,{}^cD_{0^+}^{\frac{1}{7}} x_n(\frac{2}{3}) - {}^cD_{0^+}^{\frac{1}{2}} x_n(\frac{1}{3}) = (1,0,\dots,0,\dots). \end{cases} \tag{2.13}$$

Here

$$\alpha = \frac{7}{4}, a_1 = 1, b_1 = -1, a_2 = 3, b_2 = -1, \sigma_1 = \frac{3}{4}, \sigma_2 = \frac{1}{7}, \sigma_3 = \frac{1}{2}, \xi = \frac{2}{3}, \eta = \frac{1}{3}, \lambda_1 = 2, \lambda_2 = e_1.$$

Set

$$x = (x_1, x_2, \dots, x_n, \dots),\ f = (f_1, f_2, \dots, f_n, \dots),$$
$$f_n(t, x(t)) = \frac{1}{e^{t+4}}\left(\frac{1}{n^2} + |x_n(t)|\right), t \in J.$$

For each $x_n \in \mathbb{R}$, $t \in J$ we have

$$|f_n(t, x_n)| \leq \frac{1}{e^{t+4}}(1 + |x_n(t)|).$$

Hence conditions (H1), (H2) and (H3) hold with $p(t) = \frac{1}{e^{t+4}}, t \in J$ and $\psi(x) = 1 + x$, $x \in [0, \infty)$. For any bounded set $D \subset \ell^1$, we have

$$\beta(f(t, D)) \leq \frac{1}{e^{t+4}} \beta(D), \text{ for each } t \in J.$$

Hence (H5) is satisfied.
We have

$$\|p\|_{L^\infty}(1+R)\mathscr{M} + \frac{|\lambda_2|(|v_1| + |v_0|T)}{|v_0 v_2|} \leq R,$$

thus

$$R \geq \frac{\|p\|_{L^\infty}\mathscr{M} + \frac{|\lambda_2|(|v_1|+|v_0|T)}{|v_0 v_2|}}{1 - \|p\|_{L^\infty}\mathscr{M}},$$

using the Matlab program, we can find $R \geq \frac{1072}{687}$.
We shall check that condition (2.11) is satisfied. Indeed

$$\|p\|_{L^\infty}\mathscr{M} = \frac{10}{173} < 1.$$

Consequently, theorem 2.3 implies that problem (2.13) has a solution defined on J.

2.3 Weak Solutions for fractional Langevin equations involving two fractional orders with initial value problems in Banach Spaces.

2.3.1 Introduction

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [48]. Although the existing literature on solutions of fractional Langevin equations is quite wide (see, for example, [6, 14, 15, 31, 101, 132, 133, 134, 143]). But, to the best of the author's knowledge, there is no literature to research the existence of weak solutions for fractional Langevin equations involving two fractional orders in Banach Spaces, so the research of this paper is new.

In this section, we study the existence of weak solutions for an initial value problem, posed in a given Banach space. More specifically, we pose the following fractional Langevin equations involving two fractional orders with initial value problems

$$\begin{cases} {}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+}+\gamma)x(t)=f(t,x(t)), \quad t\in I:=[0,1],\\ x^{(k)}(0)=\mu_k, \qquad 0\le k<l,\\ x^{(\alpha+k)}(0)=\nu_k, \quad 0\le k<n. \end{cases} \tag{2.14}$$

Where ${}^cD^{\alpha}_{0^+}, {}^cD^{\beta}_{0^+}$ are the Caputo fractional derivatives $m-1<\alpha\le m,\ n-1<\beta<n, l=\max\{m,n\}, m,n\in\mathbb{N}, \gamma\in\mathbb{R}, f:[0,1]\times X\longrightarrow X$ is a given function satisfying some assumptions that will be specified later, X is a Banach space with norm $\|\cdot\|, \mu_k,\nu_k\in X$.

This problem was studied recently in [134] in the scalar case using Banach contraction principal and the nonlinear alternative of Leray–Schauder. Here we extend the results of [134] to cover the abstract case.

2.3.2 Existence of solutions [2]

First of all, we define what we mean by a weak solution for the initial value problem (2.14).

Definition 2.5. By a weak solution of (2.14), we mean a function $x:I\longrightarrow X$ such that the weak fractional derivative ${}^cD^{\alpha}_{0^+}, {}^cD^{\beta}_{0^+}$ exists and are weakly continuous and satisfies problem (2.14).

For the existence of weak solutions for the initial value problem (2.14), we need the following auxiliary lemma.

Lemma 2.6 ([134])**.** *$x(t)$ is a solution of the initial problem (2.14) if and only if $x(t)$ is a solution of the integral equation :*

$$x(t)=I^{\alpha+\beta}_{0^+}f(t,x(t))-\gamma I^{\alpha}_{0^+}x(t)+Q(t). \tag{2.15}$$

2. **C. Derbazi**, H. Hammouche, M. Benchohra, J. Henderson, Weak Solutions for fractional Langevin equations involving two fractional orders with initial value problems in Banach Spaces.(submitted)

Where

$$Q(t) = \sum_{i=0}^{n-1} \frac{\nu_i + \gamma\mu_i}{\Gamma(\alpha+i+1)} t^{\alpha+i} + \sum_{j=0}^{m-1} \frac{\mu_j}{\Gamma(j+1)} t^j.$$

For simplicity of presentation, we give some notations and list some conditions as follows :

$$\mathscr{M}_{p_f} = \frac{\|p_f\|}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)},$$

$$L = \left[\left(\frac{|\gamma|}{\Gamma(\alpha)} + \frac{\|p_f\|_{L^\infty}}{\Gamma(\alpha+\beta)}\right) R + \sum_{i=0}^{n-1} \frac{\|\nu_i\| + |\gamma|\|\mu_i\|}{\Gamma(\alpha+i)} + \sum_{j=1}^{m-1} \frac{\|\mu_i\|}{\Gamma(j)}\right],$$

$$Q^* = \sup\{\|Q(t)\|, t \in I\}.$$

(H1) For each $t \in I$, the function $f(t,\cdot)$ is weakly sequentially continuous ;

(H2) For each $x \in C(I,X)$, the function $f(\cdot,x(\cdot))$ is Pettis integrable on I ;

(H3) There exist $p_f \in L^\infty(I,\mathbb{R}_+)$ such that

$$\|f(t,x)\| \le p_f(t)\|x\|, \forall (t,x) \in I \times X.$$

(H4) For each bounded set $D \subset X$, and each $t \in I$, the following inequality holds

$$\beta(f(t,D)) \le p_f(t)\beta(D).$$

Now we are in able to establish the main results.

Theorem 2.7. *Assume that assumptions (H1)-(H4) hold. If*

$$\mathscr{M}_{p_f} < 1. \tag{2.16}$$

Then the initial value problem (2.14) *has at least one solution.*

Proof.

Transform the integral equation (2.15) into a fixed point equation. Consider the operator $\mathscr{N} : C(I,X) \longrightarrow C(I,X)$ defined by :

$$\begin{aligned} \mathscr{N}x(t) &= I_{0^+}^{\alpha+\beta} f(t,x(t)) - \gamma I_{0^+}^{\alpha} x(t) + Q(t) \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,x(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds + Q(t). \end{aligned} \tag{2.17}$$

First notice that, for $x \in C(I,X)$, we have $f(\cdot,x(\cdot)) \in P(I,X)$ (assumption (H2)). Since $s \mapsto \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, s \mapsto \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}$ are $\in L^\infty(I)$ then $\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(\cdot), \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(\cdot,x(\cdot))$ for all $t \in I$ are Pettis integrable (Proposition 1.5) and thus, the operator $\mathscr{N}$ is well defined.

Let

$$R \ge \frac{Q^*}{1-\mathscr{M}_{p_f}}, \tag{2.18}$$

and consider the set

$$D = \{x \in C(I,X) : \|x\|_\infty \leq R, \|x(t_2) - x(t_1)\| \leq L|t_2 - t_1|\},$$

Notice that D is a closed, convex, bounded, and equicontinuous subset of $C(I,X)$ .We shall show that the operator $\mathscr{N}$ satisfies all the assumptions of Theorem 1.49. The proof will be given in several steps.

Step 1 :We will show that the operator $\mathscr{N}$ maps D into D.

Take $x \in D, t \in I$ and assume that $\mathscr{N}x(t) \neq 0$. Then there exists $\varphi \in X^*$ such that $\|\mathscr{N}x(t)\| = \varphi(\mathscr{N}x(t))$. Thus

$$\begin{aligned}\|\mathscr{N}x(t)\| &= \varphi\left(I_{0^+}^{\alpha+\beta} f(t,x(t)) - \gamma I_{0^+}^{\alpha} x(t) + Q(t)\right)\\ &\leq I_{0^+}^{\alpha+\beta}\varphi(f(t,x(t))) + |\gamma| I_{0^+}^{\alpha}\varphi(x(t))) + \varphi(Q(t)).\end{aligned}$$

From (H3) we get

$$\begin{aligned}\|\mathscr{N}x(t)\| &\leq \|x\|\left\{I_{0^+}^{\alpha+\beta} p_f(t) + |\gamma| I_{0^+}^{\alpha}(1)(t)\right\} + \|Q(t)\|\\ &\leq R\left\{\frac{\|p_f\|_{L^\infty}}{\Gamma(\alpha+\beta+1)} + \frac{|\gamma|}{\Gamma(\alpha+1)}\right\} + Q^*,\\ &\leq R\mathscr{M}_{p_f} + Q^* \leq R.\end{aligned}$$

Next, let $t_1, t_2 \in I$ be such that $t_1 < t_2$ and let $x \in D$ be such that

$$\mathscr{N}x(t_2) - \mathscr{N}x(t_1) \neq 0.$$

Then there exists $\varphi \in X^*$ such that

$$\|\mathscr{N}(x)(t_2) - \mathscr{N}(x)(t_1)\| = \varphi(\mathscr{N}(x)(t_2) - \mathscr{N}(x)(t_1)).$$

Then, we have

$$\begin{aligned}
\|\mathscr{N}(x)(t_2)-\mathscr{N}(x)(t_1)\| \leq \varphi\Bigg\{&\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_2}(t_2-s)^{\alpha+\beta-1}f(s,x(s))ds\\
&-\frac{\gamma}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}x(s)ds+Q(t_2)-\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}(t_1-s)^{\alpha+\beta-1}f(s,x(s))ds\\
&+\frac{\gamma}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}x(s)ds-Q(t_1)\Bigg\}\\
&\leq \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\right]\varphi(f(s,x(s))ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\varphi(f(s,x(s)))ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]\varphi(x(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\varphi(x(s))ds+\varphi(Q(t_2)-Q(t_1))\\
&\leq \left[\left(\frac{|\gamma|}{\Gamma(\alpha)}+\frac{\|p_f\|_{L^\infty}}{\Gamma(\alpha+\beta)}\right)R+\sum_{i=0}^{n-1}\frac{\|\nu_i\|+|\gamma|\|\mu_i\|}{\Gamma(\alpha+i)}+\sum_{j=1}^{m-1}\frac{\|\mu_i\|}{\Gamma(j)}\right]|t_2-t_1|\\
&=L|t_2-t_1|.
\end{aligned}$$

Hence $\mathscr{N}(D)\subset D$.

Step 2 : We will show that the operator $\mathscr{N}$ has a weakly sequentially continuous.

Let (x_n) be a sequence in D and let $x_n(t)\to x(t)$ in (E,w) for each $t\in I$. Fix $t\in J$. Since f satisfies assumption (H1), we have $f(t,x_n(t))$ converges weakly uniformly to $f(t,x(t))$. Hence the Lebesgue Dominated Convergence theorem for Pettis integral implies $\mathscr{N}x_n(t)$ converges weakly uniformly to $\mathscr{N}x(t)$ in (X,w). We do it for each $t\in I$ so $\mathscr{N}x_n\to\mathscr{N}x$. Then $\mathscr{N}:D\longrightarrow D$ is weakly sequentially continuous.

Step 3 : The implication (1.7) holds.

