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Etude de l'Existence de l'Unicité et de la Stabilité des Solutions de Certaines Equations Différentielles d'Ordre Fractionnaire

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Par
Fredj Fouad

Devant le jury composé de:

Mr Guerbati Kaddour	Pr.	Univ. Ghardaïa	Président
Mme Hammouche Hadda	Pr.	Univ. Ghardaïa	Directeur de thèse
Mr Benchohra Mouffak	Pr.	Univ. Sidi Bel Abbes	Examinateur
Mme Azzam-Laouir Dalila	Pr.	Univ. Jijel	Examinateur
Mme Abada Nadjet	MCA	E.N.S Constantine	Examinateur
Mr Boudjedaa Badredine	MCA	Centre Univ. Mila	Examinateur

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Abstract

This thesis aims to study the existence, uniqueness and stability of two types of abstract fractional differential equations in two parts. The first part explores two different hybrid differential equations with various conditions in Banach algebra. Whereas the second part investigates three random fractional differential systems in generalized Banach space with different boundary conditions. The studies in both parts were carried out using different fixed point theorems.

Résumé

Cette thèse vise à étudier l'existence, l'unicité et la stabilité de deux types d'équations différentielles fractionnaires abstraites en deux parties. La première partie explore deux équations différentielles hybrides différentes avec diverses conditions dans une algèbre de Banach. Alors que la deuxième partie étudie trois systèmes différentiels fractionnaires aléatoires dans les espaces de Banach généralisé avec différentes conditions. Les études dans les deux parties ont été menées en utilisant différents théorèmes de point fixe.

ملخص

تهدف هذه الأطروحة إلى دارسة وجود و وحدانية و استقرار حلول نوعين من المعادلات التفاضلية الكسرية المجردة في جزأين. يستكشف الجزء الأول معادلتين مختلفتين من المعادلات التفاضلية الهجينة بشروط حدية مختلفة في باناخ الجبر. بينما يبحث الجزء الثاني في ثلاثة أنظمة تفاضلية جزئية عشوائية في فضاء باناخ المعمم بشروط حدية مختلفة. أجريت الدراسات في كلا الجزأين باستخدام نظريات مختلفة للنقطة الثابتة.

Keywords: Hybrid fractional differential equation, ψ-Caputo fractional operator, hybrid boundary conditions, fixed point theorem, integral boundary conditions, Ulam-Hyres stability, random coupled system, random solution, generalized Banach spaces, random fixed point.

LIST OF SYMBOLS

We use the following notations throughout this thesis

Notations

- $\mathbb{N}$: Set of natural numbers.
- $\mathbb{R}$: Set of real numbers .
- $\mathbb{R}^n$: Space of n-dimensional real vectors.
- J: A finite interval on the half-axis $\mathbb{R}^+$.
- sup: Supremum.
- max: Maximum.
- $\Gamma(\cdot)$: Gamma function.
- I^p: The Riemann-Liouville fractional integral of order $p > 0$.
- D^p: The Riemann-Liouville fractional derivative of orde $p > 0$.
- ${}^cD^p$: The Caputo fractional derivative of orde $p > 0$.
- ${}^\rho I^p$: The Katugampola fractional integral of order $p > 0$, $\rho > 0$.
- ${}^\rho D^p$: The Katugampola fractional derivative of orde $p > 0$, $\rho > 0$.
- ${}^\rho D^{p,q}$: The Hilfer-Katugampola fractional derivative of orde $p > 0$ and type $q < 0$, with $\rho > 0$.

- $I^{p;\psi}$: The ψ-Riemann–Liouville fractional integral of order $p > 0$.
- ${}^{c}D^{p;\psi}$: The ψ-Caputo fractional derivative of order $p > 0$.
- $C(J, \mathbb{R})$: The space of continuous functions from the time interval J into $\mathbb{R}$.
- $C(J, \mathbb{R})$: The Space of n time continuously. differentiable functions on J into $\mathbb{R}$.
- $AC(J, \mathbb{R})$: Space of absolutely continuous functions on J.
- $L^{p}(J)$: Lebesgue spaces.
- $L^{\infty}(J)$: Space of functions u that are essentially bounded on J .

CONTENTS

INTRODUCTION

Fractional differential equations received great attention of many researchers working in different disciplines of science and technology. In other words, some recent publications show the importance of fractional differential equations in the mathematical modeling of many real-world phenomena. For example ecological models [30], economic models [50], physics [36], fluid mechanics [53]. In the literature, there are many studies on fractional differential equations with distinct kinds of fractional derivatives, such as Riemann-Liouville fractional derivative, Caputo fractional derivative, and Grunwald Letnikov fractional derivative, etc. For examples, see [33, 41, 44]. Very recently, a new kind of fractional derivative ψ-Caputo was introduced by Almeida in [7]. The main advantage of the derivative mentioned above is the freedom of choices of the kernels of derivative by choosing different functions ψ, which gives us some well known fractional derivatives such Caputo, Caputo-Erdelyi-Koper and Caputo Hadamard derivative. For more details on the ψ-Caputo and fractional differential equation involving ψ-Caputo, we refer the reader to a series of papers [7, 8, 19] and the references cited therein.

Recently mathematicians have shown a special interest in Fractional differential equations, which resulted in plenty of research papers that have been carried out on Fractional differential equations. That made a valuable contribution ranging from the qualitative theory of the solutions of Fractional differential equations, such as existence, uniqueness, stability and controllability to the numerical analysis. Speaking in this context, the stability analysis of functional and differential equations are important in many applications, such as optimization, numerical analysis, etc. Where computing the exact solution is rather hard. There are various kinds of stability, one of those types has recently received considerable attention from many mathematicians, so-called Ulam-Hyers stability. The source of Ulam-Hyers stability goes back to 1940 by Ulam [52], next by Hyers [27]. A variety of works have been done by many authors in regard of the Ulam-Hyers stability of Fractional differential equations, for example, the authors in [10] studied the existence and stability results for implicit Fractional differential equations. Some recent developments in Ulam's type stability are discussed by Belluot, et al. [14]. Ibrahim in [28] obtained the generalized Ulam-Hyers stability for Fractional differential equations. Some approximate analytical methods for solving Fractional differential equations can be found in [42, 39, 23, 55], also computational analyse of some fractional dynamical and biological models were investigated recently, see [34, 3].

Coupled systems of fractional differential equations supplemented with a variety of boundary conditions constitute an important field of research in view of their applications. Such systems occur naturally in many real world situations, like fractional dynamical systems [9], disease models [15], ecological effects [30], synchronization of chaotic systems [21], anomalous diffusion [48]. On the other hand, non-local and integral boundary conditions are widely used where classical boundary conditions fail to examine many physical properties of the models. Some recent works regarding coupled systems of fractional differential equations including non-local and integral boundary conditions with different approaches can be found in [6, 5, 24].

Random differential equations are found to be of great support in developing a more realistic mathematical modeling of the applied problems, which usually contain parameters or coefficients that are often unknown or inaccurate. Therefore, it is more realistic to consider such parameters as random variables whose behavior is governed by probability.Random differential equations, as natural extensions of deterministic have been studied and developed by many authors, see [1, 2, 11, 13, 22, 40, 46].

On the other hand, quadratic perturbation of nonlinear differential, also known as the hybrid differential equations had rapid progress over the last years, this is due to its importance, which lies in the fact that they include perturbations that facilitate the study of such equations by using the perturbation techniques. These equations are also considered as a particular case in dynamic systems. The starting point for this field when Dhage and Lakshmikantham [17] formulated a hybrid differential equation, where they investigated the existence and uniqueness of the solutions to the following hybrid equation

$$\begin{cases} \frac{d}{dt}\left(\dfrac{u(t)}{g(t,u(t))}\right) = f(t,u(t)); \quad \text{a.e. } t \in [t_0, t_0+T],\ t_0 \in \mathbb{R},\ T>0, \\ \\ u(t_0) = u_0, \quad u_0 \in \mathbb{R}. \end{cases}$$

Their results were based on the fixed point theorem for the product of two operators in Banach algebra.

In 2012, Zhao et al. [56] extended Dhage's work [17] to fractional order and studied the existence of solutions to the following Riemann-Liouville type hybrid fractional differential equation

$$\begin{cases} D^p_{0^+}\left(\dfrac{u(t)}{h(t,u(t))}\right) = f(t,u(t)); \quad \text{a.e. } t \in [0,T],\ 0 < p \le 1, \\ \\ u(0) = 0. \end{cases}$$

After several years, Sitho et al. [45] derived a new existence result for the following hybrid sequential integro-differential equations

$$\begin{cases} D^p_{0^+}\left[\dfrac{D^q_{0^+}u(t) - \sum_{i=1}^n I^{\eta_i}_{0^+} g_i(t,u(t))}{h(t,u(t))}\right] = f(t,u(t), I^{\gamma}_{0^+}u(t)); \quad t \in [0,T],\ 0 < p,q \le 1, \\ \\ u(0) = 0; \qquad D^q_{0^+}u(0) = 0. \end{cases}$$

Outline of the thesis

In **Chapter 1** we introduce some background material , such as fractional differential equations, generalized metric space, some random fixed point theorems and random variable.

Chapter 2 treats the main results concerning the existence and uniqueness of solutions for a class of hybrid differential equations of arbitrary fractional order of the form

$${}^cD^{p;\psi}_{0^+}\left[\frac{{}^cD^{q;\psi}_{0^+}u(t) - \sum_{i=1}^m I^{\eta_i;\psi}_{0^+} g_i(t,u(t))}{h(t,u(t))}\right] = f\left(t, u(t), {}^cD^{p;\psi}_{0^+}\left[\frac{{}^cD^{q;\psi}_{0^+}u(t) - \sum_{i=1}^m I^{\eta_i;\psi}_{0^+} g_i(t,u(t))}{h(t,u(t))}\right]\right); t \in J,$$

endowed with the hybrid fractional integral boundary conditions

$$\begin{cases} u(0)=0, \quad {}^cD_{0^+}^{q;\psi}u(0)=0, \\ a_1\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=0} + b_1\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=T} = v_1, \\ a_2{}^cD_{0^+}^{\delta;\psi}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\xi} + b_2{}^cD_{0^+}^{\delta;\psi}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=T} = v_2\ ;\ \xi\in J, \end{cases}$$

where $J=[0,T]$, $D_{0^+}^p$, $D_{0^+}^\gamma$ denotes the ψ-Caputo fractional derivative of order p,γ respectively and $2<p\le 3$, $0<\gamma\le 1$, $\gamma\in\{q,\delta\}$, $I_{0^+}^{\eta_i;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\eta_i>0$, $h\in C(J\times\mathbb{R},\mathbb{R}\setminus 0)$, $f\in C(J\times\mathbb{R}^2,\mathbb{R})$ and $g_i\in C(J\times\mathbb{R},\mathbb{R})$ with $g_i(0,0)=0$; $i=1\cdots m$. a_1,a_2,b_1,b_2,v_1,v_2 are real constants such that $b_1\neq 0$ and

$$2\big(a_2\Psi_0^{2-\delta}(\xi)+b_2\Psi_0^{2-\delta}(T)\big)-\Psi_0^1(T)(2-\delta)\big(a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)\big)\neq 0.$$

In **Chapter 3**, we study the existence, uniqueness, and Ulam-Hyers stability of solutions for the following ψ-Caputo hybrid fractional sequential integro-differential equation

$$L_\psi^p\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right] = f\big(t,u(t),\delta I_{0^+}^{\gamma;\psi}u(t)\big);\quad t\in J=[0,T],$$

endowed with the hybrid fractional integral boundary conditions

$$\begin{cases} u(0)=0, \quad {}^cD_{0^+}^{q;\psi}u(0)=0, \\ \left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=T} = \rho\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\xi}; \quad 0<\rho,\xi<T, \end{cases}$$

with

$$L_\psi^p = {}^cD_{0^+}^{p;\psi}+\lambda{}^cD_{0^+}^{p-1;\psi},$$

where $1<p\le 2$, $0<q\le 1$, ${}^cD_{0^+}^{p;\psi}$, ${}^cD_{0^+}^{q;\psi}$ denote the ψ-Caputo fractional derivative of order p,q, respectively and $I_{0^+}^{\eta_i;\psi}$, $I_{0^+}^{\gamma;\psi}$ are the ψ-Riemann-Liouville fractional integral of order $\eta_i>0$ and $\gamma>0$ respectively, $h\in C(J\times\mathbb{R},\mathbb{R}\setminus 0)$, $f\in C(J\times\mathbb{R}^2,\mathbb{R})$, $g_i\in C(J\times\mathbb{R},\mathbb{R})$ with $g_i(0,0)=0(i=1,...,m)$, and λ is appropriate positive real constants.

Chapter 4 is devoted to the existence and uniqueness results of the random coupled system of Hilfer-Katugampola fractional derivative given by

$$\begin{cases} ({}^{\rho}D_{a^+}^{p_1,q_1}u)(t,\vartheta) = f(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \\ \\ ({}^{\rho}D_{a^+}^{p_2,q_2}v)(t,\vartheta) = g(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \end{cases} ;t \in J = [a,T], \vartheta \in \Omega,$$

with the following initial conditions:

$$\begin{cases} ({}^{\rho}I_{a^+}^{1-\gamma_1}u)(a,\vartheta) = u_a(\vartheta) \\ \\ ({}^{\rho}I_{a^+}^{1-\gamma_2}v)(a,\vartheta) = v_a(\vartheta) \end{cases} ;\vartheta \in \Omega,$$

where $0 \leq a < T < \infty$, $0 < p_i < 1$, $0 \leq q_i \leq 1$ and $\gamma_i = p_i + q_i(1 - p_i)$; $i = 1,2$. $(\Omega, \mathcal{A})$ is measurable space, $u_a, v_a : \Omega \to \mathbb{R}^n$ are a measurable function, $f, g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ are given functions, ${}^{\rho}D_{a^+}^{p_i,q_i}$ is Hilfer-Katugampola fractional derivative of $p_i(0 \leq p_i \leq 1)$; $i = 1,2$ and type $q_i(0 \leq q_i \leq 1)$; $i = 1,2$ and ${}^{\rho}I_{a^+}^{1-\gamma_i}$ is generalized fractional integral of order $1 - \gamma_i(\gamma_i = p_i + q_i - p_iq_i)$.

In **Chapter 5** we investigate the existence, uniqueness and Ulam–Hyers stability results to the following nonlinear random multi-fractional equations

$$\begin{cases} {}^{c}D_{0^+}^{p_1;\psi}[{}^{c}D_{0^+}^{q_1;\psi}u(t,\vartheta) - h(t,u_t(\vartheta),v_t(\vartheta),\vartheta)] = f(t,u_t(\vartheta),v_t(\vartheta),{}^{c}D_{0^+}^{\delta_1;\psi}v(t,\vartheta),\vartheta); \\ \\ {}^{c}D_{0^+}^{p_2;\psi}[{}^{c}D_{0^+}^{q_2;\psi}v(t,\vartheta) - k(t,u_t(\vartheta),v_t(\vartheta),\vartheta)] = g(t,u_t(\vartheta),{}^{c}D_{0^+}^{\delta_2;\psi}u(t,\vartheta),v_t(\vartheta),\vartheta); \\ t \in J = [0,T], \vartheta \in \Omega, \end{cases}$$

subject to the following coupled non-local integral and boundary condition

$$\begin{cases} u(0,\vartheta) = \chi(v(\vartheta)), \quad D_{\psi}u(0,\vartheta) = 0, \quad \int_0^T v(\tau,\vartheta)d\tau = \kappa_1 u(\xi,\vartheta) \\ \\ v(0,\vartheta) = \varphi(u(\vartheta)), \quad D_{\psi}v(0,\vartheta) = 0, \quad \int_0^T u(\tau,\vartheta)d\tau = \kappa_2 v(\varrho,\vartheta) \end{cases} ;\xi,\varrho \in J,$$

where $D_{0^+}^{p_i;\psi}$ and $D_{0^+}^{\sigma_i;\psi}$ are the $\psi-$Caputo derivative of order $0 < p_i \leq 2$ and $0 < \sigma_i \leq 1$ respectively, $\sigma_i = q_i, \delta_i (i = 1,2)$. The integer order of the differential operator D_{ψ} is defined by

$$D_\psi = \frac{1}{\psi'(t)}\frac{d}{dt},$$

$(\Omega, \mathcal{A})$ is measurable space, $f, g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $h, k : J \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ are given functions , $\varphi, \chi : \mathbb{R}^n \to \mathbb{R}^n$ are given continuous function, and κ_i are real constants ; $i = 1, 2$.

Chapter 6 treats the existence and uniqueness of the following nonlinear random coupled system of ψ-Caputo fractional integro-differential equations

$$\begin{cases} {}^cD_{a^+}^{p_1;\psi} u(t,\vartheta) + \sum_{i=1}^m I_{a^+}^{\gamma_{1,i};\psi} g_{1,i}(t, u_t(\vartheta), v_t(\vartheta), \vartheta) = f_1(t, u_t(\vartheta), v_t(\vartheta), \vartheta) \\ \\ {}^cD_{a^+}^{p_2;\psi} v(t,\vartheta) + \sum_{i=1}^m I_{a^+}^{\gamma_{2,i};\psi} g_{2,i}(t, u_t(\vartheta), v_t(\vartheta), \vartheta) = f_2(t, u_t(\vartheta), v_t(\vartheta), \vartheta) \end{cases} ; t \in J, \vartheta \in \Omega.$$

$$\begin{cases} (u(t,\vartheta), v(t,\vartheta)) = (\eta_1(t,\vartheta), \eta_2(t,\vartheta)), \quad t \in [a-r, a], r > 0, \\ \\ (u(t,\vartheta), v(t,\vartheta)) = (\xi_1(t,\vartheta), \xi_2(t,\vartheta)), \quad t \in [T, T+l], l > 0, \end{cases} ; \vartheta \in \Omega,$$

where $J = [a, T]$, $D_{a^+}^{p_j;\psi}$ denotes the ψ-Caputo fractional derivative of order $1 < p_j \leq 2$, $I_{a^+}^{\gamma_{i,j};\psi}$ is the ψ-Riemann-Liouville fractional integral of orders $\gamma_{i,j} > 0$, $(\Omega, \mathcal{A})$ is measurable space, $f_i, g_{i,j} : J \times C([-r, l], \mathbb{R}^n) \times C([-r, l], \mathbb{R}^n) \times \Omega \to \mathbb{R}^n$ are given functions, $\eta_j \in C([a-r, a], \mathbb{R}^n)$ with $\eta_j(a, \vartheta) = 0$ and $\xi \in C([T, T+l], \mathbb{R}^n)$ with $\xi(T, \vartheta) = 0$; $j = 1, 2, i = 1, \cdots, m$.
We denote by $u_t(s)$ the element of $C([-r, l])$ defined by

$$u_t(s) = u(t+s), \quad s \in [-r, l].$$

CHAPTER 1

PRELIMINARIES AND BACKGROUND MATERIALS

In this chapter, we recall several definition, basic concepts, notation and elementary results that are used throughout this thesis.

1.1 Functional spaces

Let $J = [a, T] \subset \mathbb{R}$, set $C(J)$ be the space of real valued continuous functions on J endowed with the norm

$$\|f\|_{\infty} = \sup\{|f(t)| : t \in J\}.$$

$X_c^p(a, T)$ is Banach space, $(c \in \mathbb{R}, 1 \leq p < \infty)$ of complex-valued Lebesgue measurable functions f on J, equipped with the norm

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}.$$

Note that when $c = \frac{1}{p}$, the space $X_c^p(a, T)$ coincides with the $L^p(a, T)$ space, i. e

$$X_{\frac{1}{p}}^p(a, T) = L^p(a, T).$$

By $L^\infty(a,T)$, we denote the space of measurable function $f : J \to \mathbb{R}$ which are essentially bounded equipped with norm

$$\|f\|_{L^\infty} = \inf\{c > 0 : |f(t) \leq c, \text{ a.e. } t \in J\}.$$

1.2 Fractional Calculus Theory

In this section, we recall some definitions and properties of different fractional integrals and fractional differential operators that we will use throughout this thesis.

1.2.1 Hilfer-Katugampola fractional derivative

Definition 1.2.1. [32](Katugampola fractional integral)
The Katugampola fractional integral ${}^\rho I^p_{a_+}$ of order $p > 0$ left-sided is defined by

$$({}^\rho I^p_{a^+} f)(t) = \frac{1}{\Gamma(p)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p-1} s^{\rho-1} f(s) ds, \qquad t > a, \rho > 0. \tag{1.1}$$

where $\Gamma(.)$ is the Euler gamma function defined by

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt, \qquad p > 0.$$

Definition 1.2.2. [32](Katugampola fractional derivative) The Katugampola fractional derivative ${}^\rho D^p_{a_+}$ of order $p > 0$ left-sided is defined by

$$\begin{aligned} ({}^\rho D^p_{a^+} f)(t) &= \left(t^{1-\rho} \frac{d}{dt}\right)^n ({}^\rho I^{n-p}_{a^+} f)(t) \\ &= \frac{1}{\Gamma(n-p)} \left(t^{1-\rho} \frac{d}{dt}\right) \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-p-1} s^{\rho-1} f(s) ds, \quad t > a \end{aligned} \tag{1.2}$$

where $n = [p] + 1$ and $[p]$ means the integer part of p.