Now let V be a subset of D such that $\overline{V}=\overline{conv}(\mathscr{N}(V)\cup\{0\})$.
Clearly,

$$V(t)\subset\overline{conv}(\mathscr{N}(V(t))\cup\{0\}),t\in I.$$

Further, as V is bounded and equicontinuous, the function $t\to v(t)=\beta(V(t))$ is continuous on I. By assumption (H5), and the properties of the measure β,for any $t\in I$, we have

$$\begin{aligned}
v(t) &\leq \beta(\overline{conv}(\mathcal{N}(V)(t)\cup\{0\})) \leq \beta(\mathcal{N}(V)(t)) \\
&\leq \beta(I_{0^+}^{\alpha+\beta}f(t,V(t))+\gamma I_{0^+}^{\alpha}V(t)+Q(t)) \\
&\leq I_{0^+}^{\alpha+\beta}\beta(f(t,V(t)))+|\gamma| I_{0^+}^{\alpha}\beta(V(t)) \\
&\leq I_{0^+}^{\alpha+\beta}(p_f(t)v(t))+|\gamma| I_{0^+}^{\alpha}(v(t)) \\
&\leq \|v\|_\infty \left\{\frac{\|p_f\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{|\gamma|}{\Gamma(\alpha+1)}\right\}, \\
&\leq \|v\|_\infty \mathcal{M}_{p_f}.
\end{aligned}$$

which gives

$$\|v\|_\infty \leq \|v\|_\infty \mathcal{M}_{p_f}.$$

This means that

$$\|v\|_\infty(1-\mathcal{M}_{p_f}) \leq 0.$$

By (2.16) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in I$, and then $V(t)$ is relatively weakly compact in X. Applying Theorem 1.49 we conclude that $\mathcal{N}$ has a fixed point which is a solution of the problem (2.14).

□

2.3.3 An Example

In this section we give an example to illustrate the usefulness of our main result. Let

$$X = c_0 = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \to 0 (n \to \infty)\},$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|x\|_\infty = \sup_{n\geq 1}|x_n|.$$

Example 2.8. Consider the following fractional Langevin problem posed in c_0 :

$$\begin{cases} {}^cD_{0^+}^{\frac{1}{4}}({}^cD_{0^+}^{\frac{1}{2}}+\frac{1}{10})x(t) = f(t,x(t)), \quad t \in I := [0,1] \\ x(0) = \mu_0 = (0,0,\dots,0,\dots) \\ x^{(\frac{1}{2})}(0) = \nu_0 = \left(\frac{1}{2},\frac{1}{4},\dots,\frac{1}{2^n},\dots\right). \end{cases} \tag{2.19}$$

Note that, this problem is a particular case of IVP (2.14), where

$$\alpha = \frac{1}{2}, \beta = \frac{1}{4}, \gamma = \frac{1}{10},$$

and $f : J \times c_0 \longrightarrow c_0$ given by

$$f(t,x) = \frac{1}{(t^2+2)^2}\left\{\frac{n}{n+1}\ln(|x_n|+1)\right\}_{n\geq 1}, \quad \text{for } t \in I, x = \{x_n\}_{n\geq 1} \in c_0.$$

It is clear that condition (H1) and (H2) holds, and as

$$\begin{aligned}\|f(t,x)\|_\infty &= \frac{1}{(t^2+2)^2}\left\|\frac{n}{n+1}\ln(|x_n|+1)\right\| \\ &\leq \frac{1}{(t^2+2)^2}\|x\|_\infty,\end{aligned}$$

for each $t \in J$ and $x \in c_0$, condition (H3) follows with $p_f(t) = \frac{1}{(t^2+2)^2}, t \in I$.
On the other hand, for any bounded set $D \subset c_0$, we have

$$\beta(f(t,D)) \leq \frac{1}{(t^2+2)^2}\beta(D), \text{ for each } t \in I.$$

Hence (H4) is satisfied.
We shall check that condition (2.11) is satisfied. Indeed

$$\mathscr{M}_{p_f} = 0.3849 < 1.$$

and

$$\frac{Q^*}{1-\mathscr{M}_{p_f}} = 0.9172.$$

Then r can be chosen as $r = 1 \geq 0.9172$. Consequently, Theorem 2.7 implies that problem (2.19) has at least one solution $x \in C(I, c_0)$.

Chapitre 3

Fractional differential equations in Banach algebras

3.1 Introduction

The aim of this chapter is to prove the existence of solutions for a class of hybrid fractional differential equations in the Banach algebra of all continuous functions on a bounded interval. We also present examples to show the validity of conditions and efficiency of our results. This exposition is divided into three parts. The first one deals with the existence of solutions for fractional hybrid differential equations with three-point boundary hybrid conditions. The second one deals with results connecting the existence of solution fractional hybrid differential equations with p-Laplacian operator. Our approach mainly depends on a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [59], while the last part of this chapter is devoted to the existence of solutions for fractional hybrid differential equations with deviating arguments under hybrid conditions with the help of a technique associated with the measure of noncompactness and generalized Darbo fixed point theorem in a Banach algebra. Moreover, an example is given at the end of each section to illustrate the validity of our main results. The chapter is inspired in the papers [56, 57, 58].

3.2 Fractional hybrid differential equations with three-point boundary hybrid conditions

3.2.1 Introduction

In recent years, hybrid fractional differential equations have achieved a great deal of interest and attention from several researchers. By hybrid differential equations, we mean that the terms in the equation are perturbed either linearly or quadratically or through the combination of first and second types. Perturbation taking place in the form of the sum or difference of terms in an equation is called linear. On the other hand, if the equation is perturbed through the product or quotient of the terms in it, then it is called quadratic perturbation. So the study of the hybrid differential equation is more general and covers several dynamic systems for some developments on the existence results of hybrid fractional differential equations, we can refer to [20, 21, 81, 82, 104, 108, 128, 129, 137] and the references therein.

The main theme of this section is to discuss the existence of solutions on a bounded interval $J := [0,T]$ for the following hybrid differential equation with three-point boundary hybrid conditions

$$\begin{cases} {}^cD^{\alpha}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = h(t,x(t)),\ 1<\alpha\leq 2, t\in J:=[0,T], \\ a_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} + b_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_1, \\ a_2\,{}^cD^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta} + b_2\,{}^cD^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_2, 0<\eta<T, \end{cases} \tag{3.1}$$

where ${}^cD^{\alpha}_{0^+}, {}^cD^{\beta}_{0^+}$ denote the Caputo fractional derivatives of order α and β, respectively, $0<\beta\leq 1$, $a_i, b_i, c_i, i=1,2$ are real constants such that $a_1+b_1\neq 0, a_2\eta^{1-\beta}+b_2T^{1-\beta}\neq 0$ $g\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\}), f,h\in C(J\times\mathbb{R},\mathbb{R})$.

This results can be consedered as a generalization of [66]. For example, if we choose $f(t,x(t))=0, g(t,x(t))=1$ as constant functions, then our problem (3.1) will reduce to boundary value problems for fractional order differential equations of the type :

$$\begin{cases} {}^cD^{\alpha}_{0^+}x(t) = h(t,x(t)),\ 1<\alpha\leq 2, t\in J:=[0,T], \\ a_1x(0)+b_1x(T)=\lambda_1, \\ a_2\,{}^cD^{\beta}_{0^+}x(\eta)+b_2\,{}^cD^{\beta}_{0^+}x(T)=\lambda_2, 0<\eta<T. \end{cases} \tag{3.2}$$

3.2.2 Existence of solutions [1]

By $E=C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J:=[0,T]$ into $\mathbb{R}$ with the norm

$$\|x\| = \sup_{t\in J}|x(t)|,$$

and a multiplication in E by

$$(xy)(t) = x(t)y(t).$$

Clearly, E is a Banach algebra with respect to the above supremum norm and the multiplication in it. In this section, we give our main existence result for problem (3.1). Before stating Let us define what we mean by a solution to the problem (3.1).

Definition 3.1. A function $x\in C(J,\mathbb{R})$, is said to be a solution of (3.1) if it satisfies the equation ${}^cD^{\alpha}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = h(t,x(t))$ on J, and the condition

$$a_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} + b_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_1,$$

$$a_2\,{}^cD^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta} + b_2\,{}^cD^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T} = \lambda_2,\ 0<\eta<T.$$

1. **C. Derbazi**, H. Hammouche, M. Benchohra and Y. Zhou, Fractional hybrid differential equations with three-point boundary hybrid conditions, Adv. Difference Equ. **2019**, Paper No. 125, 11 pp.

The integral form that is equivalent to problem (3.1) is given by the following.

Lemma 3.2. *Let h be a continuous function on J then the hybrid linear differential equation*

$$ {}^{c}D^{\alpha}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]=h(t),\quad t\in J=:[0,T], 1<\alpha\leq 2, \tag{3.3}$$

with boundary conditions

$$\begin{aligned} a_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0}+b_1\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T}&=\lambda_1,\\ a_2\,{}^{c}D^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\eta}+b_2\,{}^{c}D^{\beta}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=T}&=\lambda_2,\ 0<\eta<T, \end{aligned}\tag{3.4}$$

is equivalent to

$$\begin{aligned} x(t)=f(t,x(t))+g(t,x(t))\Bigg[&I^{\alpha}_{0^+}h(t)-\frac{b_1}{a_1+b_1}I^{\alpha}_{0^+}h(T)+\frac{\lambda_1}{a_1+b_1}\\ &+\frac{(b_1T-(a_1+b_1)t)\Gamma(2-\beta)(a_2I^{\alpha-\beta}_{0^+}h(\eta)+b_2I^{\alpha-\beta}_{0^+}h(T)-\lambda_2)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})}\Bigg]. \end{aligned}\tag{3.5}$$

Proof. Applying the Riemann-Liouville fractional integral operator of order α to both sides of (3.3) and using Lemmas 1.12, 1.24, we have

$$\frac{x(t)-f(t,x(t))}{g(t,x(t))}=I^{\alpha}_{0^+}h(t)-c_0-c_1t,\ \forall\, c_0,c_1\in\mathbb{R}. \tag{3.6}$$

Consequently, the general solution of (3.3) is

$$x(t)=g(t,x(t))(I^{\alpha}_{0^+}h(t)-c_0-c_1t)+f(t,x(t)),\quad \forall c_0,c_1\in\mathbb{R}. \tag{3.7}$$

Applying the boundary conditions (3.4) in (3.6) we find that

$$\begin{aligned} -a_1c_0+b_1(I^{\alpha}_{0^+}h(T)-c_0-c_1T)&=\lambda_1,\\ a_2I^{\alpha-\beta}_{0^+}h(\eta)+b_2I^{\alpha-\beta}_{0^+}h(T)-\frac{a_2\eta^{1-\beta}+b_2T^{1-\beta}}{\Gamma(2-\beta)}c_1&=\lambda_2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} c_0&=-\frac{b_1T\Gamma(2-\beta)(a_2I^{\alpha-\beta}_{0^+}h(\eta)+b_2I^{\alpha-\beta}_{0^+}h(T)-\lambda_2)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})}+\frac{b_1}{a_1+b_1}I^{\alpha}_{0^+}h(T)-\frac{\lambda_1}{a_1+b_1},\\ c_1&=\frac{\Gamma(2-\beta)(a_2I^{\alpha-\beta}_{0^+}h(\eta)+b_2I^{\alpha-\beta}_{0^+}h(T)-\lambda_2)}{a_2\eta^{1-\beta}+b_2T^{1-\beta}}. \end{aligned}$$

Substituting the value of c_0,c_1 in (3.7) we get (3.5). Conversely, it is clear that if x satisfies equation (3.5), then equations (3.3)–(3.4) hold.

□

For developing the existence result, we consider some assumptions which are the following.

(H1) The functions $g : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and $h, f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous such that

(H2) There exist two positive functions ϕ_0, ϕ_1 with bound $\|\phi_0\|$ and $\|\phi_0\|$ respectively, such that

$$|f(t,x) - f(t,y)| \leq \phi_0(t)|x-y|, \tag{3.8}$$

and

$$|g(t,x) - g(t,y)| \leq \phi_1(t)|x-y|, \tag{3.9}$$

for each $(t,x,y) \in J \times \mathbb{R} \times \mathbb{R}$.

(H3) There exist a functions $p \in L^\infty(J, \mathbb{R}_+)$ and there exist continuous nondecreasing function $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that

$$|h(t,x)| \leq p(t)\psi(|x|), \tag{3.10}$$

for each $t \in J$ and all $x \in \mathbb{R}$.