Lemma 1.2.3. *[38] Let $x > a$, ${}^\rho I^p_{a^+}$ and ${}^\rho D^p_{a^+}$, according to (1.1) and (1.2) respectively, we have:*

$$\left[{}^\rho I^p_{a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu-1}\right](x) = \frac{\Gamma(\mu)}{\Gamma(p+\mu)} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{p+\mu-1}, \quad p \geq 0,\ \mu > 0,$$

$$\left[{}^\rho D^p_{a^+} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu-1}\right](x) = 0 \qquad 0 < p < 1.$$

Lemma 1.2.4. *[38] Let $p, \beta > 0$, $1 \leq p \leq \infty$, $0 < a < T < \infty$, $\rho, c \in \mathbb{R}$ and $\rho \geq c$. Then, for $f \in X_c^p(a,T)$ the semigroup property is valid. That is*

$$\left({}^{\rho}I_{a^+}^{p}\, {}^{\rho}I_{a^+}^{\beta} f\right)(u) = \left({}^{\rho}I_{a^+}^{p+\beta} f\right)(u).$$

Lemma 1.2.5. *[38] Let $0 < p < 1$, $0 \leq \gamma < 1$. If $\varphi \in C_\gamma$ and ${}^{\rho}I_{a^+}^{p}\varphi \in C_\gamma^1(J)$, then*

$$\left({}^{\rho}I_{a^+}^{p}\, {}^{\rho}D_{a^+}^{p}\varphi\right)(u) = \varphi(u) - \frac{\left({}^{\rho}I_{a^+}^{1-p}\varphi\right)(a)}{\Gamma(p)} \left(\frac{u^\rho - a^\rho}{\rho}\right)^{p-1}, \tag{1.3}$$

for all $u \in (a, T]$.

Definition 1.2.6. [38] the Hilfer-Katugampola fractional derivative left-sided, with respect of order $0 < p < 1$ and type $0 \leq \beta \leq 1$, with $\rho > 0$ is defined by

$$\left({}^{\rho}D_{a^+}^{p,\beta} f\right)(t) = \left({}^{\rho}I_{a^+}^{\beta(1-p)} \left(t^{1-\rho}\frac{d}{dt}\right) {}^{\rho}I_{a^+}^{(1-\beta)(1-p)} f\right)(t).$$

property 1. [38]The operator ${}^{\rho}D_{a^+}^{p,\beta}$ can be written as

$${}^{\rho}D_{a^+}^{p,\beta} = {}^{\rho}I_{a^+}^{\beta(1-p)}\delta_\rho {}^{\rho}I_{a^+}^{1-\gamma} = {}^{\rho}I_{a^+}^{\beta(1-p)}\, {}^{\rho}D_{a^+}^{\gamma}; \qquad \gamma = p + \beta(1-p).$$

where $\delta_\rho = \left(t^{\rho-1}\frac{d}{dt}\right)$.

property 2. The fractional derivative ${}^{\rho}D_{a^+}^{p,\beta}$ is considered as interpolation with conve-nient parametersof the following fractional derivative:

1. Hilfer fractional derivative when $\rho \to 1$ [25].
2. Hilfer-Hadamard fractional derivative when $\rho \to 0$ [31].
3. Generalized fractional derivative (Katugampola derivative) when $\beta = 0$ [32].
4. Caputo-type fractional derivative when $\beta = 1$ [33].
5. Riemann-liouville fractional derivative when $\beta = 0$, $\rho \to 1$ [33].
6. Hadamard fractional derivative when $\beta = 0$, $\rho \to 0$ [33].
7. Caputo fractional derivative when $\beta = 1$, $\rho \to 1$ [33].

8. Caputo-Hadamard fractional derivative when $\beta = 1$, $\rho \to 0$ [33].

9. Liouville fractional derivative when $\beta = 0$, $\rho \to 1$, $a = 0$ [33].

10. Weyl fractional derivative when $\beta = 0$, $\rho \to 1$, $a = -\infty$ [26].

1.2.2 the ψ-Caputo derivative

Let an increasing function $\psi : J \to \mathbb{R}$ satisfy $\psi'(t) \neq 0$ for all $t \in J$. For effortlessness, we set $\Psi^k(t,s) := \psi'(s)(\psi(t) - \psi(s))^k$ and $\Psi^k_a(t) = (\psi(t) - \psi(a))^k$.

Definition 1.2.7. [8] The ψ-Riemann–Liouville fractional integral of a function $f : [a,T] \to \mathbb{R}$ is defined by

$$I^{p;\psi}_{a^+} f(t) = \frac{1}{\Gamma(p)} \int_a^t \Psi^{p-1}(t,s) f(s) ds, \qquad 0 < a < s < t.$$

Example 1.2.8. We set $f(t) = \Psi^q_a(t)$, with $q > 0$,

$$I^{p;\psi}_{a^+} f(t) = \frac{\Gamma(q+1)}{\Gamma(p+q+1)} \Psi^{p+q}_a(t).$$

Definition 1.2.9. [8] The ψ-Caputo fractional derivative of order $p > 0$ for a function $f \in C^n[0,\infty)$ is defined by

$${}^cD^{p;\psi}_{a^+} f(t) = \frac{1}{\Gamma(n-p)} \int_a^t \Psi^{n-p-1}(t,s) D^n_\psi f(s) ds, \qquad 0 < a < s < t,$$

where $n = [p] + 1$ and $D^n_\psi = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n$.

Lemma 1.2.10. *[8] Let $p > 0$. The following holds*

- *If $f \in C(J, \mathbb{R})$, then*

$${}^cD^{p;\psi}_{a^+} I^{p;\psi}_{a^+} f(t) = f(t), \quad t \in J.$$

- *If $f \in C^n(J, \mathbb{R})$, $n - 1 < p < n$, then*

$$I^{p;\psi}_{a^+} {}^cD^{p;\psi}_{a^+} f(t) = f(t) - \sum_{k=0}^{n-1} c_k \Psi^k_a(t), \quad t \in J,$$

where $c_k = \frac{D_\psi^k f(a)}{k!}$.

1.3 Random operators

The purpose of this section is to present some definitions and notions regarding random operator that are essential for the study of the problems in this thesis.

We shall denote by $(\Omega, \mathcal{E})$ a measurable space, where Ω is nonempty set of $\mathbb{R}^n$, $\mathcal{E}$ is a σ-algebra of the Borel subsets of $\mathbb{R}^n$.

Definition 1.3.1. Let $(\Omega, \mathcal{E})$ and $(\Omega, \mathcal{G})$ be two measurable spaces. A mapping $F : (\Omega, \mathcal{E}) \to (\Omega, \mathcal{G})$ is said to be measurable if the σ-algebra of Borel $Q^{-1}(\mathcal{G}) \subset \mathcal{E}$ i.e.

$$Q^{-1}(G) \subset \mathcal{E} \qquad \text{for all } G \subset \mathcal{G}.$$

Definition 1.3.2. [46] A function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is called jointly measurable if $f(\cdot, v)$ is measurable for all $v \in \mathbb{R}^n$ and $f(u, .)$ is continuous for all $u \in \Omega$.

Definition 1.3.3. [46] A function $f : J \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory if the following conditions are satisfied:

(i) $(t, u) \to f(t, u, v)$ is jointly measurable for any $v \in \mathbb{R}^n$, and

(ii) $v \to f(t, u, v)$ is continuous for any $t \in J$ and $u \in \Omega$.

Definition 1.3.4. [22] Let X be a metric space. A mapping $Q : \Omega \times X \to X$ is Said to be random operator if, for any $u \in X$, $Q(., u)$ is measurable.

Definition 1.3.5. A random operator $Q(\vartheta)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $Q(\vartheta, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $\vartheta \in \Omega$

Definition 1.3.6. [22] A random fixed point of a random operator Q is a measurable function $f : \Omega \to X$ such that

$$f(u) = Q(u, f(u)), \qquad \text{for all } u \in \Omega.$$

1.4 Generalized Banach space

In this section, we introduce some definitions, notions and notations on generalized metrics spaces in the sense of Perov. Then we give definition of generalized Banach space. We are also interested in the study of some properties of a convergent matrix.

1.4.1 Generalized metric space

First, we give some notions in $\mathbb{R}^n$. Let $u, v \in \mathbb{R}^n$ with $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$, then

- By $u \leq v$ we mean $u_i \leq v_i$ for all $i = 1, \cdots, n$.
- The set $\mathbb{R}^n_+$ is defined by $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n : u_i > 0, \forall i, 1 \leq i \leq n\}$.
- If $c \in \mathbb{R}$, then $u \leq c$ means $u_i \leq c$ for each $i = 1, \cdots, n$.
- The absolute value of u is $|u| = (|u_1|, \cdots, |u_n|)$.
- The maximum of u is $\max(u, v) = (\max(u_1, v_1), \cdots, \max(u_n, v_n))$.

Definition 1.4.1. [46] Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^n_+$ which satisfies the following properties

(i) $d(u, v) \geq 0$ for all $u, v \in X$ if $d(u, v) = 0$, then $u = v$;

(ii) $d(u, v) = d(u, v)$ for all $u, v \in X$;

(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Definition 1.4.2. We call the pair (X, d) a generalized metric space, if and only if d_i; $i = 1, ..., n$, are metrics on X, with

$$d(u, v) = \begin{pmatrix} d_1(u, v) \\ \cdot \\ \cdot \\ \cdot \\ d_n(u, v) \end{pmatrix}.$$

The mapping d is called a generalized metric in the Perov's sense.

Definition 1.4.3. Let (X,d) be a generalized metric space. Let $u_0 \in X$ and $r = (r_1, \cdots, r_n) \in \mathbb{R}^n_+$, we denote by

$$B(u_0,r) = \{u \in X : d(u_0,u) < r\} = \{u \in X : d_i(u_0,u) < r_i,\ i = 1, \cdots, n\}$$

the open ball centered at u_0 with radius r, and

$$\overline{B}(u_0,r) = \{u \in X : d(u_0,u) \leq r\} = \{u \in X : d_i(u_0,u) \leq r_i,\ i = 1, \cdots, n\}$$

the closed ball centered at u_0 with radius r.

Remark 1.4.4. We mention that for a generalized metric space the notions of the open set, closed set, convergence, Cauchy sequence and completeness are similar to those in the usual metric space.

Definition 1.4.5. The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by the norm $\|\cdot\|$ (i.e., $d(u,v) := \|u-v\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space.

1.4.2 Convergent matrix

Definition 1.4.6. [51] A matrix $M = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less then one. In other words this means that all the eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$; where I denotes the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.

Theorem 1.4.7. *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R})$. The following assertions are equivalent:*

1. *The matrix M converge to zero.*
2. $M^k \to 0$ *as* $k \to 0$.
3. *The matrix $(I - M)$ is nonsingular and*

$$(I-M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots.$$

4. *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 1.4.8. Let $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a nonsingular matrix. We say the matrix A has the absolute value property if $A^{-1}|A| \leq I$, where $|A| = (|a_{ij}|)_{1 \leq i,j \leq n}$.

Example 1.4.9. Some examples of matrices $A \in \mathcal{A}_{n\times n}(\mathbb{R})$ convergent to zero which also the property $(I-A)^{-1}|I-A| \leq I$ are as follows:

1. $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a,b \in \mathbb{R}_+$ and $max(a,b) < 1$.
2. $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a,b,c \in \mathbb{R}_+$ and $a+b<1, c<1$.
3. $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a,b,c \in \mathbb{R}_+$ and $|a-b|<1, a>1, b>0$.

Definition 1.4.10. A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ is said to be order preserving(or positive) if $l_0 \leq l_1$ and $k_0 \leq k_1$ imply

$$M\begin{pmatrix} l_0 \\ k_0 \end{pmatrix} \leq M\begin{pmatrix} l_1 \\ k_1 \end{pmatrix}$$

in the sense of components.

Lemma 1.4.11. *Let*

$$M = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

where $a,b,c,d \geq 0$ and $\det M > 0$. Then M^{-1} is order preserving.

Definition 1.4.12. [22] Let (X,d) be a generalized metric space. An operator $Q: X \to X$ is said to be contractive if there exist a matrix M convergent to zero such that

$$d(Q(u),Q(v)) \leq Md(u,v) \qquad \text{for all } u,v \in X.$$

1.5 Fixed points theory

1.5.1 Fixed Point Theorems in Banach Spaces

Theorem 1.5.1. *[47] Let S be a non-empty closed convex subset of a Banach space E, then any contraction mapping Q of S into itself has a unique fixed point.*

Theorem 1.5.2. *[29](Itoh) Let X be a nonempty, closed convex bounded subset of the separable Banach space E and let $Q: \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $Q(\vartheta)u = u$ has a random solution.*

1.5.2 Fixed Point Theorems in Banach Algebra

Let $E = C(J, R)$ be the Banach space of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and a multiplication in E by $\|u\| = \sup_{t \in J} |u(t)|$ and $(uv)(t) = u(t)v(t)$, for all $t \in J$. Clearly E is a Banach algebra with above defined supremum norm and multiplication in it.

Lemma 1.5.3. *[16](Dhage) Let S be a nonempty, convex, closed and bounded set such that $S \subseteq E$, and let $A : E \to E$ and $B : S \to E$ be two operators which satisfy the following:*

(1) A is contraction,

(2) B is completely continuous, and

(3) $u = Au + Bv$, for all $v \in S \Rightarrow u \in S$.

Then there exists a solution of the operator equation $u = Au + Bu$.

1.5.3 Random Fixed Point Theorems in Generalized Banach Spaces

Theorem 1.5.4. *[46] Let $(\Omega, \mathcal{B})$ be a measurable space, let X be a real separable generalized Banach space and let $Q : \Omega \times X \to X$ be a continuous random operator. Let $M(\vartheta) \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ be a random variable matrix such that for every $\vartheta \in \Omega$, the matrix $M(\vartheta)$ converges to 0 and*

$$d(Q(\vartheta, u_1), Q(\vartheta, u_2)) \leq M(\vartheta) d(u_1, u_2)$$

for each $u_1, u_2 \in X$, $\vartheta \in \Omega$.
Then there exists a random variable $u : \Omega \to X$ which is the unique random fixed point of Q.

Theorem 1.5.5. *[46, 22, 40] Let $(\Omega, \mathcal{B})$ be a measurable space, let X be a separable generalized Banach space and let $F : \Omega \times X \to X$ be a completely continuous random operator, then either of the following holds:*

(i) the random equation $Q(x, u) = u$ has a random solution i.e., there is a measurable function $u : \Omega \to X$ such that

$$Q(x, u(x)) = u(x) \qquad \text{for all } x \in \Omega.$$

(ii) the set $H = \{u : \Omega \to X \text{ is measurable} |\ \sigma(x) Q(x, u) = x\}$ with $0 < \sigma(x) < 1$ is unbounded on Ω.

Theorem 1.5.6. *[46, 22] Let X be generalized Banach space, let K be a separable closed convex subset of X, let $Q : \Omega \times K \to K$ be a continuous random operator. Suppose that $Q(u, K)$ is compact for every $u \in \Omega$. Then Q has a random fixed point $u : \Omega \to K$.*

Lemma 1.5.7. *[46] Let X be a separable generalized Banach space. Suppose that $A, B : \Omega \times X \to X$ are random operators such that:*

1. *A is a continuous random and $M(\vartheta)$-contraction operator,*
2. *B is a completely continuous random operator,*
3. *the matrix $I - M$ has the absolute value property. if*

$$\mathcal{N} = \left\{ u : \Omega \to X \text{ is measurable } \middle| \mu(\vartheta)A(u,\vartheta) + \mu(\vartheta)B(\frac{u}{\mu(\vartheta)},\vartheta) = u \right\}$$

is bounded for all measurable mappings $\mu : \Omega \to \mathbb{R}$ with $0 < \mu(\vartheta) < 1$ on Ω, then the random equation

$$u = A(u,\vartheta) + B(u,\vartheta), \qquad u \in X,$$

has at least one solution.

Finally, we give helpful result, we need in the study of our problems.

Lemma 1.5.8. *[4](Gronwall Inequality) Let $p > 0, u(t), v(t)$ be nonnegative functions and $w(t)$ be nonnegative and nondecreasing function for $t \in J, w(t) \leq C$, where C is a constant. If*

$$u(t) \leq v(t) + w(t) \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{p-1} s^{\rho-1} u(s) ds,$$

then

$$u(t) \leq v(t) + w(t) \int_a^t \sum_{k=1}^{\infty} \frac{(w(t)\Gamma(p))^k}{\Gamma(kp)} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{kp-1} s^{\rho-1} v(s) ds, \quad t \in J.$$

CHAPTER 2

HYBRID IMPLICIT MULTI-FRACTIONAL DIFFERENTIAL EQUATION

2.1 Introduction

In this chapter, we study the existence and uniqueness results of a fractional hybrid boundary value problem with multiple fractional derivatives of $\psi-$Caputo with different orders. Using a useful generalization of Krasnoselskii's fixed point theorem, we have established results of at least one solution, while the uniqueness of the solution is derived by Banach's fixed point. Nowadays, many researchers have shown interest in quadratic perturbations of nonlinear differential equations. Some recent works regarding hybrid differential equations can be found in [37, 56, 18, 45] and the references cited therein. Dhage and Lakshmikantham [17] discussed the existence and uniqueness theorems of the solution to the ordinary first-order hybrid differential equation with perturbation of the first type

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{u(t)}{g(t,u(t)}\right) = f(t,u(t)); \qquad \text{a.e } t \in [t_0, t_0+T], \\ \\ u(t_0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases}$$

where $t_0, T \in \mathbb{R}$ with $T > 0$, $g : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R} \setminus 0$ and $f : [t_0, t_0+T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Using fixed point theorem in Banach algebra, the authors obtained the existence results.

In [20], Dong et al, established the existence and the uniqueness of solutions for the following implicit fractional differential equation

$$\begin{cases} {}^cD^p u(t) = f(t, u(t), {}^cD^p u(t)); \quad t \in [0,T], \quad 0 < p \leq 1, \\ \\ u(0) = u_0, \end{cases}$$

where ${}^cD^p$ is the Caputo fractional derivative, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Sitho et al. [45] studied existence results for the initial value problems of hybrid fractional sequential integro-differential equations:

$$\begin{cases} D^p \left[\dfrac{D^q u(t) - \sum_{i=1}^{n} I^{\eta_i} g_i(t, u(t))}{h(t, x(t))} \right] = f(t, u(t), I^\gamma x(t)); \quad t \in J, \\ \\ u(0) = 0, \qquad D^q u(0) = 0, \end{cases}$$

where D^p, D^q denotes the Riemann-Liouville fractional derivative of order p, q respectively and $0 < p, q \leq 1$, I^{η_i} is the Riemann-Liouville fractional integral of order $\eta_i > 0$, $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus 0)$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ and $g_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $g_i(0,0) = 0; i = 1 \cdots m$.

In 2019, Derbazi et al [18]. proved the existence of solutions for the fractional hybrid boundary value problem

$${}^cD^p \left[\frac{u(t) - g(t, u(t))}{h(t, u(t))} \right] = f(t, u(t)); \quad t \in J,$$

with the fractional hybrid boundary value conditions

$$\begin{cases} a_1 \left[\dfrac{u(t) - g(t, u(t))}{h(t, u(t))} \right]_{t=0} + b_1 \left[\dfrac{u(t) - g(t, u(t))}{h(t, u(t))} \right]_{t=T} = v_1, \\ \\ a_2^c D^\delta \left[\dfrac{u(t) - g(t, u(t))}{h(t, u(t))} \right]_{t=\xi} + b_2^c D^\delta \left[\dfrac{u(t) - g(t, u(t))}{h(t, u(t))} \right]_{t=T} = v_2 \, ; \xi \in J, \end{cases}$$

where $1 < p \leq 2$, $0 < \delta \leq 1$, $\xi \in J$ and $a_1, a_2, b_1, b_2, v_1, v_2$ are real constants. Moreover, two fractional derivatives appeared in the above problem are of Caputo type.

The above findings motivated us to investigate the existence and uniqueness of solutions for a class of hybrid differential equations of arbitrary fractional order of the form[1]

$$ {}^{c}D_{0^+}^{p;\psi}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]=f\left(t,u(t),{}^{c}D_{0^+}^{p;\psi}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]\right);\ t\in J, \tag{2.1}$$

endowed with the hybrid fractional integral boundary conditions

$$\begin{cases} u(0)=0, \quad {}^{c}D^{q;\psi}u(0)=0, \\ a_1\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=0}+b_1\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=T}=v_1, \\ a_2{}^{c}D_{0^+}^{\delta;\psi}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\xi}+b_2{}^{c}D_{0^+}^{\delta;\psi}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=T}=v_2\ ;\ \xi\in J, \end{cases} \tag{2.2}$$

where $J=[0,T]$, $D_{0^+}^{p}$, $D_{0^+}^{\gamma}$ denotes the ψ-Caputo fractional derivative of order p,γ respectively and $2<p\leq 3$, $0<\gamma\leq 1$, $\gamma\in\{q,\delta\}$, $I_{0^+}^{\eta_i;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\eta_i>0$, $h\in C(J\times\mathbb{R},\mathbb{R}\setminus 0)$, $f\in C(J\times\mathbb{R}^2,\mathbb{R})$ and $g_i\in C(J\times\mathbb{R},\mathbb{R})$ with $g_i(0,0)=0$; $i=1,\cdots,m$. a_1,a_2,b_1,b_2,v_1,v_2 are real constants such that $b_1\neq 0$ and

$$2\left(a_2\Psi_0^{2-\delta}(\xi)+b_2\Psi_0^{2-\delta}(T)\right)-\Psi_0^{1}(T)(2-\delta)\left(a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)\right)\neq 0.$$

Main result

Lemma 2.1.1. *Let $2<p<3$, $0<q<1$. For any functions $F\in C(J,\mathbb{R})$, $H\in C(J,\mathbb{R}\setminus 0)$ and $G_i\in C(J,\mathbb{R})$ with $G_i(0)=0$; $i=1,...,m$, the following linear fractional boundary value problem*

$$D_{0^+}^{p;\psi}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]=F(t);\quad 2<p\leq 3,\ 0<q\leq 1,\ t\in J, \tag{2.3}$$

[1]**F. Fredj**, H. Hammouche, :On existence results for hybrid ψ-Caputo multi-fractional differential equations with hybrid conditions(accepted).