(H4) There exists a number $r > 0$ such that

$$r \geq \frac{g_0\Lambda + f_0}{1 - \|\phi_0\|\Lambda - \|\phi_1\|}. \tag{3.11}$$

and

$$\|\phi_0\|\Lambda + \|\phi_1\| < 1. \tag{3.12}$$

where $f_0 = \sup_{t\in J}|f(t,0)|, g_0 = \sup_{t\in J}|g(t,0)|$, and

$$\begin{aligned}\Lambda = \|p\|\psi(r)\left[\frac{T^\alpha}{\Gamma(\alpha+1)}\left(1 + \frac{|b_1|}{|a_1+b_1|}\right) + \frac{(|a_1|+2|b_1|)\Gamma(2-\beta)(|a_2|\eta^{\alpha-\beta} + |b_2|T^{\alpha-\beta})T}{|a_1+b_1||a_2\eta^{1-\beta} + b_2T^{1-\beta}|\Gamma(\alpha-\beta+1)}\right] \\ + \frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|}|\lambda_2| + \frac{|\lambda_1|}{|a_1+b_1|}\end{aligned} \tag{3.13}$$

Our main existence result is based on a hybrid fixed point theorem for a sum of three operators due to Dhage [59], which we have provided in Lemma 1.48.

Theorem 3.3. *Assume that conditions (H1)-(H4) hold. Then the problem* (3.1) *has at least one solution on* J .

Proof. Define a subset S of E defined by

$$S = \{x \in E : \|x\|_E \leq r\}.$$

Clearly, S is a closed, convex and bounded subset of the Banach space E. By Lemma 3.2, the boundary value problem (3.1) is equivalent to the equation

$$\begin{aligned}x(t) = f(t,x(t)) + g(t,x(t))\Bigg[I^\alpha_{0^+}h(s,x(s))(t) - \frac{b_1}{a_1+b_1}I^\alpha_{0^+}h(s,x(s))(T) + \frac{\lambda_1}{a_1+b_1} \\ + \frac{(b_1T-(a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})}(a_2I^{\alpha-\beta}_{0^+}h(s,x(s))(\eta) + b_2I^{\alpha-\beta}_{0^+}h(s,x(s))(T) - \lambda_2)\Bigg], \; t \in J.\end{aligned} \tag{3.14}$$

Define three operators $\mathscr{A},\mathscr{C} : E \longrightarrow E$ and $\mathscr{B} : S \longrightarrow E$ by

$$\begin{aligned}
&\mathscr{A}x(t) = g(t,x(t)), t \in J.\\
&\mathscr{B}x(t) = I_{0^+}^{\alpha}h(s,x(s))(t) - \frac{b_1}{a_1+b_1}I_{0^+}^{\alpha}h(s,x(s))(T) + \frac{(b_1T-(a_1+b_1)t)\Gamma(2-\beta)}{(a_1+b_1)(a_2\eta^{1-\beta}+b_2T^{1-\beta})}\times\\
&\qquad (a_2I_{0^+}^{\alpha-\beta}h(s,x(s))(\eta) + b_2I_{0^+}^{\alpha-\beta}h(s,x(s))(T) - \lambda_2) + \frac{\lambda_1}{a_1+b_1}, t \in J,
\end{aligned}$$

and

$$\mathscr{C}x(t) = f(t,x(t)), t \in J.$$

Then the integral Equation (3.14) can be written in the operator form as

$$x(t) = \mathscr{A}x(t)\mathscr{B}x(t) + \mathscr{C}x(t),\ t \in J.$$

We shall show that the operators $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$ satisfy all the conditions of Lemma 1.48. This will be achieved in the following series of steps.

Step 1 : First, we show that $\mathscr{A}$ and $\mathscr{C}$ are Lipschitzian on E. Let $x,y \in E$, then by (H2), for $t \in J$, we have

$$|\mathscr{A}x(t) - \mathscr{A}y(t)| = |g(t,x(t)) - g(t,y(t))| \leq \phi_0(t)|x(t) - y(t)|,$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \|\phi_0\|\|x-y\|,$$

for all $x,y \in E$. Therefore, $\mathscr{A}$ is a Lipschitzian on E with Lipschitz constant $\|\phi_0\|$.
Now for $\mathscr{C} : E \longrightarrow E, x,y \in E$, we have

$$|\mathscr{C}x(t) - \mathscr{C}y(t)| = |f(t,x(t)) - f(t,y(t))| \leq \phi_1(t)|x(t) - y(t)|,$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathscr{C}x - \mathscr{C}y\| \leq \|\phi_1\|\|x-y\|,$$

Hence,$\mathscr{C} : E \longrightarrow E$ is a Lipschitzian on E with Lipschitz constant $\|\phi_1\|$.

Step 2 : we show that $\mathscr{B}$ is is completely continuous operator on S into E. First, we show that $\mathscr{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then for each $t \in J$

$$|\mathscr{B}x_n(t) - \mathscr{B}x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds}$$
$$+ \frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds}$$
$$\frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)} \Big(|a_2| \int_0^\eta (\eta-s)^{\alpha-\beta-1} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds}$$
$$+ |b_2| \int_0^T (T-s)^{\alpha-\beta-1} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds} \Big)$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{s\in[0,T]} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds}$$
$$+ \frac{|b_1|}{|a_1+b_1|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \sup_{s\in[0,T]} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds}$$
$$\frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)} \Bigg(|a_2| \int_0^\eta (\eta-s)^{\alpha-\beta-1} \sup_{s\in[0,T]} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds}$$
$$+ |b_2| \int_0^T (T-s)^{\alpha-\beta-1} \sup_{s\in[0,T]} |h(s,x_n(s)) - h(s,x(s))| \mathrm{ds} \Bigg)$$
$$\leq \left[\frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b_1|}{|a_1+b_1|}\right) + \frac{(|a_1|+2|b_1|)\Gamma(2-\beta)(|a_2|\eta^{\alpha-\beta}+|b_2|T^{\alpha-\beta})T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta+1)} \right] \times$$
$$\|h(\cdot,x_n(\cdot)) - h(\cdot,x(\cdot))\|.$$

Since h is continuous, we have $\|\mathscr{B}x_n - \mathscr{B}x\| \to 0$ as $n \to \infty$. Next we will prove that the set $\mathscr{B}(S)$ is a uniformly bounded in S. For any $x \in S$, we have

$$\begin{aligned} |\mathscr{B}x(t)| \leq{}& I_{0^+}^\alpha |h(s,x(s))|(t) + \frac{|b_1|}{|a_1+b_1|} I_{0^+}^\alpha |h(s,x(s))|(T) \\ &+ \frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)} (|a_2| I_{0^+}^\alpha |h(s,x(s))|(\eta) \\ &+ |b_2| I_{0^+}^\alpha |h(s,x(s))|(T) + |\lambda_2|) + \frac{|\lambda_1|}{|a_1+b_1|}. \end{aligned}$$

Using (3.34) we can write

$$\begin{aligned} |\mathscr{B}x(t)| \leq{}& I_{0^+}^\alpha p(s)\psi(|x(s)|)(t) + \frac{|b_1|}{|a_1+b_1|} I_{0^+}^\alpha p(s)\psi(|x(s)|)(T) \\ &+ \frac{(|b_1|T + (|a_1|+|b_1|)T)\Gamma(2-\beta)}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|} (|a_2| I_{0^+}^\alpha p(s)\psi(|x(s)|)(\eta) \\ &+ |b_2| I_{0^+}^\alpha p(s)\psi(|x(s)|)(T) + |\lambda_2|) + \frac{|\lambda_1|}{|a_1+b_1|} \\ &\leq \|p\|\psi(r) \left[\frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b_1|}{|a_1+b_1|}\right) + \frac{(|a_1|+2|b_1|)\Gamma(2-\beta)(|a_2|\eta^{\alpha-\beta}+|b_2|T^{\alpha-\beta})T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta+1)} \right] \\ &+ \frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|} |\lambda_2| + \frac{|\lambda_1|}{|a_1+b_1|}. \end{aligned}$$

Thus $\|\mathscr{B}x\| \leq \Lambda$ with Λ given in (3.31). for all $x \in S$. This shows that $\mathscr{B}$ is uniformly bounded on S. Furthermore, we have

$$|(\mathscr{B}x)'(t)| \leq I_{0^+}^{\alpha-1}|h(s,x(s))|(t) + \frac{\Gamma(2-\beta)}{|a_2\eta^{1-\beta}+b_2T^{1-\beta}|}\times \\ (|a_2|I_{0^+}^{\alpha}|h(s,x(s))|(\eta)+|b_2|I_{0^+}^{\alpha}|h(s,x(s))|(T)+|\lambda_2|).$$

Some computations give

$$|(\mathscr{B}x)'(t)| \leq \|p\|\psi(r)\left[\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{\Gamma(2-\beta)(|a_2|\eta^{\alpha-\beta}+|b_2|T^{\alpha-\beta})}{|a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta+1)}\right]+\frac{\Gamma(2-\beta)}{|a_2\eta^{1-\beta}+b_2T^{1-\beta}|}|\lambda_2| := L.$$

Now, for $t_1, t_2 \in J$ with $t_1 < t_2$, we get

$$|\mathscr{B}x(t_2)-\mathscr{B}x(t_1)| \leq \int_{t_1}^{t_2}\left|(\mathscr{B}x)'(s)\right| \mathrm{ds} \leq L(t_2-t_1).$$

Therefore, $\mathscr{B}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathscr{B}$ is completely continuous.
Step 3 : The hypothesis (c) of Lemma 1.48 is satisfied.
Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathscr{A}x\mathscr{B}y + \mathscr{C}x$. Then we have

$$\begin{aligned}
|x(t)| &\leq |\mathscr{A}x(t)|\,|\mathscr{B}y(t)|+|\mathscr{C}x(t)| \\
&\leq |g(t,x(t))|\left\{I_{0^+}^{\alpha}|h(s,y(s))|(t)+\frac{|b_1|}{|a_1+b_1|}I_{0^+}^{\alpha}|h(s,y(s))|(T)\right. \\
&+\frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}(|a_2|I_{0^+}^{\alpha}|h(s,y(s))|(\eta) \\
&\left.+|b_2|I_{0^+}^{\alpha}|h(s,y(s))|(T)+|\lambda_2|)+\frac{|\lambda_1|}{|a_1+b_1|}\right\}+|f(t,x(t))|, \\
&\leq (|g(t,x(t))-g(t,0))|+|g(t,0))|)\left\{I_{0^+}^{\alpha}p(s)\psi(|y(s)|)(t)+\frac{|b_1|}{|a_1+b_1|}I_{0^+}^{\alpha}p(s)\psi(|y(s)|)(T)\right. \\
&+\frac{(|a_1|+2|b_1|)\Gamma(2-\beta)T}{|a_1+b_1||a_2\eta^{1-\beta}+b_2T^{1-\beta}|\Gamma(\alpha-\beta)}(|a_2|I_{0^+}^{\alpha}p(s)\psi(|y(s)|)(\eta) \\
&\left.+|b_2|I_{0^+}^{\alpha}p(s)\psi(|y(s)|)(T)+|\lambda_2|)+\frac{|\lambda_1|}{|a_1+b_1|}\right\}+|f(t,x(t))-f(t,0))|+|f(t,0)|, \\
|x(t)| &\leq (\|\phi_0\||x(t)|+g_0)\Lambda+\|\phi_1\||x(t)|+f_0.
\end{aligned}$$

Thus,

$$|x(t)| \leq \frac{g_0\Lambda+f_0}{1-\|\phi_0\|\Lambda-\|\phi_1\|}.$$

Taking the supremum over t,

$$\|x\| \leq \frac{g_0\Lambda+f_0}{1-\|\phi_0\|\Lambda-\|\phi_1\|} \leq r.$$

Step 4 : Finally we show that $\delta M+\xi < 1$, that is, (d) of Lemma 1.48 holds.
Since

$$M = \|B(S)\| = \sup_{x\in S}\left\{\sup_{t\in J}|Bx(t)|\right\} \leq \Lambda,$$

and so

$$\|\phi_0\|M + \|\phi_1\| \leq \|\phi_0\|\Lambda + \|\phi_1\| < 1,$$

with $\delta = \|\phi_0\|, \xi = \|\phi_1\|$. Thus all the conditions of Lemma 1.48 are satisfied and hence the operator equation $x = \mathscr{A}x\mathscr{B}x + \mathscr{C}x$ has a solution in S. As a result, problem (3.1) has a solution on J.