supplemented with the following conditions

$$\begin{cases} u(0)=0, \quad {}^cD_{0^+}^{q;\psi}u(0)=0, \\ a_1\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=0}+b_1\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=T}=v_1, \\ a_2{}^cD_{0^+}^{\delta;\psi}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=\xi}+b_2{}^cD_{0^+}^{\delta;\psi}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=T}=v_2\,;\xi\in J, \end{cases} \tag{2.4}$$

has a unique solution, which is given by

$$\begin{aligned} u(t) &= I_{0^+}^{q;\psi}\Big(H(s)I_{0^+}^{p;\psi}F(s)\Big)(t)+\sum_{i=1}^m I_{0^+}^{\eta_i+q;\psi}G_i(s)(t) \\ &+ I_{0^+}^{q;\psi}\left(H(s)\Big(\Psi_0^1(s)\Omega_3-\Psi_0^2(s)\Omega_2\Big)\left(\frac{v_1}{b_1}-I_{0^+}^{p;\psi}F(s)\right)\right)(t) \\ &+ \Omega_1\Big(v_2-a_2I_{0^+}^{p-\delta;\psi}F(\xi)-b_2I_{0^+}^{p-\delta;\psi}F(T)\Big)I_{0^+}^{q;\psi}\left(H(s)\Big(\Psi_0^2(s)-\Psi_0^1(T)\Psi_0^1(s)\Big)\right)(t), \end{aligned} \tag{2.5}$$

where

$$\Omega_1=\frac{\Gamma(3-\delta)}{2\big(a_2\Psi_0^{2-\delta}(\xi)+b_2\Psi_0^{2-\delta}(T)\big)-\Psi_0^1(T)(2-\delta)\big(a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)\big)},$$

$$\Omega_2=\frac{a_2\Psi_0^{1-\delta}(\xi)+b_2\Psi_0^{1-\delta}(T)}{\Gamma(2-\delta)\Omega_1}, \quad \Omega_3=1+\Omega_2\Psi_0^1(T).$$

Proof. Applying the ψ-Caputo fractional integral of order p to both sides of equation in (2.3) and using Lemma 1.2.10, we get

$$\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}=I_{0^+}^{p;\psi}F(t)+c_0+c_1\Psi_0^1(t)+c_2\Psi_0^2(t), \tag{2.6}$$

where $c_0,c_1,c_2\in\mathbb{R}$.

Next, applying ψ-Caputo fractional integral of order q to both sides (2.6), we get

$$\begin{aligned} u(t) &= I_{0^+}^{q;\psi}\Big(H(s)I_{0^+}^{p;\psi}F(s)\Big)(t) + \sum_{i=1}^{m} I_{0^+}^{\eta_i+q;\psi}G_i(s)(t) \\ &\quad + I_{0^+}^{q;\psi}\Big(H(s)\Big(c_0 + c_1\Psi_0^1(s) + c_2\Psi_0^2(s)\Big)\Big)(t) + c_3; \qquad c_3 \in \mathbb{R}. \end{aligned} \tag{2.7}$$

With the help of conditions $u(0) = 0$ and ${}^cD^{q;\psi}u(0) = 0$, we find, $c_3 = 0$ and $c_0 = 0$ respectively. Applying the boundary conditions (2.4), and from (2.6), we obtain

$$c_1\Psi_0^1(T) + c_2\Psi_0^2(T) = \frac{v_1}{b_1} - I_{0^+}^{p;\psi}F(T),$$

and

$$\begin{aligned} &\frac{c_1}{\Gamma(2-\delta)}\Big(a_2\Psi_0^{1-\delta}(\xi) + b_2\Psi_0^{1-\delta}(T)\Big) + \frac{2c_2}{\Gamma(3-\delta)}\Big(a_2\Psi_0^{2-\delta}(\xi) - b_2\Psi_0^{2-\delta}(T)\Big) \\ &\qquad = v_2 - a_2 I_{0^+}^{p-\delta;\psi}F(\xi) - b_2 I_{0^+}^{p-\delta;\psi}F(T). \end{aligned}$$

Solving the resulting equations for c_1 and c_2, we find that

$$c_1 = \Big(\frac{v_1}{b_1} - I_{0^+}^{p;\psi}F(T)\Big)\Omega_3 - \Big(v_2 - a_2 I_{0^+}^{p-\delta;\psi}F(\xi) - b_2 I_{0^+}^{p-\delta;\psi}F(T)\Big)\Omega_1\Psi_0^1(T),$$

$$c_2 = \Big(v_2 - a_2 I^{p-\delta;\psi}F(\xi) - b_2 I_{0^+}^{p-\delta;\psi}F(T)\Big)\Omega_1 - \Big(\frac{v_1}{b_1} - I_{0^+}^{p;\psi}F(T)\Big)\Omega_2.$$

Inserting c_0, c_1, c_2 and c_3 in (2.7), which leads to the solution of system (2.5) . □

2.2 Existence of solutions

In this subsection, we prove the existence of a solution for the system (2.1)-(2.2) by applying a generalization of Krasnoselskii's fixed point theorem.

Theorem 2.2.1. *Assume that the following hypotheses hold*

(H1) The functions $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus 0)$ and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ are continuous, there exist bounded functions $L, M : J \to [0, \infty)$, such that

$$|h(t, u(t)) - h(t, \overline{u}(t))| \leq L(t)|u(t) - \overline{u}(t)|,$$

and

$$|f(t,u(t),\overline{u}(t)) - f(t,v(t),\overline{v}(t))| \leq M(t)\big(|u(t)-v(t)| + |\overline{u}(t)-\overline{v}(t)|\big),$$

for $t \in J$ *and* $u,v,\overline{u},\overline{v} \in \mathbb{R}$.

(H2) There exist function $\varphi_i, \chi, \vartheta \in C(J,\mathbb{R})$ *such that*

$$|g_i(t,u(t))| \leq \varphi_i(t) \quad \text{for each } (t,u) \in J \times \mathbb{R},$$

$$|h(t,u(t))| \leq \chi(t) \quad \text{for each } (t,u) \in J \times \mathbb{R},$$

$$|f(t,u(t),\overline{u}(t))| \leq \vartheta(t) \quad \text{for each } (t,u,\overline{u}) \in J \times \mathbb{R}^2.$$

(H3) There exists $0 < \Lambda, M^* < 1$, *where*

$$\begin{aligned}\Lambda =& \frac{\Psi_0^p(T)}{\Gamma(p+1)}\left(\frac{\chi^* M^*}{1-M^*} + \vartheta^* L^*\right)\left(\frac{\Psi_0^q(T)}{\Gamma(q+1)} + \frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right)\\ &+ |\Omega_1|(q+2)\frac{\Psi_0^{q+2}(T)}{\Gamma(q+3)}\Bigg(|v_2|L^* + \frac{|a_2|\Psi_0^{p-\delta}(\xi) + |b_2|\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\\ &\times\left(\frac{\chi^* M^*}{1-M^*} + \vartheta^* L^*\right)\Bigg) + \frac{|v_1|L^*}{|b_1|}\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right).\end{aligned} \tag{2.8}$$

where: $L^* = \sup_{t\in J}|L(t)|$, $M^* = \sup_{t\in J}|M(t)|$, $\chi^* = \sup_{t\in J}|\chi(t)|$, $\vartheta^* = \sup_{t\in J}|\vartheta(t)|$ *and* $\varphi_i^* = \sup_{t\in J}|\varphi_i(t)|$; $i = 1,2,\cdots,m$.

Then the problem (2.1)-(2.2) *has at least one solution on* J.

Proof. First, we choose $r > 0$ so that

$$\begin{aligned}r \geq& \chi^*\vartheta^*\frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^*\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right)\left(\frac{|v_1|}{|b_1|} + \frac{\Psi_0^p(T)}{\Gamma(p+1)}\vartheta^*\right)\\ &+ |\Omega_1|\frac{(2+q)\Psi_0^{q+2}(T)}{\Gamma(q+3)}\left(|v_2| + |a_2|\frac{\Psi_0^{p-\delta}(\xi)}{\Gamma(p-\delta+1)}\vartheta^* + |b_2|\frac{\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\vartheta^*\right)\\ &+ \sum_{i=1}^{n}\frac{\varphi_i^*}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(T).\end{aligned}$$

Now, we set $E = C(J, R)$, and we define $B_r \subset E$ as

$$B_r = \{u \in E : \|u\| \leq r\}.$$

Clearly B_r is a closed, convex and bounded subset of the Banach space E.

Let

$${}^cD_{0^+}^{p;\psi}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t) - \sum_{i=1}^{m} I_{0^+}^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right] = F_u(t)$$

supplemented with the conditions (3.2), then by lemma 3.1.1, we get

$$\begin{aligned} u(t) &= I_{0^+}^{q;\psi}\Big(h(s,u(s))I_{0^+}^{p;\psi}F_u(s)\Big)(t) + \sum_{i=1}^{m} I_{0^+}^{\eta_i+q;\psi} g_i(s,u(s))(t) + \\ &+ I_{0^+}^{q;\psi}\left(h(s,u(s))\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\left(\frac{v_1}{b_1} - I_{0^+}^{p;\psi}F_u(s)\right)\right)(t) \\ &+ \Omega_1\Big(v_2 - a_2 I_{0^+}^{p-\delta;\psi}F_u(\xi) - b_2 I_{0^+}^{p-\delta;\psi}F_u(T)\Big) I_{0^+}^{q;\psi}\left(h(s,u(s))\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\right)(t), \end{aligned}$$

where $F_u(t) = f(t,u(t),F_u(t))$.

Let us define two operators $C_p, C_{p-\delta} : E \to E$ and $D : E \to E$ such that

$$C_p u(t) = \frac{1}{\Gamma(p)}\int_0^t \Psi^{p-1}(t,s)F_u(s)ds, \quad t \in J;$$

and

$$C_{p-\delta}u(t) = \frac{1}{\Gamma(p-\delta)}\int_0^t \Psi^{p-\delta-1}(t,s)F_u(s)ds; \quad t \in J,$$

and

$$Du(t) = h(t,u(t)); \quad t \in J.$$

Then, using assumptions (H1)-(H2) , we have

$$|C_{p-\delta}u(t) - C_{p-\delta}\overline{u}(t)| \leq \frac{1}{\Gamma(p-\delta)}\int_0^t \Psi^{p-\delta-1}(t,s)|F_u(s) - F_{\overline{u}}(s)|ds, \tag{2.9}$$

and

$$\begin{aligned} |F_u(t) - F_{\overline{u}}(t)| &\leq |f(t,u(t),F_u(t)) - f(t,\overline{u}(t),F_{\overline{u}}(t))| \\ &\leq M(t)\big(|u(t) - \overline{u}(t)| + |F_u(t) - F_{\overline{u}}(t)|\big) \\ &\leq \frac{M(t)}{1-M(t)}|u(\cdot) - \overline{u}(\cdot)|. \end{aligned} \tag{2.10}$$

By replacing (2.10) in (2.9) and proceeding with supermum over t, we obtain

$$\|C_{p-\delta}u(t) - C_{p-\delta}\overline{u}(t)\|_\infty \leq \frac{M^*\Psi_0^{p-\delta}(T)}{(1-M^*)\Gamma(p-\delta+1)}\|u(\cdot) - \overline{u}(\cdot)\|_\infty,$$

and

$$\|Du(t) - D\overline{u}(t)\|_\infty \leq L^*\|u(\cdot) - \overline{u}(\cdot)\|_\infty,$$

$$\|C_{p-\delta}u(t)\|_\infty \leq \frac{\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\vartheta^*,$$

and

$$\|Du(t)\|_\infty \leq \chi^*.$$

Now, we define two more operators $A : E \to E$ and $B : B_r \to E$ such that

$$\begin{aligned} Au(t) = I_{0^+}^{q;\psi}\Big(Du(s)C_pu(s)\Big)(t) + I_{0^+}^{q;\psi}\Bigg(Du(s)\bigg(\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\Big(\frac{v_1}{b_1} - C_pu(s)\Big)(t) \\ + \Omega_1\Big(v_2 - a_2C_{p-\delta}u(\xi) - b_2C_{p-\delta}u(T)\Big)I_{0^+}^{q;\psi}\bigg(Du(s)\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\bigg)(t), \end{aligned}$$

and

$$Bu(t) = \sum_{i=1}^{m} I_{0^+}^{\eta_i+q;\psi} g_i(s,u(s))(t).$$

We need to show that the two operators A and B satisfy all the conditions of Lemma 3.2. This can be achieved in the following steps.