□

3.2.3 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following boundary value problem :

Example 3.4.

$$\begin{cases} {}^cD^{\frac{3}{2}}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right] = \frac{e^{-2t}}{\sqrt{(9+t)}}\sin x(t). \quad \forall t \in J := [0,1] \\ \left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=0} + 2\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = \frac{1}{2}, \\ 3\,{}^cD^{\frac{1}{2}}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=\frac{1}{2}} + 0.25\,{}^cD^{\frac{1}{2}}_{0^+}\left[\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right]_{t=1} = 1 \end{cases} \tag{3.15}$$

In this case we take

$$\alpha = \frac{3}{2}, \beta = \frac{1}{2}, a_1 = 1, a_2 = 3, \lambda_1 = \frac{1}{2}, b_1 = 2, b_2 = \frac{1}{4}, \lambda_2 = 1, \eta = \frac{1}{2}, T = 1.$$

$$f(t,x(t)) = \frac{t^2}{100}\left(\frac{1}{2}(x(t) + \sqrt{x^2(t)+1} + e^{-t})\right)$$

$$g(t,x(t)) = \frac{\sqrt{\pi}e^{-2\pi t}\cos(\pi t)}{(7\pi+15e^t)^2}\frac{x(t)}{1+x(t)} + \frac{t}{10}$$

$$h(t,x(t)) = \frac{e^{-2t}}{\sqrt{(9+t)}}\sin x(t).$$

We can show that

$$|f(t,x) - f(t,y)| \leq \frac{t^2}{100}|x-y|, \quad |g(t,x) - g(t,y)| \leq \frac{\sqrt{\pi}e^{-2\pi t}}{(7\pi+15e^t)^2}|x-y|$$

$$|h(t,x)| \leq p(t)\psi(|x|),$$

where

$$\psi(|x|) = |x|, \quad p(t) = e^{-2t},$$

hence, we have

$$\phi_0(t) = \frac{t^2}{100}, \phi_1(t) = \frac{\sqrt{\pi}e^{-2\pi t}}{(7\pi+15e^t)^2}.$$

Then,

$$\|\phi_0\| = \frac{1}{100}, \|\phi_1\| = \frac{\sqrt{\pi}}{(7\pi+15)^2}, \|p\| = 1, f_0 = \sup_{t\in J}|f(t,0)| = \frac{1}{100}, g_0 = \sup_{t\in J}|g(t,0)| = \frac{1}{10},$$

using the Matlab program, it follows by (3.35), (3.36) that the constant r satisfies the inequality $0.0146 < r < 21.8589$. As all the conditions of Theorem 3.3 are satisfied, therefore the problem (3.15) has at least one solution on J.

3.3 Fractional Hybrid Differential Equations with p-Laplacian operator.

3.3.1 Introduction

A p-Laplacian differential equation was first introduced by Leibenson [100] when he studied the turbulent flow in a porous medium. Since then, fractional differential equation and the differential equation with a p-Laplacian operator are widely applied in different fields of physics and natural phenomena, for example, non-Newtonian mechanics, fluid mechanics, viscoelasticity mechanics, combustion theory, mathematical biology, the theory of partial differential equations. Hence, there have been many published papers which are devoted to the existence of solutions of boundary value problems for the p-Laplacian operator equations, see [32, 65, 85, 86, 91, 92, 103, 127] and their references. However, to our knowledge, there are few literatures deal with the existence of solutions to hybrid fractional differential equations (HFDEs for short) with p-Laplacian operator (see [92]). Based on the reason mentioned, in this paper, we study the existence of solutions for the following boundary value problem of hybrid fractional differential equations with p-Laplacian operator

$$\begin{cases} {}^cD^{\beta}_{0^+}\left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)\right]\right) = h(t,x(t)),\ t\in I:=[0,1], \\ \left.\left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)^{(i)}\right|_{t=0} = 0, \quad i = 0,2,3\dots,n-1 \\ \left.\left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)\right|_{t=1} = 0, \\ \left.\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)^{(j)}\right|_{t=0} = 0, \quad \text{for } j = 2,3\dots,n-1 \\ {}^cD^{\mu}_{0^+}\left[\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0, \quad x(0)=0, \end{cases} \tag{3.16}$$

where ${}^cD^{\nu}_{0^+}$ is the Caputo fractional derivative of order $\nu\in\{\alpha,\beta,\mu\}$ such that $n-1<\alpha,\beta\le n$, $0<\mu\le 1, I^{\theta}_{0^+}$ is the Riemann–Liouville fractional integral of order $\theta>0, \theta\in$

$\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$, $\phi_p(u)$ is a p-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$ for $p > 1$, $\phi_p^{-1} = \phi_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $h \in C(I \times \mathbb{R}, \mathbb{R})$, $f_k \in C(I \times \mathbb{R}, \mathbb{R})$, $0 < \sigma_k$, $k = 1, 2, \ldots, m$.

3.3.2 Existence Results [2]

In this section, we give our main existence result for problem (3.16). Before stating. Let us defining what we mean by a solution of the problem (3.16).

Definition 3.5. A function $x \in C(I, \mathbb{R})$, is said to be a solution of (3.16) if it satisfies the equation ${}^cD^{\beta}_{0^+}\left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right) = h(t, x(t))$ on I, and the condition

$$\begin{cases} \left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)^{(i)}\Bigg|_{t=0} = 0, \quad i = 0, 2, 3 \ldots, n-1 \\ \left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)\Bigg|_{t=1} = 0, \\ \left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)^{(j)}\Bigg|_{t=0} = 0, \quad \text{for } j = 2, 3 \ldots, n-1 \\ {}^cD^{\mu}_{0^+}\left[\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0, \quad x(0) = 0, \end{cases} \tag{3.17}$$

The integral form that is equivalent to problem (3.16) is given by the following.

Lemma 3.6. *Let h be continuous function on I. The solution of the boundary value problem*

$${}^cD^{\beta}_{0^+}\left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right) = h(t), \quad t \in I := [0,1], \tag{3.18}$$

with boundary conditions

$$\begin{cases} \left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)^{(i)}\Bigg|_{t=0} = 0, \quad i = 0, 2, 3 \ldots, n-1 \\ \left(\phi_p\left[{}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)\Bigg|_{t=1} = 0, \\ \left(\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right)^{(j)}\Bigg|_{t=0} = 0, \quad for\ j = 2, 3 \ldots, n-1 \\ {}^cD^{\mu}_{0^+}\left[\frac{x(t)-\sum_{k=1}^m I^{\sigma_k}_{0^+} f_k(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0, \quad x(0) = 0, \end{cases} \tag{3.19}$$

2. **C. Derbazi**, H. Hammouche, M. Benchohra and S. K. Ntouyas, Existence results for fractional hybrid differential equations with p–Laplacian operator. (submitted).

is given by

$$x(t) = \left\{ \delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} h(\tau) \mathrm{d}\tau - s \int_0^1 (1-\tau)^{\beta-1} h(\tau) \mathrm{d}\tau \right) \mathrm{ds} \right.$$
$$\left. -\delta_2 t \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} h(\tau) \mathrm{d}\tau - s \int_0^1 (1-\tau)^{\beta-1} h(\tau) \mathrm{d}\tau \right) \mathrm{ds} \right\} g(t, x(t))$$
$$+ \sum_{k=1}^m \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k - 1} f_k(s, x(s)) \mathrm{ds},$$

where

$$\begin{aligned} \delta_1 &= \frac{(\Gamma(\beta))^{1-q}}{\Gamma(\alpha)}, \\ \delta_2 &= \frac{\Gamma(2-\mu)(\Gamma(\beta))^{1-q}}{\Gamma(\alpha-\mu)}. \end{aligned} \tag{3.20}$$

Proof. **Step 1.** We know the following BVP

$$\begin{cases} {}^cD_{0^+}^{\alpha} \left[\frac{x(t) - \sum_{k=1}^m I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))} \right] = h(t) \\ \left(\frac{x(t) - \sum_{k=1}^m I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))} \right)^{(j)} \Bigg|_{t=0} = 0, \quad \text{for } j = 2, 3 \ldots, n-1 \\ \\ {}^cD_{0^+}^{\mu} \left[\frac{x(t) - \sum_{k=1}^m I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))} \right]_{t=1} = 0, \quad x(0) = 0, \end{cases} \tag{3.21}$$

has a unique solution satisfying

$$\begin{aligned} x(t) = g(t,x(t)) &\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \mathrm{ds} - \frac{\Gamma(2-\mu)t}{\Gamma(\alpha-\mu)} \int_0^1 (1-s)^{\alpha-\mu-1} h(s) \mathrm{ds} \right] \\ &+ \sum_{k=1}^m \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k-1} f_k(s,x(s)) \mathrm{ds}. \end{aligned} \tag{3.22}$$

In fact, applying the Riemann–Liouville fractional integral operator of order α to both sides of (3.21) and using Lemmas 1.12, 1.24, we have

$$\frac{x(t) - \sum_{k=1}^m I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))} = I_{0^+}^{\alpha} h(t) + \sum_{k=1}^n d_k t^{k-1}. \tag{3.23}$$

By the use of $\left(\frac{x(t) - \sum_{k=1}^m I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))} \right)^{(j)} \Bigg|_{t=0} = 0$, for $j = 2, 3 \ldots, n-1$ we get $d_3 = d_4 = \cdots = d_n = 0$ and hence (3.23) takes the form

$$\frac{x(t) - \sum_{k=1}^m I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))} = I_{0^+}^{\alpha} h(t) + d_1 + d_2 t. \tag{3.24}$$

From condition $x(0)=0$ yields $d_1=0$. Applying Caputo's fractional derivative of order μ to (3.24), we get

$$^cD^{\mu}_{0^+}\left[\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right]=I^{\alpha-\mu}_{0^+}h(t)+d_2\frac{t^{1-\mu}}{\Gamma(2-\mu)}.$$

The boundary condition $^cD^{\mu}_{0^+}\left[\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right]_{t=1}=0$, implies

$$d_2=-\Gamma(2-\mu)I^{\alpha-\mu}_{0^+}h(1).$$

Thus, (3.24) becomes

$$\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}=I^{\alpha}_{0^+}h(t)-\Gamma(2-\mu)tI^{\alpha-\mu}_{0^+}h(1),$$

which implies that

$$\begin{aligned}x(t)=g(t,x(t))&\left[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s)\mathrm{d}s-\frac{\Gamma(2-\mu)t}{\Gamma(\alpha-\mu)}\int_0^1(1-s)^{\alpha-\mu-1}h(s)\mathrm{d}s\right]\\&+\sum_{k=1}^{m}\frac{1}{\Gamma(\sigma_k)}\int_0^t(t-s)^{\sigma_k-1}f_k(s,x(s))\mathrm{d}s.\end{aligned}$$

Step 2. Let $y={}^cD^{\alpha}_{0^+}\left(\frac{x(t)-\sum_{k=1}^{m}I^{\sigma_k}_{0^+}f_k(t,x(t))}{g(t,x(t))}\right)$ and $z=\phi_p(y)$. It is easy to know that $y=\phi_q(z)$. Then, the solution of the following boundary value problem

$$\begin{cases}^cD^{\beta}_{0^+}z(t)=h(t),\ t\in I,\\ z(0)=z''(0)=z^{(3)}(0)=\cdots=z^{(n-1)},\quad z(1)=0,\end{cases}\tag{3.25}$$

can be written as

$$z(t)=I^{\beta}_{0^+}h(t)-tI^{\beta}_{0^+}h(1).\tag{3.26}$$

In fact, applying the Riemann-Liouville fractional integral operator of order β to both sides of (3.25) and using Lemmas 1.12, 1.24, we have

$$z(t)=I^{\beta}_{0^+}h(t)+\sum_{k=1}^{n}c_kt^{k-1},\tag{3.27}$$

By the use of $z^{(i)}(0)=0,\quad$ for $i=0,2,3\dots,n-1$ we get $c_1=c_3=\cdots=c_n=0$ and hence (3.27) takes the form

$$z(t)=I^{\beta}_{0^+}h(t)+c_2t,\tag{3.28}$$

and $z(1)=0$, implies

$$c_2=-I^{\beta}_{0^+}h(1).$$

Therefore, we have

$$z(t) = I_{0^+}^{\beta} h(t) - t I_{0^+}^{\beta} h(1). \tag{3.29}$$

Combining with the expression of y, we know that the solution of (3.18)–(3.19) satisfies

$$\begin{cases} {}^cD_{0^+}^{\alpha}\left[\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))}\right] = \phi_p^{-1}\left(I_{0^+}^{\beta} h(t) - t I_{0^+}^{\beta} h(1)\right), \ t \in I, \\ \left.\left(\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))}\right)^{(j)}\right|_{t=0} = 0, \quad \text{for } j = 2,3\dots,n-1 \\ \\ {}^cD_{0^+}^{\mu}\left[\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\sigma_k} f_k(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0, \quad x(0) = 0. \end{cases} \tag{3.30}$$

As we have stated in **Step 1**, we can easily get the solution of BVP (3.30) as follows :

$$\begin{aligned} x(t) = &\left\{\delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q\left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\mathrm{d}\tau - s\int_0^1 (1-\tau)^{\beta-1} h(\tau)\mathrm{d}\tau\right)\mathrm{ds}\right. \\ &\left.-\delta_2 t \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q\left(\int_0^s (s-\tau)^{\beta-1} h(\tau)\mathrm{d}\tau - s\int_0^1 (1-\tau)^{\beta-1} h(\tau)\mathrm{d}\tau\right)\mathrm{ds}\right\} g(t,x(t)) \\ &+\sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k-1} f_k(s,x(s))\mathrm{ds}. \end{aligned}$$

□

For simplicity of presentation, we give some notations :

$$\begin{aligned} \mathscr{M}_h = &(\Gamma(\beta+1))^{1-q} \max\{1, 2^{q-2}\} \|p_h\| \psi(r) \left[\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)} + \frac{\Gamma(q)}{\Gamma(\alpha+q)}\right. \\ &\left.+\frac{\Gamma(2-\mu)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha-\mu+\beta(q-1)+1)} + \frac{\Gamma(2-\mu)\Gamma(q)}{\Gamma(\alpha-\mu+q)}\right]. \end{aligned} \tag{3.31}$$

Now we list some hypotheses as follows :

(H1) The functions $g : I \times \mathbb{R} \longrightarrow \mathbb{R}\setminus\{0\}$ h, and $f_k : I \times \mathbb{R} \longrightarrow \mathbb{R}$, $k = 1,2,3\dots,m$, are continuous.

(H2) There exist positive functions $\mu_g(t)$ and $\lambda_k(t)$, $k = 1,2,3\dots,m$, with bound $\|\mu_g\|$ and $\|\lambda_k\|$, $k = 1,2,3\dots,m$, respectively, such that

$$|f_k(t,x) - f_k(t,y)| \leq \lambda_k(t)|x-y|, \quad k = 1,2,3\dots,m, \tag{3.32}$$

and

$$|g(t,x) - g(t,y)| \leq \mu_g(t)|x-y|, \text{ for each } (t,x,y) \in I \times \mathbb{R} \times \mathbb{R}. \tag{3.33}$$

(H3) There exist a function $p_h \in L^{\infty}(J, \mathbb{R}_+)$ and a continuous nondecreasing function ψ : $[0,\infty) \longrightarrow (0,\infty)$ such that

$$|h(t,x)| \leq \phi_p(p_h(t)\psi(\|x\|)), \text{ for each } t \in I \text{ and all } x \in \mathbb{R}. \tag{3.34}$$

(H4) There exists a number $r > 0$ such that

$$r \geq \frac{G_0 \mathscr{M}_h + \sum_{k=1}^{m} \frac{F_k}{\Gamma(\sigma_k+1)}}{1 - \|\mu_g\| \mathscr{M}_h - \sum_{k=1}^{m} \frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}}, \tag{3.35}$$

and

$$\|\mu_g\| \mathscr{M}_h + \sum_{k=1}^{m} \frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)} < 1, \tag{3.36}$$

where

$$G_0 = \sup_{t\in I} |g(t,0)|, F_k = \sup_{t\in I} |f_k(t,0)|, \quad k = 1,2,3\ldots,m. \tag{3.37}$$

Theorem 3.7. *Assume that conditions (H1)-(H4) hold. Then the problem (3.16) has at least one solution on I.*

Proof. Define the closed, convex and bounded subset S of the Banach space E as

$$S = \{x \in E : \|x\|_E \leq r\}.$$

By Lemma 3.9, the boundary value problem (3.16) is equivalent to the equation

$$\begin{aligned} x(t) = &\left\{\delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q\left(\int_0^s (s-\tau)^{\beta-1} h(\tau,u(\tau))\mathrm{d}\tau - s\int_0^1 (1-\tau)^{\beta-1} h(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{ds}\right. \\ &\left. -\delta_2 t \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q\left(\int_0^s (s-\tau)^{\beta-1} h(\tau,u(\tau))\mathrm{d}\tau - s\int_0^1 (1-\tau)^{\beta-1} h(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{ds}\right\} g(t,x(t)) \\ &+ \sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k-1} f_k(s,x(s))\mathrm{ds}. \end{aligned} \tag{3.38}$$

Define three operators $\mathscr{A}, \mathscr{C} : E \longrightarrow E$ and $\mathscr{B} : S \longrightarrow E$ by

$$\begin{aligned} &\mathscr{A}x(t) = g(t,x(t)),\ t \in I. \\ &\mathscr{B}x(t) = \delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q\left(\int_0^s (s-\tau)^{\beta-1} h(\tau,x(\tau))\mathrm{d}\tau - s\int_0^1 (1-\tau)^{\beta-1} h(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{ds} \\ &-\delta_2 t \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q\left(\int_0^s (s-\tau)^{\beta-1} h(\tau,x(\tau))\mathrm{d}\tau - s\int_0^1 (1-\tau)^{\beta-1} h(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{ds},\ t \in I, \end{aligned}$$

and

$$\mathscr{C}x(t) = \sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k-1} f_k(s,x(s))\mathrm{ds},\ t \in I.$$

Then the integral Equation (3.38) can be written in the operator form as

$$x(t) = \mathscr{A}x(t)\mathscr{B}x(t) + \mathscr{C}x(t),\ \ t \in I.$$

We shall show that the operators $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ satisfy all conditions of Lemma 1.48. This will be achieved in the following series of steps.

Step 1 : First we show that $\mathscr{A}$ and $\mathscr{C}$ are Lipschitzian on E. Let $x,y \in E$, then by (H2), for $t \in I$, we have

$$|\mathscr{A}x(t) - \mathscr{A}y(t)| = |g(t,x(t)) - g(t,y(t))| \leq \mu_g(t)|x(t) - y(t)|,$$

for all $t \in I$. Taking supremum over t, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \|\mu_g\|\|x - y\|,$$

for all $x,y \in E$. Therefore, $\mathscr{A}$ is a Lipschitzian on E with Lipschitz constant $\|\mu_g\|$.
Now for $\mathscr{C} : E \longrightarrow E, x,y \in E$, we have

$$\begin{aligned}
|\mathscr{C}x(t) - \mathscr{C}y(t)| &= \left|\sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k - 1} f_k(s,x(s))\mathrm{ds} - \sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k - 1} f_k(s,y(s))\mathrm{ds}\right| \\
&\leq \sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k - 1} |f_k(s,x(s)) - f_k(s,y(s))|\mathrm{ds} \\
&\leq \sum_{k=1}^{m} \frac{1}{\Gamma(\sigma_k)} \int_0^t (t-s)^{\sigma_k - 1} \lambda_k(s)|x(s) - y(s)|\mathrm{ds} \\
&\leq \|x - y\| \sum_{k=1}^{m} \frac{\|\lambda_k\|}{\Gamma(\sigma_k + 1)},
\end{aligned}$$

for all $t \in I$. Taking supremum over t, we obtain

$$\|\mathscr{C}x - \mathscr{C}y\| \leq \sum_{k=1}^{m} \frac{\|\lambda_k\|}{\Gamma(\sigma_k + 1)} \|x - y\|,$$

Hence, $\mathscr{C} : E \longrightarrow E$ is a Lipschitzian on E with Lipschitz constant $\sum_{k=1}^{m} \frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}$.
Step 2 : We show that $\mathscr{B}$ is a completely continuous operator from S into E. Firstly, observe that continuity of $\mathscr{B}$ follows from the continuity of h and $\phi_q(\cdot)$.

Next we will prove that the set $\mathscr{B}(S)$ is a uniformly bounded in S. For any $x \in S$, we have

$$
\begin{aligned}
&|\mathscr{B}x(t)| \leq \\
&\delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau + s\int_0^1 (1-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau \right) \mathrm{ds} \\
&+ \delta_2 \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau + s\int_0^1 (1-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau \right) \mathrm{ds} \\
&\leq \delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} \phi_p(p_h(\tau)\psi(|x(\tau)|)) \mathrm{d}\tau \right. \\
&\left. + s\int_0^1 (1-\tau)^{\beta-1} \phi_p(p_h(\tau)\psi(|x(\tau)|)) \mathrm{d}\tau \right) \mathrm{ds} \\
&+ \delta_2 \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} \phi_p(p_h(\tau)\psi(|x(\tau)|)) \mathrm{d}\tau \right. \\
&\left. + s\int_0^1 (1-\tau)^{\beta-1} \phi_p(p_h(\tau)\psi(|x(\tau)|)) \mathrm{d}\tau \right) \mathrm{ds} \\
&\leq \delta_1 \|p_h\| \psi(\|x\|) \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} \mathrm{d}\tau + s\int_0^1 (1-\tau)^{\beta-1} \mathrm{d}\tau \right) \mathrm{ds} \\
&+ \delta_2 \|p_h\| \psi(\|x\|) \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} \mathrm{d}\tau + s\int_0^1 (1-\tau)^{\beta-1} \mathrm{d}\tau \right) \mathrm{ds} \\
&\leq \|p_h\| \psi(\|x\|) \left[\delta_1 \int_0^t (t-s)^{\alpha-1} \phi_q \left(\frac{s^\beta}{\beta} + \frac{s}{\beta} \right) \mathrm{ds} + \delta_2 \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q \left(\frac{s^\beta}{\beta} + \frac{s}{\beta} \right) \mathrm{ds} \right] \\
&\leq \frac{\|p_h\| \psi(\|x\|)}{\beta^{q-1}} \left[\delta_1 \int_0^t (t-s)^{\alpha-1} \left(s^\beta + s \right)^{q-1} \mathrm{ds} + \delta_2 \int_0^1 (1-s)^{\alpha-\mu-1} \left(s^\beta + s \right)^{q-1} \mathrm{ds} \right] \\
&\leq \frac{\max\{1, 2^{q-2}\} \|p_h\| \psi(\|x\|)}{\beta^{q-1}} \left[\delta_1 \int_0^t (t-s)^{\alpha-1} \left(s^{\beta(q-1)} + s^{q-1} \right) \mathrm{ds} \right. \\
&\left. + \delta_2 \int_0^1 (1-s)^{\alpha-\mu-1} \left(s^{\beta(q-1)} + s^{q-1} \right) \mathrm{ds} \right] \\
&\leq (\Gamma(\beta+1))^{1-q} \max\{1, 2^{q-2}\} \|p_h\| \psi(r) \left[\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)} + \frac{\Gamma(q)}{\Gamma(\alpha+q)} \right. \\
&\left. + \frac{\Gamma(2-\mu)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha-\mu+\beta(q-1)+1)} + \frac{\Gamma(2-\mu)\Gamma(q)}{\Gamma(\alpha-\mu+q)} \right].
\end{aligned}
$$

Thus $\|\mathscr{B}x\| \leq \mathscr{M}_h$ with $\mathscr{M}_h$ given in (3.31) for all $x \in S$. This shows that $\mathscr{B}$ is uniformly bounded on S. Furthermore, we have

$$
\begin{aligned}
&|(\mathscr{B}x)'(t)| \leq (\alpha-1)\delta_1 \int_0^t (t-s)^{\alpha-2} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau \right. \\
&\left. + s\int_0^1 (1-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau \right) \mathrm{ds} + \delta_2 \int_0^1 (1-s)^{\alpha-\mu-1} \phi_q \left(\int_0^s (s-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau \right. \\
&\left. + s\int_0^1 (1-\tau)^{\beta-1} |h(\tau,x(\tau))| \mathrm{d}\tau \right) \mathrm{ds}.
\end{aligned}
$$

Some computations give

$$|(\mathscr{B}x)'(t)| \le (\Gamma(\beta+1))^{1-q}\max\{1,2^{q-2}\}\|p_h\|\psi(r)\left[\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1))}+\frac{\Gamma(q)}{\Gamma(\alpha-1+q)}\right.$$
$$\left.+\frac{\Gamma(2-\mu)\Gamma(\beta(q-1)+1)}{\Gamma(\alpha-\mu+\beta(q-1)+1)}+\frac{\Gamma(2-\mu)\Gamma(q)}{\Gamma(\alpha-\mu+q)}\right]:=L.$$

Now, for $t_1,t_2\in I$ with $t_1<t_2$, we get

$$|\mathscr{B}x(t_2)-\mathscr{B}x(t_1)| \le \int_{t_1}^{t_2}\left|(\mathscr{B}x)'(s)\right|\mathrm{d}s \le L(t_2-t_1).$$

Therefore, $\mathscr{B}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathscr{B}$ is completely continuous.