Step.1 First we show that A is a contraction mapping. Let $u(t), \overline{u}(t) \in B_r$, then we have

$$\begin{aligned} &|Au(t) - A\overline{u}(t)| \\ &\leq I_{0^+}^{q;\psi}\Big(\big|Du(s)C_pu(s) - D\overline{u}(s)C_p\overline{u}(s)\big|\Big(1 + \big|\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\big|\Big) \end{aligned}$$

$$+ \frac{|v_1|}{|b_1|}\left|\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\right|\left|Du(s) - D\overline{u}(s)\right|\Big)(t)$$

$$+ |\Omega_1| I_{0^+}^{q;\psi}\Big(\left|\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\right|\Big(|v_2|\left|Du(s) - D\overline{u}(s)\right| + |a_2|\big|Du(s)C_{p-\delta}u(\xi)$$

$$- D\overline{u}(s)C_{p-\delta}\overline{u}(\xi)\big| + |b_2|\left|Du(s)C_{p-\delta}u(T) - D\overline{u}(s)C_{p-\delta}\overline{u}(T)\right|\Big)\Big)(t)$$

$$\leq I_{0^+}^{q;\psi}\Big(\Big(\left|Du(s)\right|\left|C_p u(s) - C_p\overline{u}(s)\right| + \left|C_p\overline{u}(s)\right|\left|Du(s) - D\overline{u}(s)\right|\Big)$$

$$\times \Big(1 + \left|\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\right|\Big) + \frac{|v_1|}{|b_1|}\left|\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\right|\left|Du(s) - D\overline{u}(s)\right|\Big)(t)$$

$$+ |\Omega_1| I_{0^+}^{q;\psi}\Big(\left|\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\right|\Big(\left|Du(s) - D\overline{u}(s)\right|\big(|v_2| + |a_2|\left|C_{p-\delta}\overline{u}(\xi)\right| + |b_2|\left|C_{p-\delta}\overline{u}(T)\right|\big)\Big)$$

$$+ \left|Du(s)\right|\Big(|a_2|\left|C_{p-\delta}u(\xi) - C_{p-\delta}\overline{u}(\xi)\right| + |b_2|\left|C_{p-\delta}u(T) - C_{p-\delta}\overline{u}(T)\right|\Big)\Big)(t).$$

Using the hypotheses (H1)-(H2) and taking the supremum over t, we get

$$\|Au(\cdot) - A\overline{u}(\cdot)\|_\infty \leq \Lambda\|u(\cdot) - \overline{u}(\cdot)\|_\infty. \tag{2.11}$$

Hence by (2.8) the operator A is a contraction mapping.

Step 2. Next, we prove that the operator B satisfies condition (2) of Lemma 3.2, that is, the operator B is compact and continuous on B_r. Therefore first, we show that the operator B is continuous on B_r.

Let $u_n(t)$ be a sequence of functions in B_r converging to a function $u(t) \in B_r$. Then, by the Lebesgue dominant convergence theorem, for all $t \in J$, we have

$$\begin{aligned}\lim_{n\to\infty} Bu_n(t) &= \lim_{n\to\infty}\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^t \Psi^{\eta_i+q-1}(t,s)g_i(s,u_n(s))ds\\ &= \sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^t \Psi^{\eta_i+q-1}(t,s)\lim_{n\to\infty} g_i(s,u_n(s))ds\\ &= \sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^t (s)\Psi^{\eta_i+q-1}(t,s)g_i(s,u(s))ds.\end{aligned}$$

Hence $\lim_{n\to\infty} Bu_n(t) = Bu(t)$. Thus B is a continuous operator on B_r. Further, we show that the operator B is uniformly bounded on B_r, for any $u \in B_r$, we have

$$\begin{aligned}\|Bu(t)\|_\infty &\le \sup_{t\in J}\left\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_a^t \Psi^{\eta_i+q-1}(t,s)|g_i(s,u(s))|ds\right\}\\ &\le \sum_{i=1}^{m}\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}\varphi_i^* \le r.\end{aligned}$$

Therefore $Bu(t) \le r$, for all $t \in J$, which shows that B is uniformly bounded on B_r. Now, we show that the operator B is equi-continuous. Let $t_1, t_2 \in J$ with $t_1 > t_2$.
Then for any $u(t) \in B_r$, we have

$$\begin{aligned}&|Bu(t_1) - Bu(t_2)\\ &\le \sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\left|\int_0^{t_2}\Big(\Psi^{\eta_i+q-1}(t_1,s) - \Psi^{\eta_i+q-1}(t_2,s)\Big)g_i(s,u(s))ds\right|\\ &+\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\left|\int_{t_2}^{t_1}\Psi^{\eta_i+q-1}(t_1,s)g_i(s,u(s))ds\right|\\ &\le \sum_{i=1}^{m}\frac{\varphi_i^*}{\Gamma(\eta_i+q+1)}\Big(2|\psi(t_1)-\psi(t_2)|^{\eta_i+q} + \big|\Psi_0^{\eta_i+q}(t_2) - \Psi_0^{\eta_i+q}(t_1)\big|\Big).\end{aligned}$$

As $t_2 \to t_1$, so the right-hand side tends to zero. Thus B is equi-continuous. Therefore, it follows from the Arzelá–Ascoli theorem that B is a compact operator on B_r. We conclude that B is completely continuous.

Step 3. Condition (3) of Lemma 3.2 holds. For any $\overline{u} \in B_r$, we have

$$\begin{aligned}\|u(t)\|_\infty &= \|Au(t) + B\overline{u}(t)\|_\infty\\ &\le \|Au(t)\|_\infty + \|B\overline{u}(t)\|_\infty\\ &\le \sup_{t\in J}\left\{\left|I_{0^+}^{q;\psi}\Big(Du(s)C_p u(s)\Big)(t) + I_{0^+}^{q;\psi}\Big(Du(s)\Big(\Big(\Psi_0^1(s)\Omega_3 - \Psi_0^2(s)\Omega_2\Big)\Big(\frac{v_1}{b_1} - C_p u(s)\Big)(t)\right.\right.\\ &\left.\left.+\Omega_1\Big(v_2 - a_2 C_{p-\delta}u(\xi) - b_2 C_{p-\delta}u(T)\Big)I_{0^+}^{q;\psi}\Big(Du(s)\Big(\Psi_0^2(s) - \Psi_0^1(T)\Psi_0^1(s)\Big)\Big)(t)\right|\right\}\\ &+\sup_{t\in J}\left\{\sum_{i=1}^{m} I_{0^+}^{\eta_i+q;\psi}\big|g_i(s,\overline{u}(s))\big|(t)\right\}\\ &\le \chi^*\vartheta^*\frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \chi^*\left(\frac{|\Omega_3|\Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)}\right)\left(\frac{|v_1|}{|b_1|} + \frac{\Psi_0^p(T)}{\Gamma(p+1)}\vartheta^*\right)\end{aligned}$$

$$+|\Omega_1|\frac{(2+q)\Psi_0^{q+2}(T)}{\Gamma(q+3)}\Big(|v_2|+|a_2|\frac{\Psi_0^{p-\delta}(\xi)}{\Gamma(p-\delta+1)}\vartheta^*+|b_2|\frac{\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)}\vartheta^*\Big)$$
$$+\sum_{i=1}^{n}\frac{\varphi_i^*}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(T).$$

Which implies $\|u\|_\infty \leq r$, and so $u \in B_r$. Hence all the conditions of Lemma 3.2 are satisfied. Therefore, the operator equation $u(t) = Au(t) + Bu(t)$ has at least one solution in B_r. Consequently, there exists a solution of problem (2.1)-(2.2) in J. Thus the proof is completed. □

2.3 Uniqueness of solutions

In the next result, we proved the uniqueness of solutions for the problem (2.1)-(2.2) based on Banach's fixed point theorem.

Theorem 2.3.1. *Assume that (H1)-(H2) and the following hypotheses hold.*

(H4) The function $g_i \in C(J\times\mathbb{R},\mathbb{R})$ is continuous, and there exist bounded function $K_i : J \to (0,\infty)$, such that

$$|g_i(t,u(t)) - g_i(t,\overline{u}(t))| \leq K_i(t)|u(t)-\overline{u}(t)|.$$

If

$$\Lambda+\sum_{i=1}^{m}K_i^*\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}<1,$$

with $K_i^* = \sup_{t\in J}|K_i(t)|; i = 1,2,\cdots,m$.

Then the problem (2.1)-(2.2) has a unique solution.