Step 3 : The hypothesis (c) of Lemma 1.48 is satisfied.

Let $x\in E$ and $y\in S$ be arbitrary elements such that $x=\mathscr{A}x\mathscr{B}y+\mathscr{C}x$. Then we have

$$\begin{aligned}
&|x(t)| \le |\mathscr{A}x(t)|\,|\mathscr{B}y(t)|+|\mathscr{C}x(t)| \le |g(t,x(t))|\times\\
&\left[\delta_1\int_0^t(t-s)^{\alpha-1}\phi_q\left(\int_0^s(s-\tau)^{\beta-1}|h(\tau,y(\tau))|\mathrm{d}\tau+s\int_0^1(1-\tau)^{\beta-1}|h(\tau,y(\tau))|\mathrm{d}\tau\right)\mathrm{d}s\right.\\
&\left.+\delta_2\int_0^1(1-s)^{\alpha-\mu-1}\phi_q\left(\int_0^s(s-\tau)^{\beta-1}|h(\tau,y(\tau))|\mathrm{d}\tau+s\int_0^1(1-\tau)^{\beta-1}|h(\tau,y(\tau))|\mathrm{d}\tau\right)\mathrm{d}s\right]\\
&+\sum_{k=1}^m\frac{1}{\Gamma(\sigma_k)}\int_0^t(t-s)^{\sigma_k-1}|f_k(t,x(t))|\mathrm{d}s\\
&\le (|g(t,x(t))-g(t,0))|+|g(t,0))|)\mathscr{M}_h\\
&+\sum_{k=1}^m\frac{1}{\Gamma(\sigma_k)}\int_0^t(t-s)^{\sigma_k-1}(|f_k(t,x(t))-f_k(t,0))|+|f(t,0)|)\mathrm{d}s,\\
&\le (\|\mu_g\||x(t)|+G_0)\mathscr{M}_h+\sum_{k=1}^m\frac{|x(t)|\|\lambda_k\|+F_k}{\Gamma(\sigma_k+1)}.
\end{aligned}$$

Thus,

$$|x(t)| \le \frac{G_0\mathscr{M}_h+\sum_{k=1}^m\frac{F_k}{\Gamma(\sigma_k+1)}}{1-\|\mu_g\|\mathscr{M}_h-\sum_{k=1}^m\frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}}.$$

Taking the supremum over t, we get

$$\|x\| \le \frac{G_0\mathscr{M}_h+\sum_{k=1}^m\frac{F_k}{\Gamma(\sigma_k+1)}}{1-\|\mu_g\|\mathscr{M}_h-\sum_{k=1}^m\frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}} \le r.$$

Step 4 : Finally we show that $\delta M+\xi<1$, that is, (d) of Lemma 1.48 holds.

Since

$$M=\|\mathscr{B}(S)\|=\sup_{x\in S}\left\{\sup_{t\in I}|\mathscr{B}x(t)|\right\}\le\mathscr{M}_h,$$

and so

$$\|\mu_g\|M+\sum_{k=1}^m\frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}\le\|\mu_g\|\mathscr{M}_h+\sum_{k=1}^m\frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}<1,$$

with

$$\delta = \|\mu_g\|, \xi = \sum_{k=1}^{m} \frac{\|\lambda_k\|}{\Gamma(\sigma_k+1)}.$$

Thus all conditions of Lemma 1.48 are satisfied and hence the operator equation $x = \mathscr{A}x\mathscr{B}x + \mathscr{C}x$ has a solution in S. As a result, problem (3.16) has a solution on I.

□

3.3.3 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following boundary value problem :

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2}}\left(\phi_5\left[{}^cD_{0^+}^{\frac{11}{4}}\left(\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\frac{2k+1}{2}} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right) = \phi_p\left(\frac{1}{(t^2+2)^2}(1+\sin x(t))\right), \quad t \in I := [0,1] \\ \left(\phi_5\left[{}^cD_{0^+}^{\frac{11}{4}}\left(\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\frac{2k+1}{2}} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)^{(i)}|_{t=0} = 0, \quad i = 0,2, \\ \left(\phi_5\left[{}^cD_{0^+}^{\frac{11}{4}}\left(\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\frac{2k+1}{2}} f_k(t,x(t))}{g(t,x(t))}\right)\right]\right)|_{t=1} = 0, \\ \left(\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\frac{2k+1}{2}} f_k(t,x(t))}{g(t,x(t))}\right)^{(2)}|_{t=0} = 0, \\ {}^cD_{0^+}^{\frac{1}{2}}\left[\frac{x(t)-\sum_{k=1}^{m} I_{0^+}^{\frac{2k+1}{2}} f_k(t,x(t))}{g(t,x(t))}\right]_{t=1} = 0, \quad x(0) = 0. \end{cases} \tag{3.39}$$

In this case we take

$$n=3,\ p=5,\ q=\frac{5}{4},\quad \alpha=\frac{11}{4},\ \beta=\frac{5}{2},\ \mu=\frac{1}{2}, \sigma_k=\frac{2k+1}{2}, k=1,2,\ldots,10,$$

$$f_k(t,x(t)) = \frac{1}{2(t^2+k)^2}\left(x(t)+\sqrt{x^2(t)+1}+e^{-t}\right), k=1,2,\ldots,10,$$

$$g(t,x(t)) = \frac{e^{-2\pi t}\cos(\pi t)}{(e^t+9)^2}\frac{x(t)}{1+x(t)}+\frac{1}{10}$$

$$h(t,x(t)) = \frac{1}{(t^2+2)^2}(1+\sin x(t)).$$

We can show that

$$|f_k(t,x)-f_k(t,y)| \le \frac{1}{(t^2+k)^2}|x-y|, k=1,2,\ldots,10,$$

$$|g(t,x)-g(t,y)| \leq \frac{1}{(e^t+9)^2}|x-y|$$

hence, we have

$$\lambda_k(t) = \frac{1}{(t^2+k)^2},\ \mu_g(t) = \frac{1}{(e^t+9)^2}.$$

Then,

$$\|\lambda_k\| = \frac{1}{k^2},\ \|\mu_g\| = \frac{1}{100}$$

$$F_k = \sup_{t\in I}|f_k(t,0)| = \frac{1}{k^2},\ G_0 = \sup_{t\in I}|g(t,0)| = \frac{1}{10}.$$

On the other hand, For each $x \in \mathbb{R}$, $t \in J$ we have

$$\begin{aligned} |f(t,x)| &= \left|\phi_p\left(\frac{1}{(t^2+2)^2}(1+\sin x)\right)\right| \\ &\leq \phi_p\left(\frac{1}{(t^2+2)^2}(1+|x|)\right). \end{aligned}$$

Therefore, assumption (H3) of the Theorem 3.7 is satisfied with $p_h(t) = \frac{1}{(t^2+2)^2}, t \in I$, and $\psi(x) = x+1,\ x \in [0,\infty)$. Using the Matlab program, we see that equations (3.35), (3.36) are followed with a number $r \in (0, 6.0086) \cup (119.6157, 135.6242)$. As all conditions of Theorem 3.7 are satisfied, the problem (3.39) has at least one solution on I.

3.4 Fractional Hybrid Differential Equations with deviating arguments under hybrid conditions

3.4.1 Introduction

In this section, we use the technique based upon measures of noncompactness in conjunction with a generalization of Darbo's fixed point theorem with a view to studying the existence of solutions for a hybrid fractional differential equation involving the Caputo fractional derivative with deviating argument of the form

$$\begin{cases} {}^cD^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right] = g(t,x(t),x(\rho(t))), t \in I := [0,1], \\ \left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=1} = 0, \ {}^cD^{\beta}_{0^+}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=\eta} = 0, \ x^{(2)}(0) = 0. \end{cases} \tag{3.40}$$

Where $2 < \alpha \leq 3, 0 < \beta \leq 1$ are a real number, ${}^cD^{\alpha}_{0^+}, {}^cD^{\beta}_{0^+}$ are the Caputo fractional derivative, $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\}), g \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, φ and ρ are functions from $[0,1]$ into itself. The result is illustrated with a suitable example. To see more applications about the usefulness of MNC in the study of some classes of nonlinear integral equations in Banach algebra the reader can be referred to [65, 47, 62, 88, 89].

3.4.2 Existence of Solutions [3]

Let us start by defining what we mean by a solution of the problem (3.40).

Definition 3.8. By a solution of the problem (3.40) we mean a function $x \in C(J,\mathbb{R})$ that satisfies the conditions $\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=1} = {}^cD^{\beta}_{0^+}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=\eta} = x^{(2)}(0) = 0$ and the equation ${}^cD^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right] = g(t,x(t),x(\rho(t)))$ on I

For the existence of solutions for the problem (3.40) we need the following auxiliary lemma

Lemma 3.9. *Let $2 < \alpha \leq 3$ and suppose that $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $h \in C(I)$. Then the fractional hybrid BVP*

$${}^cD^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right] = h(t), \quad 0 < t < 1, \tag{3.41}$$

with boundary conditions

$$\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=1} = {}^cD^{\beta}_{0^+}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=\eta} = x^{(2)}(0) = 0, \tag{3.42}$$

where $\varphi : [0,1] \longrightarrow [0,1]$ has the following unique solution :

$$x(t) = f(t,x(t),x(\varphi(t)))\left\{I^{\alpha}_{0^+}h(t) - I^{\alpha}_{0^+}h(1) + \frac{1-t}{v_0}I^{\alpha-\beta}_{0^+}h(\eta)\right\}. \tag{3.43}$$

3. **C. Derbazi**, H. Hammouche, M. Benchohra, Fractional Hybrid Differential Equations with deviating arguments under hybrid conditions. (submitted).

where

$$v_0 = \frac{\eta^{1-\beta}}{\Gamma(2-\beta)}$$

Proof. Applying the Riemann-Liouville fractional integral operator of order α to both sides of (3.41) and using Lemmas 1.12, 1.24, we have

$$\frac{x(t)}{f(t,x(t),x(\varphi(t)))} = I_{0^+}^{\alpha}h(t) + c_0 + c_1 t + c_2 t^2, \ \forall\, c_0, c_1, c_2 \in \mathbb{R}.$$

By the use of initial condition $x^{(2)}(0) = 0$, we get $c_2 = 0$,
and so,

$$\frac{x(t)}{f(t,x(t),x(\varphi(t)))} = I_{0^+}^{\alpha}h(t) + c_0 + c_1 t, \ \forall\, c_0, c_1 \in \mathbb{R}.$$

Consequently, the general solution of (3.41) is

$$x(t) = f(t,x(t),x(\varphi(t)))(I_{0^+}^{\alpha}h(t) + c_0 + c_1 t) \quad \forall \quad c_0, c_1 \in \mathbb{R}. \tag{3.44}$$

Then, by using Lemmas 1.12 and 1.24, we may obtain

$$\begin{aligned} {}^cD_{0^+}^{\beta}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=\eta} &= I_{0^+}^{\alpha-\beta}h(\eta) + c_1\frac{\Gamma(2)}{\Gamma(2-\beta)}\eta^{1-\beta} \\ \left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=1} &= I_{0^+}^{\alpha}h(1) + c_0 + c_1, \end{aligned}$$

which together with the boundary condition ${}^cD_{0^+}^{\beta}\left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=\eta} = \left[\frac{x(t)}{f(t,x(t),x(\varphi(t)))}\right]_{t=1} = 0$, implies that

$$\begin{aligned} c_1 &= -\frac{1}{v_0}I_{0^+}^{\alpha-\beta}h(\eta), \\ c_0 &= \frac{1}{v_0}I_{0^+}^{\alpha-\beta}h(\eta) - I_{0^+}^{\alpha}h(1). \end{aligned}$$

Substituting the value of c_0, c_1 in (3.44) we get (3.43).