Proof. According to lemma 2.1.1, we define the operator $Q : E \to E$ by

$$Qu(t) = Au(t) + Bu(t)$$

First, we show that $Q(B_r) \subset B_r$. As in the previous proof (**step 3**) of Theorem 3.2.1 ,we can obtain

For $u \in B_r$ and $t \in J$

$$\begin{aligned}\|Qu\|_\infty \le \chi^* \vartheta^* \frac{\Psi_0^{p+q}(T)}{\Gamma(p+1)\Gamma(q+1)} + \|\chi\| \left(\frac{|\Omega_3| \Psi_0^{q+1}(T)}{\Gamma(q+2)} + \frac{2|\Omega_2|\Psi_0^{q+2}(T)}{\Gamma(q+3)} \right) \left(\frac{|v_1|}{|b_1|} + \frac{\Psi_0^p(T)}{\Gamma(p+1)} \vartheta^* \right) \\ + |\Omega_1| \frac{(2+q)\Psi_0^{q+2}(T)}{\Gamma(q+3)} \left(|v_2| + |a_2| \frac{\Psi_0^{p-\delta}(\xi)}{\Gamma(p-\delta+1)} \vartheta^* + |b_2| \frac{\Psi_0^{p-\delta}(T)}{\Gamma(p-\delta+1)} \vartheta^* \right) \\ + \sum_{i=1}^{n} \frac{\varphi_i^*}{\Gamma(\eta_i+q+1)} \Psi_0^{\eta_i+q}(T) \le r,\end{aligned}$$

This shows that $Q(B_r) \subset B_r$. Next, we prove that the operator Q is a contraction. For $u, \overline{u} \in B_r$

$$\|Qu(\cdot) - Q\overline{u}(\cdot)\|_\infty \le \|Au(\cdot) - A\overline{u}(\cdot)\|_\infty + \|Bu(\cdot) - B\overline{u}(\cdot)\|_\infty$$

and

$$\begin{aligned}&\|Bu(\cdot) - B\overline{u}(\cdot)\|_\infty \\ &\le \sup_{t \in J} \left\{ \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i+q)} \int_0^t \psi'(s) \Psi^{\eta_i+q-1}(t,s) \big| g_i(s,u(s)) - g_i(s,\overline{u}(s)) \big| ds \right\} \\ &\le \sum_{i=1}^{m} \frac{K_i^*}{\Gamma(\eta_i+q+1)} \Psi_0^{\eta_i+q}(T) \|u(\cdot) - \overline{u}(\cdot)\|_\infty.\end{aligned} \tag{2.12}$$

from (2.11) and (2.12), we get

$$\|Qu(t) - Q\overline{u}(t)\|_\infty \le \left(\Lambda + \sum_{i=1}^{m} \frac{K_i^*}{\Gamma(\eta_i+q+1)} \Psi_0^{\eta_i+q}(T) \right) \|u(\cdot) - \overline{u}(\cdot)\|_\infty.$$

This implies that Q is a contractive operator. Consequently, by Theorem 2.3.1, we conclude that Q has a unique fixed point, which is a solution of problem (2.1)-(2.2). This completes the proof. □

2.4 Example

Consider the following fractional hybrid differential equation

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{2};t}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]=f\left(t,u(t),{}^cD_{0^+}^{\frac{5}{2};t}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right);t\in[0,1] \\ u(0)=0,\quad {}^cD_{0^+}^{\frac{3}{4};t}u(0)=0, \\ 2\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=0}+\frac{2}{7}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=1}=\frac{7}{2}, \\ \frac{7}{13}{}^cD_{0^+}^{\frac{4}{5};t}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\frac{4}{5}}+\frac{1}{2}{}^cD_{0^+}^{\frac{4}{5};t}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]_{t=1}=2\,, \end{cases} \tag{2.13}$$

where

$$\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))(s)=I_{0^+}^{\frac{1}{3};t}\left(\frac{\sin^2u(s)}{8(s+1)^2}\right)(t)+I_{0^+}^{\frac{3}{2};t}\left(\frac{1}{2\pi\sqrt{81+s^2}}\frac{|u(s)|}{2+|u(s)|}\right)(t)$$
$$+I_{0^+}^{\frac{7}{3};t}\left(\frac{\sin u(s)}{3\pi\sqrt{49+s^2}}\right)(t),$$

and

$$h(t,u(t))=\frac{e^{-3t}\cos u(t)}{2t+40}+\frac{1}{80}(t^3+1),$$

and

$$f\left(t,u(t),{}^cD_{0^+}^{\frac{5}{2};t}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right)$$
$$=\frac{1}{60\sqrt{t+81}}\left(\frac{|u(t)|}{3+|u(t)|}-\arctan\left({}^cD_{0^+}^{\frac{5}{2};t}\left[\frac{{}^cD_{0^+}^{\frac{3}{4};t}u(t)-\sum_{i=1}^{3}I_{0^+}^{\eta_i;t}g_i(t,u(t))}{h(t,u(t))}\right]\right)\right)$$

Here

$$p=\frac{5}{2},\qquad q=\frac{3}{4},\qquad m=3,\qquad \eta_1=\frac{1}{3},\qquad \eta_2=\frac{3}{2},\qquad \eta_3=\frac{7}{3},\qquad \delta=\frac{4}{5},\qquad a_1=2,$$
$$a_2=\frac{7}{13},\qquad b_1=\frac{2}{7},\qquad b_2=1/2,\qquad v_1=\frac{7}{2},\qquad v_2=2,\qquad \xi=\frac{4}{5},$$
$$g_1=\frac{\sin^2 u(t)}{8(t+1)^2},\qquad g_2=\frac{1}{2\pi\sqrt{81+t^2}}\frac{|u(t)|}{2+|u(t)|},\qquad g_3=\frac{\sin u(t)}{3\pi\sqrt{49+t^2}}.$$

The hypothesis (H1), (H2) and (H4) is satisfied with the following positives functions:

$$L(t)=\frac{e^{-3}}{2t+40},\quad M(t)=\vartheta(t)=\frac{1}{60\sqrt{t+81}},\quad \varphi_1(t)=K_1(t)=\frac{1}{8(t+1)^2},$$
$$\varphi_2(t)=K_2(t)=\frac{1}{2\pi\sqrt{81+t^2}},\quad \varphi_3(t)=K_3(t)=\frac{1}{3\pi\sqrt{49+t^2}}$$

and

$$\chi(t)=\frac{e^{-3}}{2t+40}+\frac{1}{80}(t^3+1),$$

which gives

$$L^*=\frac{1}{40},\ M^*=\vartheta^*=\frac{1}{540},\ \chi^*=\frac{3}{80},\ \varphi_1^*=K_1^*=\frac{1}{8},\ \varphi_2^*=K_2^*=\frac{1}{18\pi},\ \varphi_3^*=K_3^*=\frac{1}{21\pi}.$$

With the given data, we find that

$$\Omega_1\simeq 1.81820508,\quad \Omega_2\simeq 0.60797139,\quad \Omega_3\simeq 1.60797139,$$

and the hypothesis (H3) is satisfied by

$$\Lambda\simeq 0.48820986<1.$$

By Theorem 2.2.1 , the problem (2.13) has a solution on $[0,1]$.

Also, we have

$$\Lambda+\sum_{i=1}^{3}\frac{K_i^*}{\Gamma(\eta_i+\frac{7}{4})}\Psi_0^{\eta_i+\frac{3}{4}}(T)\simeq 0.61782704<1.$$

In the view of Theorem 2.3.1 problem (2.13) has an unique solution.

CHAPTER 3

HYBRID FRACTIONAL SEQUENTIAL INTEGRO-DIFFERENTIAL EQUATION

3.1 Introduction

This Chapter study the existence, uniqueness, and Ulam-Hyres stability of solutions for the following ψ-Caputo hybrid fractional sequential integro-differential equation[1](for short ψ-Caputo HFSIDE)

$$L_{\psi}^{p}\left[\frac{{}^{c}D_{0^{+}}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^{+}}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]=f\big(t,u(t),\delta I_{0^{+}}^{\gamma;\psi}u(t)\big);\quad t\in J=[0,T];\tag{3.1}$$

endowed with the hybrid fractional integral boundary conditions

$$\begin{cases} u(0)=0, \quad {}^{c}D_{0^{+}}^{q;\psi}u(0)=0, \\ \left[\frac{{}^{c}D_{0^{+}}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^{+}}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=T}=\rho\left[\frac{{}^{c}D_{0^{+}}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^{+}}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]_{t=\xi}; \quad 0<\rho,\xi<T. \end{cases}\tag{3.2}$$

[1]**F. Fredj**, H. Hammouche, M. S. Abdo, W. Albalawi and A. H. Almaliki, : A study on ψ-Caputo type hybrid multi fractional differential equations with hybrid boundary conditions. Journal of Mathematics, ID 9595398(2022).

with
$$L_{\psi}^{p} = {}^{c}D_{0^+}^{p;\psi} + \lambda {}^{c}D_{0^+}^{p-1;\psi},$$
where $1 < p \leq 2$, $0 < q \leq 1$, ${}^{c}D_{0^+}^{p;\psi}$, ${}^{c}D_{0^+}^{q;\psi}$ denote the ψ-Caputo fractional derivative of order p,q, respectively and $I_{0^+}^{\eta_i;\psi}$, $I_{0^+}^{\gamma;\psi}$ are the ψ-Riemann-Liouville fractional integral of order $\eta_i > 0$ and $\gamma > 0$ respectively, $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus 0)$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $g_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $g_i(0,0) = 0$; $(i = 1, ..., m)$, and λ is appropriate positive real constants.

Using an advantageous generalization of Krasnoselskii's fixed point theorem, we establish results of at least one solution, whereas the uniqueness of the solution is derived from Banach's fixed point. Besides, the Ulam-Hyers stability for the analyzed problem is investigated by applying the techniques of nonlinear functional analysis.

Main result

Lemma 3.1.1. *Let $1 < p \leq 2$, $0 < q \leq 1$. For any functions $F \in C(J, \mathbb{R})$, $H \in C(J, \mathbb{R} \setminus 0)$ and $G_i \in C(J, \mathbb{R})$ with $G_i(0) = 0$; $i = 1, ..., 2$, the following linear fractional boundary value problem*

$$L_{\psi}^{p}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t) - \sum_{i=1}^{m} I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right] = F(t); \quad t \in J, \tag{3.3}$$

supplemented with the conditions

$$\begin{cases} u(0) = 0, \quad {}^{c}D_{0^+}^{q;\psi}u(0) = 0, \\ \left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t) - \sum_{i=1}^{m} I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=T} = \rho\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t) - \sum_{i=1}^{m} I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=\xi}; \quad 0 < \rho, \xi < T. \end{cases} \tag{3.4}$$

has a unique solution, that is

$$\begin{aligned} u(t) &= I_{0^+}^{q;\psi}\left(H(s)\int_0^s I_{0^+}^{p-1;\psi}F(\tau)\,\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau\right)(t) \\ &+ \left(\int_0^T I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau \right. \\ &\left. -\rho\int_0^{\xi} I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\right) \\ &\times \frac{1}{\Delta}I_{0^+}^{q;\psi}\left(H(s)\left(1-e^{-\lambda\Psi_0(s)}\right)\right)(t)+\sum_{i=1}^{m} I_{0^+}^{\eta_i+q;\psi}G_i(t), \end{aligned} \tag{3.5}$$

where $\Delta = \left(1+\rho e^{-\lambda\Psi_0(\xi)} - e^{-\lambda\Psi_0(T)}-\rho\right)\neq 0$, *and* $\lambda\in\mathbb{R}^+$.