□

We study the Problem (3.40) under the following assumptions :

(H1) $f \in C(I\times\mathbb{R}\times\mathbb{R}, \mathbb{R}\setminus\{0\})$ and $g \in C(I\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$.

(H2) The functions $\varphi, \rho : [0,1] \longrightarrow [0,1]$ are continuous.

(H3) The function f satisfy

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq (\max(|x_1 - x_2|, |y_1 - y_2|) + 1)^k - 1,$$

for any $t \in [0,1]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $k, \in (0,1)$.

(H4) There exist continuous nondecreasing function $\psi : [0,\infty) \longrightarrow [0,\infty)$ and function $p \in C(I,\mathbb{R}^+)$ such that

$$|g(t,x,y)| \leq p(t)\left(\psi_1(|x|) + \psi_2(|y|)\right),$$

for each $(t,x,y) \in I\times\mathbb{R}\times\mathbb{R}$.

Notice that assumption (H1) gives us the existence nonnegative constant K_1 such that

$$K_1 = \sup\{|f(t,0,0)| : t \in [0,1]\}.$$

(H5) There exists $r_0 > 0$ such that

$$r_0 \geq \left[(r_0+1)^k - 1 + K_1\right] \|p\| (\psi_1(r_0) + \psi_2(r_0)) \left\{ \frac{2}{\Gamma(\alpha+1)} + \frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)} \right\}, \tag{3.45}$$

and

$$(\|p\|(\psi_1(r_0) + \psi_2(r_0)) \left\{ \frac{2}{\Gamma(\alpha+1)} + \frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)} \right\} \leq 1.$$

Theorem 3.10. *Under assumptions (H1)-(H5), problem (3.40) has at least one solution in $C(I)$.*

Proof. In view of Lemma 3.9 we consider the operator $\mathscr{T}$ defined on $C(I)$ by

$$\begin{aligned}\mathscr{T}x(t) = f(t, x(t), x(\varphi(t))) \{ & I_{0^+}^{\alpha} g(s, x(s), x(\rho(s)))(t) - I_{0^+}^{\alpha} g(s, x(s), x(\rho(s)))(1) \\ & - \frac{t-1}{v_0} I_{0^+}^{\alpha-\beta} g(s, x(s), x(\rho(s)))(\eta) \Big\}, \ t \in I.\end{aligned}$$

Notice that the fixed point problem $\mathscr{T}x = x$ is equivalent to problem (2.1). Next we introduce two operators $\mathscr{F}, \mathscr{G}$ defined on $C(I)$ by

$$\mathscr{F}x(t) = f(t, x(t), x(\varphi(t))),$$

and

$$\mathscr{G}x(t) = I_{0^+}^{\alpha} g(s, x(s), x(\rho(s)))(t) - I_{0^+}^{\alpha} g(s, x(s), x(\rho(s)))(1) - \frac{t-1}{v_0} I_{0^+}^{\alpha-\beta} g(s, x(s), x(\rho(s)))(\eta),$$

for any $x \in C(I)$ and $t \in I$.
Observe that $\mathscr{T}x = (\mathscr{F}x) \cdot (\mathscr{G}x)$ for any $x \in C(I)$.
We split the proof into several steps.
Step 1 : $\mathscr{T}$ applies $C(I)$ into itself.
In order to show that $\mathscr{T}x \in C(I)$ it is sufficient to show that $\mathscr{F}x, \mathscr{G}x \in C(I)$ for any $x \in C(I)$. Obviously the conditions of Theorem 3.10 guarantee that if $x \in C(I)$ then $\mathscr{F}x \in C(I)$. Next, we will prove that if $x \in C(I)$ then $\mathscr{G}x \in C(I)$.

To do this, let $t \in I$ be fixed and $\{t_n\}$ be a sequence in I such that $t_n \to t$ as $n \to \infty$. Without loss of generality, we may assume $t_n > t$. Then, we get

$$\begin{aligned}|\mathscr{G}x(t_n) - \mathscr{G}x(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^{t_n} |(t_n - s)^{\alpha-1} - (t-s)^{\alpha-1}| |g(s, x(s), x(\rho(s)))| \mathrm{ds} \\ & + \frac{1}{\Gamma(\alpha)} \int_t^{t_n} |t-s|^{\alpha-1} |g(s, x(s), x(\rho(s)))| \mathrm{ds} \\ & + \frac{|t - t_n|}{|v_0|\Gamma(\alpha-\beta)} \int_0^{\eta} (\eta - s)^{\alpha-\beta-1} |g(s, x(s), x(\rho(s)))| \mathrm{ds}.\end{aligned}$$

In view of (H4) we obtain

$$
\begin{aligned}
|\mathscr{G}x(t_n)-\mathscr{G}x(t)| &\le \frac{\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{\Gamma(\alpha)}\int_0^{t_n}|(t_n-s)^{\alpha-1}-(t-s)^{\alpha-1}|\mathrm{ds}\\
&+\frac{\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{\Gamma(\alpha)}\int_t^{t_n}|t-s|^{\alpha-1}\mathrm{ds}\\
&+\frac{|t_n-t|\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{|v_0|\Gamma(\alpha-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}\mathrm{ds}\\
&\le \frac{\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{\Gamma(\alpha)}\Big[\int_0^{t}\left|(t_n-s)^{\alpha-1}-(t-s)^{\alpha-1}\right|\mathrm{d}s\\
&+\int_t^{t_n}\left|(t_n-s)^{\alpha-1}-(t-s)^{\alpha-1}\right|\mathrm{d}s+\int_t^{t_n}|t-s|^{\alpha-1}\mathrm{d}s\Big]\\
&+\frac{|t_n-t|\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{|v_0|\Gamma(\alpha-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}\mathrm{d}s\\
&= \frac{\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{\Gamma(\alpha)}\Big[\int_0^{t}\left[(t_n-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]\mathrm{d}s\\
&+\int_t^{t_n}(t_n-s)^{\alpha-1}\mathrm{d}s+\int_t^{t_n}(s-t)^{\alpha-1}\mathrm{d}s+\int_t^{t_n}(s-t)^{\alpha-1}\mathrm{d}s\Big]\\
&+\frac{|t_n-t|\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))}{|v_0|\Gamma(\alpha-\beta)}\int_0^{\eta}(\eta-s)^{\alpha-\beta-1}\mathrm{d}s\\
&\le \|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))\left(\frac{2}{\Gamma(\alpha+1)}(t_n-t)^{\alpha}+\frac{(t_n-t)}{\Gamma(\alpha)}+\frac{(t_n-t)}{|v_0|\Gamma(\alpha-\beta+1)}\eta^{\alpha-\beta}\right),
\end{aligned}
$$

where we have used the fact that $t_n^{\alpha}-t^{\alpha}\le\alpha(t_n-t)$. From the last estimate, we deduce that $(\mathscr{G}x)(t_n)\to(\mathscr{G}x)(t)$ when $n\to\infty$.Therefore, $\mathscr{G}x\in C(I)$. This proves that if $x\in C(I)$. then $\mathscr{T}x\in C(I)$.

Step 2 : An estimate of $\|\mathscr{T}x\|$for $x\in C(I)$.

Fix $x\in C(I)$. and $t\in C(I)$. In view of assumptions, we have

$$
\begin{aligned}
&|(\mathscr{T}x)(t)|=|(\mathscr{F}x)(t)||(\mathscr{G}x)(t)|\\
&=|f(t,x(t),x(\varphi(t)))|\,|I_{0^+}^{\alpha}g(s,x(s),x(\rho(s)))(t)-I_{0^+}^{\alpha}g(s,x(s),x(\rho(s)))(1)\\
&-\frac{t-1}{v_0}I_{0^+}^{\alpha-\beta}g(s,x(s),x(\rho(s)))(\eta)\Big|\\
&\le|f(t,x(t),x(\varphi(t)))|-f(t,0,0)|+|f(t,0,0)|)\\
&\times\left\{I_{0^+}^{\alpha}|g(s,x(s),x(\rho(s)))|(t)+I_{0^+}^{\alpha}|g(s,x(s),x(\rho(s)))|(1)+\frac{1}{|v_0|}I_{0^+}^{\alpha-\beta}|g(s,x(s),x(\rho(s)))|(\eta)\right\}\\
&\le[(\max(|x(t)|,|x(\varphi(t))|)+1)^k-1+K_1]\\
&\times\{I_{0^+}^{\alpha}p(s)[\psi_1(|x(s)|)+\psi_2(x(\rho(s)))](t)+I_{0^+}^{\alpha}p(s)[\psi_1(|x(s)|)+\psi_2(x(\rho(s)))](1)\\
&+\frac{1}{|v_0|}I_{0^+}^{\alpha-\beta}p(s)[\psi_1(|x(s)|)+\psi_2(x(\rho(s)))](\eta)\Big\}\\
&\le[(\max(\|x\|,\|x\|)+1)^k-1+K_1]\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))\left[\frac{2}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right].
\end{aligned}
$$

Therefore,

$$\|\mathscr{T}x\| \leq [(\|x\|+1)^k - 1 + K_1]\|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))\left[\frac{2}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right].$$

By assumption (H5), we infer that the operator $\mathscr{T}$ applies B_{r_0} into itself. Moreover, from the last estimates, it follows that

$$\|\mathscr{F}B_{r_0}\| \leq (r_0+1)^k - 1 + K_1$$

and

$$\|\mathscr{G}B_{r_0}\| \leq \|p\|(\psi_1(\|x\|)+\psi_2(\|x\|))\left[\frac{2}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right].$$

Step 3 : The operators $\mathscr{F}$ and $\mathscr{G}$ are continuous on the ball B_{r_0} .
In fact, firstly we prove that F is continuous on B_{r_0}. To do this, we fix $\varepsilon > 0$ and we take $x, y \in B_{r_0}$ with $\|x-y\| \leq \varepsilon$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)| &= |f(t,x(t),x(\varphi(t))) - f(t,y(t),y(\varphi(t)))| \\
&\leq (\max(|x(t)-y(t)|,|x(\varphi(t))-y(\varphi(t))|)+1)^k - 1 \\
&\leq (\max(\|x-y\|,\|x-y\|)+1)^k - 1 \\
&= (\|x-y\|+1)^k - 1 \leq (\varepsilon+1)^k - 1,
\end{aligned}$$

and since $(\varepsilon+1)^k - 1 \to 0$ when $\varepsilon \to 0$, we have proved that F is continuous in B_{r_0} .

Next, we prove that $\mathscr{G}$ is continuous in B_{r_0} . In order to do this, we fix $\varepsilon > 0$ and we take $x, y \in B_{r_0}$ with $\|x-y\| \leq \varepsilon$. Then, for $t \in [0,1]$, we get

$$\begin{aligned}
|(\mathscr{G}x)(t)-(\mathscr{G}y)(t)| &\leq I_{0^+}^{\alpha}|g(s,x(s),x(\rho(s))) - g(s,y(s),x(\rho(s)))|(t) \\
&+ I_{0^+}^{\alpha}|g(s,x(s),x(\rho(s))) - g(s,y(s),x(\rho(s)))|(1) \\
&+ \frac{1}{|v_0|}I_{0^+}^{\alpha-\beta}|g(s,x(s),x(\rho(s))) - g(s,y(s),x(\rho(s)))|(\eta) \\
&\leq \omega_g(I,\varepsilon)\left[\frac{2}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right],
\end{aligned}$$

where $\omega_g(I,\varepsilon) = \sup\{|g(t,x_1,x_2)-g(t,y_1,y_2)| : t \in I = [0,1], x_i, y_i \in [-r_0,r_0], 1 \leq i \leq 2, |x_i - y_i| \leq \varepsilon\}$. Since g is uniformly continuous on the compact $[0,1]\times[-r_0,r_0]\times[-r_0,r_0]$, we have $\omega_g(I,\varepsilon) \to 0$ as $\varepsilon \to 0$ and, therefore, the last inequality proves that operator G is continuous on B_{r_0}.
Consequently, since $\mathscr{T} = \mathscr{F}\cdot\mathscr{G}$, it follows that $\mathscr{T}$ is continuous on B_{r_0}.
Step 4 : Estimates of $\omega_0(\mathscr{F}X)$ and $\omega_0(\mathscr{G}X)$ for $\emptyset \neq X \subset B_{r_0}$. Firstly, we estimate $\omega_0(\mathscr{F}X)$.
For $\varepsilon > 0$ given, since $\varphi : [0,1] \to [0,1]$ is uniformly continuous, we can find $\delta > 0$ (which can be taken with $\delta < \varepsilon$) such that, for $|t_1 - t_2| < \delta$ we have $|\varphi(t_2)-\varphi(t_1)| < \varepsilon$.
Now, we take $x \in X$ and $t_1, t_2 \in [0,1]$ with $|t_1-t_2| \leq \delta < \varepsilon$. Then

$$\begin{aligned}
|(\mathscr{F}x)(t_2)-(\mathscr{F}x)(t_1)| &= |f(t_2,x(t_2),x(\varphi(t_2))) - f(t_1,x(t_1),x(\varphi(t_1)))| \\
&\leq |f(t_2,x(t_2),x(\varphi(t_2))) - f(t_2,x(t_1),x(\varphi(t_1)))| \\
&+ |f(t_2,x(t_1),x(\varphi(t_1))) - f(t_1,x(t_1),x(\varphi(t_1)))| \\
&\leq [(\max(|x(t_2)-x(t_1)|,|x(\varphi(t_2))-y(\varphi(t_1))|)+1)^k - 1] + \omega(f,\varepsilon) \\
&\leq [(\omega(X,\varepsilon)+1)^k - 1] + \omega(f,\varepsilon),
\end{aligned}$$

where $\omega(f,\varepsilon)$ denotes the quantity

$$\omega(f,\varepsilon)=\sup\{|f(t_2,x,y)-f(t_1,x,y)|:t_1,t_2\in[0,1],|t_1-t_2|\leq\varepsilon,x,y\in[-r_0,r_0]\}.$$

Therefore,

$$\omega(\mathscr{F}X,\varepsilon)\leq[(\omega(X,\varepsilon)+1)^k-1]+\omega(f,\varepsilon).$$

Since $f(t,x,y)$ is uniformly continuous on the compact $[0,1]\times[-r_0,r_0]\times[-r_0,r_0]$, $\omega(f,\varepsilon)\to 0$ when $\varepsilon\to 0$, and, consequently, from the last inequality, we infer

$$\omega_0(\mathscr{F}X)\leq(\omega_0(X)+1)^k-1.$$

Next, we estimate $\omega_0(\mathscr{G}X)$. Fix $\varepsilon>0$, and we take $x\in X$ and $t_1,t_2\in[0,1]$ with $|t_1-t_2|\leq\varepsilon$. Without loss of generality, we can suppose that $t_1<t_2$. Then, we have

$$\begin{aligned}|\mathscr{G}x(t_2)-\mathscr{G}x(t_1)|&=\int_{t_1}^{t_2}\left|(\mathscr{G}x)'(s)\right|\mathrm{ds}\\&\leq\|p\|(\psi_1(r_0)+\psi_2(r_0))\left[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right](t_2-t_1)\\&\leq\|p\|(\psi_1(r_0)+\psi_2(r_0))\left[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right]\varepsilon.\end{aligned}$$

Therefore,

$$\omega(\mathscr{G}x,\varepsilon)\leq\|p\|(\psi_1(r_0)+\psi_2(r_0))\left[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right]\varepsilon,$$

and this gives us $\omega_0(\mathscr{G}X)=0$.

Step 5 : An estimate of $\omega_0(\mathscr{T}X)$ for $\emptyset\neq X\subset B_{r_0}$. Taking into account that $\omega_0(XY)\leq\|X\|\omega_0(Y)+\|Y\|\omega_0(X)$ from the estimates obtained in **Steps 2** and **4**, we deduce

$$\begin{aligned}\omega_0(TX)&=\omega_0(\mathscr{F}X\cdot\mathscr{G}X)\leq\|\mathscr{F}X\|\omega_0(\mathscr{G}X)+\|\mathscr{G}X\|\omega_0(\mathscr{F}X)\\&\leq\|\mathscr{F}B_{r_0}\|\omega_0(\mathscr{G}X)+\|\mathscr{G}B_{r_0}\|\omega_0(\mathscr{F}X)\\&\leq[(\omega_0(X)+1)^k-1]\|p\|(\psi_1(r_0)+\psi_2(r_0))\left[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right].\end{aligned}$$

By assumption (H5), $\|p\|(\psi_1(r_0)+\psi_2(r_0))\left[\frac{1}{\Gamma(\alpha)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right]\leq 1$, and from the last estimate, we infer that

$$\omega_0(\mathscr{T}X)\leq(\omega_0(X)+1)^k-1,$$

or, equivalently,

$$\omega_0(\mathscr{T}X)+1\leq(\omega_0(X)+1)^k.$$

Therefore, the contractive condition appearing in Theorem 1.43 is satisfied with $\varphi(t)=t+1$, where $\varphi\in\Theta$. By Theorem 1.43, the operator T has at least one fixed point in B_{r_0}. Which is a solution of the problem (3.40). This completes the proof.

3.4.3 An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following boundary value problem :

Example 3.11.

$$\begin{cases} {}^{c}D_{0^+}^{\frac{5}{2}}\left[\dfrac{x(t)}{\frac{1}{\mu}\left(\sqrt[5]{1+|\sin x(t)|}+\sqrt[5]{1+\frac{|x(t^2)|}{1+|x(t^2)|}}\right)}\right]=g(t,x(t),x(\rho(t))),\ t\in I=[0,1] \\ \left[\dfrac{x(t)}{\frac{1}{\mu}\left(\sqrt[5]{1+|\sin x(t)|}+\sqrt[5]{1+\frac{|x(t^2)|}{1+|x(t^2)|}}\right)}\right]_{t=1}={}^{c}D_{0^+}^{\frac{1}{2}}\left[\dfrac{x(t)}{\frac{1}{\mu}\left(\sqrt[5]{1+|\sin x(t)|}+\sqrt[5]{1+\frac{|x(t^2)|}{1+|x(t^2)|}}\right)}\right]_{t=\frac{1}{4}}=0, \\ x^{(2)}(0)=0. \end{cases} \tag{3.46}$$

In this case we take

$$\begin{aligned} \alpha &= \frac{3}{2},\beta=\frac{1}{2},\eta=\frac{1}{4}, \\ f(t,x,y) &= \frac{1}{\mu}\left(\sqrt[5]{1+|\sin x|}+\sqrt[5]{1+\frac{|y|}{1+|y|}}\right), \\ g(t,x,y) &= \frac{e^{-2t}}{\sqrt{(9+t)}}\left(\frac{\sin x}{4}+\frac{3}{8}y\left(1+\frac{y}{\sqrt{1+y^2}}\right)\right), \\ \varphi(t) &= t^2,\rho(t)=\sqrt{t}, \\ K_1 &= \sup\{|f(t,0,0)|:t\in[0,1]\}=\frac{2}{\mu}. \end{aligned}$$

It is easy to check that

$$\begin{aligned} |g(t,x,y)| &= \left|\frac{e^{-2t}}{\sqrt{(9+t)}}\left(\frac{\sin x}{4}+\frac{3}{8}y\left(1+\frac{y}{\sqrt{1+y^2}}\right)\right)\right| \\ &\le p(t)(\psi_1(|x|)+\psi_2(|y|)), \end{aligned}$$

with $p(t)=e^{-2t}$, $\psi_1(|x|)=\frac{|x|}{4}$, $\psi_2(|y|)=\frac{3|y|}{4}$.
On the other hand, for any $t\in[0,1]$ and $x,y,x_1,y_1\in\mathbb{R}$ we have

$|f(t,x,y)-f(t,x_1,y_1)| \leq$

$$\begin{aligned}
&\left|\frac{1}{\mu}\left(\sqrt[5]{1+|\sin x|}+\sqrt[5]{1+\frac{|y|}{1+|y|}}\right)-\frac{1}{\mu}\left(\sqrt[5]{1+|\sin x_1|}+\sqrt[5]{1+\frac{|y_1|}{1+|y_1|}}\right)\right| \\
&\leq \frac{1}{\mu}\left|\sqrt[5]{1+|\sin x|}-\sqrt[5]{1+|\sin x_1|}\right|+\frac{1}{\mu}\left|\sqrt[5]{1+\frac{|y|}{1+|y|}}-\sqrt[5]{1+\frac{|y_1|}{1+|y_1|}}\right| \\
&\leq \frac{1}{\mu}\left|\left(\sqrt[5]{1+|\sin x|}-1\right)-\left(\sqrt[5]{1+|\sin x_1|}-1\right)\right| \\
&+\frac{1}{\mu}\left|\left(\sqrt[5]{1+\frac{|y|}{1+|y|}}-1\right)-\left(\sqrt[5]{1+\frac{|y_1|}{1+|y_1|}}-1\right)\right|.
\end{aligned}$$

Applying Lemma 1.28 we get
$|f(t,x,y)-f(t,x_1,y_1)| \leq$

$$\begin{aligned}
&\leq \frac{1}{\mu}\left(\sqrt[5]{1+||\sin x|-|\sin x_1||}-1\right)+\frac{1}{\mu}\left(\sqrt[5]{1+\left|\frac{|y|}{1+|y|}-\frac{|y_1|}{1+|y_1|}\right|}-1\right) \\
&\leq \frac{1}{\mu}\left(\sqrt[5]{1+|x-x_1|}-1\right)+\frac{1}{\mu}\left(\sqrt[5]{1+|y-y_1|}-1\right) \\
&\leq \frac{2}{\mu}\sqrt[5]{\max(|x-x_1|,|y-y_1|)+1}-1.
\end{aligned}$$

Therefore, assumption (H3) of the Theorem 3.10 is satisfied when $\mu \geq 2$ with $k = 1/5$. Observe that in this case the inequality involved in (3.45) has the form :

$$r_0\left[(r_0+1)^{\frac{1}{5}}-1+\frac{2}{\mu}\right]\left\{\frac{2}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right\} \leq r_0.$$

For $\mu = 2$, we have

$$r_0 \leq \frac{1}{\left[\frac{2}{\Gamma(\alpha+1)}+\frac{\eta^{\alpha-\beta}}{|v_0|\Gamma(\alpha-\beta+1)}\right]^5}-1,$$

which is satisfied by $r_0 = 7$. Thus, all the conditions of Theorem 3.10 are satisfied, and consequently problem (3.46) has at least one solution $x(t) \in C[0,1]$.

□

Conclusion and Perspective

In this thesis, we have considered the problem of existence of solutions for various classes of initial value problem and boundary value problem for nonlinear fractional differential equations involving the Caputo fractional-order in Banach Spaces. The results are based on the technique of measures of noncompactness. and the argument of fixed points. Some appropriate fixed point theorems have been used, in particular ; Mönch's fixed point theorem, Darbo's fixed point theorem and Dhage fixed point theorem.

For the perspective, it would be interesting to extend the results of the present thesis by considering differential inclusions involving new formulations of fractional derivatives and integrals have been presented recently (namely, Caputo–Hadamard, Caputo-Fabrizio, ψ–Caputo and ψ–Hilfer are just a few). Also, we will study the problem of stability for a class of boundary value problem for nonlinear fractional differential equations.

Another possible future work is to use a novel technique developed by G. García [69] based on the so called degree of nondensifiability (briefly, DND), which is not a measure of noncompactness but can be used as an alternative tool in certain fixed point problems, where such measures do not work out. For more applications about the usefulness of DND in the study of existence of solutions for certain integral equations, the reader can be referred to [68, 69, 70, 71].

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