Proof. Applying the ψ-Riemann-Liouville fractional integral of order $p-1$ to both sides of (3.3), and using Lemma 1.2.10, we arrive

$$\begin{aligned} &{}^cD_{0^+}^{1;\psi}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]+\lambda\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right] \\ &= I_{0^+}^{p-1;\psi}F(t)+c_0; \qquad c_0\in\mathbb{R}, \end{aligned} \tag{3.6}$$

by multiplying $\psi'(t)e^{\lambda\Psi_0(t)}$ to both sides of (3.6), we find that

$$\begin{aligned} &e^{\lambda\Psi_0(t)}\frac{d}{dt}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right] \\ &\qquad + \lambda\psi'(t)e^{\lambda\Psi_0(t)}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I^{\eta_i;\psi}G_i(t)}{H(t)}\right] \\ &= \psi'(t)e^{\lambda\Psi_0(t)}I_{0^+}^{p-1;\psi}F(t)+c_0\psi'(t)e^{\lambda\Psi_0(t)}. \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \frac{d}{dt}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}.e^{\lambda\Psi_0(t)}\right] &= e^{\lambda\Psi_0(t)}\frac{d}{dt}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right] \\ &+ \lambda\psi'(t)e^{\lambda\Psi_0(t)}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we find that

$$\frac{d}{dt}\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}.e^{\lambda\Psi_0(t)}\right]=I_{0^+}^{p-1;\psi}F(t)\psi'(t)e^{\lambda\Psi_0(t)}+c_0\psi'(t)e^{\lambda\Psi_0(t)}.$$

Integrating from 0 to t, and using the fact that $G_i(0)=0(i=1,...,m)$, and from the condition ${}^cD_{0^+}^{q;\psi}u(0)=0$ in (3.4), we have

$$\begin{aligned}\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}.e^{\lambda\Psi_0(t)}&=\int_0^t I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{\lambda\Psi_0(\tau)}d\tau\\&+\frac{c_0}{\lambda}\left(e^{-\lambda\Psi_0(t)-1}\right)+c_1;\quad c_1\in\mathbb{R}.\end{aligned}$$

By multiplying $e^{-\lambda\Psi_0(t)}$ to both sides, we get

$$\begin{aligned}\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}&=\int_0^t I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(t)-\Psi_0(\tau))}d\tau\\&+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(t)}\right)+c_1e^{-\lambda\Psi_0(t)}.\end{aligned}\tag{3.9}$$

Next, applying ψ-Riemann-Lipuville fractional integral of order q to both sides of (3.9), and using Lemma 1.2.10, we get

$$\begin{aligned}u(t)=I_{0^+}^{q;\psi}&\left(H(s)\left(\int_0^s I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(s)}\right)\right.\right.\\&\left.\left.+c_1e^{-\lambda\Psi_0(s)}\right)\right)(t)+\sum_{i=1}^{m}I_{0^+}^{\eta_i+q;\psi}G_i(s)(t)+c_2;\quad c_2\in\mathbb{R}.\end{aligned}\tag{3.10}$$

With the help of conditions ${}^cD_{0^+}^{q;\psi}u(0)=0$, $u(0)=0$ and $G_i(0)=0(i=1,...,m)$, we find $c_1=c_2=0$. Then, we apply the third condition of (3.4) in (3.10), we obtain

$$\begin{aligned}&\int_0^T I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(T)}\right)\\&\qquad=\rho\int_0^\xi I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau+\frac{c_0}{\lambda}\left(1-e^{-\lambda\Psi_0(\xi)}\right).\end{aligned}$$

Some computations give us

$$c_0=\frac{\lambda}{\Delta}\bigg(\int_0^T I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau-\rho\int_0^{\xi} I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\bigg).$$

Inserting c_0, c_1 and c_2 in (3.10), which leads to the solution (3.5). Conversely, by Lemma 1.2.10 and by taking ${}^cD_{0^+}^{q;\psi}$ on both sides of (3.10), we obtain

$$\begin{aligned}\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}&=\int_0^t I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(t)-\Psi_0(\tau))}d\tau\\&+\bigg(\int_0^T I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau\\&-\rho\int_0^{\xi} I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\bigg)\frac{1-e^{-\lambda\Psi_0(t)}}{\Delta}.\end{aligned}\tag{3.11}$$

Next, operating ${}^cD_{0^+}^{p;\psi}+\lambda{}^cD_{0^+}^{p-1;\psi}$ on both sides of above equation, with the help of Lemma 1.2.10, we get

$$\begin{aligned}&\left[{}^cD_{0^+}^{p;\psi}+\lambda{}^cD_{0^+}^{p-1;\psi}\right]\left[\frac{{}^cD_{0^+}^{q;\psi}u(t)-\sum_{i=1}^m I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]\\&=\left[{}^cD_{0^+}^{p;\psi}+\lambda{}^cD_{0^+}^{p-1;\psi}\right]\bigg(\int_0^t I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(t)-\Psi_0(\tau))}d\tau+\bigg(\int_0^T I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)\\&\times e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau-\rho\int_0^{\xi} I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\bigg)\frac{1-e^{-\lambda\Psi_0(t)}}{\Delta}\bigg)\\&=\left[{}^cD_{0^+}^{p;\psi}+\lambda{}^cD_{0^+}^{p-1;\psi}\right]\int_0^t I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(t)-\Psi_0(\tau))}d\tau-\frac{c_0}{\lambda}{}^cD_{0^+}^{p;\psi}e^{-\lambda\Psi_0(t)}\\&-c_0{}^cD_{0^+}^{p-1;\psi}e^{-\lambda\Psi_0(t)}\\&={}^cD_{0^+}^{p-1;\psi}\left[\frac{1}{\psi'(t)}\frac{d}{dt}+\lambda\right]\int_0^t I_{0^+}^{p-1;\psi}F(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(t)-\Psi_0(\tau))}d\tau\\&={}^cD_{0^+}^{p-1;\psi}I_{0^+}^{p-1;\psi}F(t)\\&=F(t).\end{aligned}$$

Now, it remains to review the boundary conditions (3.4) of our problem . Substitution $t=0$ in (3.5) with the fact that $G_i(0)=0; i=1,\cdots,m$, leads to $u(0)=0$. Next, we apply ${}^cD_{0^+}^{q;\psi}$ on (3.5), then we substitute $t=0$, it follows that ${}^cD_{0^+}^{q;\psi}u(0)=0$. We substitute $t=T$

and $t=\xi$, we find that the two resulting equations are equal and from it we get that

$$\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=T}=\rho\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}G_i(t)}{H(t)}\right]_{t=\xi}.$$

This means that $u(t)$ satisfies (3.3) and (3.4). Therefore, $u(t)$ is solution of problem (3.3)-(3.4). □

Lemma 3.1.2. *For $F\in C(J,\mathbb{R})$ and $H\in C(J,\mathbb{R}\setminus 0)$, we have*

$$(1)\left|\int_0^t\frac{\Psi^{q-1}(t,s)}{\Gamma(q)}H(s)\int_0^s\int_0^\tau\frac{\Psi^{p-2}(\tau,\sigma)}{\Gamma(p-1)}F(\sigma)d\sigma\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau ds\right|$$
$$\le\frac{\Psi_0^{p+q-1}(T)}{\lambda\Gamma(p+q)\Gamma(q)}(1-e^{-\lambda\Psi_0(t)})\|H\|_\infty\|F\|_\infty.$$

$$(2)\left|\int_0^T\int_0^\tau\frac{\Psi^{p-2}(t,\sigma)}{\Gamma(p-1)}F(\sigma)d\sigma\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau\right|\le\frac{\Psi_0^{p-1}(T)}{\lambda\Gamma(p)}(1-e^{-\lambda\Psi_0(T)})\|F\|_\infty.$$

$$(3)\left|\int_0^\xi\int_0^\tau\frac{\psi'^{p-2}(t,\sigma)}{\Gamma(p-1)}F(\sigma)d\sigma\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\right|\le\frac{\Psi_0^{p-1}(\xi)}{\lambda\Gamma(p)}(1-e^{-\lambda\Psi_0(\xi)})\|F\|_\infty.$$

Proof. To prove the property *(1), we have*

$$\int_0^\tau\frac{\Psi^{p-2}(t,\sigma)}{\Gamma(p-1)}d\sigma=\frac{\Psi_0^{p-1}(\tau)}{\Gamma(p)},$$

and

$$\int_0^s\frac{\Psi_0^{p-1}(\tau)}{\Gamma(p)}\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau\le\frac{\Psi_0^{p-1}(s)}{\lambda\Gamma(p)}\int_0^s\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau$$
$$=\frac{\Psi_0^{p-1}(s)}{\lambda\Gamma(p)}(1-e^{-\lambda\Psi_0(s)}).$$

From the above integrals and left side of *(1)*, we get

$$\left|\int_0^t\frac{\Psi^{q-1}(t,s)}{\Gamma(q)}H(s)\int_0^s\int_0^\tau\frac{\Psi^{p-2}(t,\sigma)}{\Gamma(p-1)}F(\sigma)d\sigma\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau ds\right|$$
$$\le\|H\|_\infty\|F\|_\infty\int_0^t\frac{\Psi^{q-1}(t,s)}{\Gamma(q)}\frac{\Psi_0^{p-1}(s)}{\lambda\Gamma(p)}(1-e^{-\lambda\Psi_0(s)})ds$$

$$\leq \|H\|_\infty \|F\|_\infty \frac{\left(1-e^{-\lambda \Psi_0(T)}\right)}{\lambda \Gamma(p)} \int_0^t \frac{\Psi^{q-1}(t,s)}{\Gamma(q)} \Psi_0^{p-1}(s) ds$$
$$\leq \frac{\Psi_0^{p+q-1}(T)}{\lambda \Gamma(p+q)} \left(1-e^{-\lambda \Psi_0(T)}\right) \|H\|_\infty \|F\|_\infty.$$

The proofs of properties *(2)* and *(3)* are similar to the proof of *(1)*. □

Now, we consider the following assumptions:

(H1) $h \in C(J \times \mathbb{R}, \mathbb{R} \setminus 0)$, $F \in C(J \times \mathbb{R}^2, \mathbb{R})$, and there exist positive and bounded functions $L(t)$ and $M(t)$, such that

$$|h(t,u) - h(t,\overline{u})| \leq L(t)|u - \overline{u}|,$$

and

$$|f(t,u_1,u_2) - f(t,\overline{u}_1,\overline{u}_2)| \leq M(t)\big(|u_1 - \overline{u}_1| + |u_2 - \overline{u}_2|\big),$$

for $t \in J$ and $u_1, u_2, \overline{u}_1, \overline{u}_2 \in \mathbb{R}$.

(H2) There exist functions $\varphi_i, \chi, \vartheta \in C(J, \mathbb{R})$ such that

$$|g_i(t,u)| \leq \varphi_i(t) \quad \text{for each } t,u \in J \times \mathbb{R},$$
$$|h(t,u)| \leq \chi(t) \quad \text{for each } t,u \in J \times \mathbb{R},$$
$$|f(t,u,\overline{u})| \leq \vartheta(t) \quad \text{for each } t,u,\overline{u} \in J \times \mathbb{R} \times \mathbb{R}.$$

(H3) There exist constants $0 < \Lambda, \Upsilon < 1$, such that

$$\frac{1-e^{-\lambda \Psi_0(T)}}{\lambda} \Big(\Lambda \chi^* M^* + \Upsilon L^* \vartheta^*\Big) < 1, \tag{3.12}$$

where

$$\Lambda = \frac{\Psi_0^{p+q-1}(T)}{\Gamma(p+q)} + \frac{\delta \Gamma(\gamma-1) \Psi_0^{\gamma+p+q-2}(T)}{\Gamma(\gamma+p+q-1)} + \frac{\Psi_0^q(T)}{\Delta \Gamma(q+1)}$$
$$\times \left(\left(1-e^{-\lambda \Psi_0(T)}\right)\left(\frac{\Psi_0^{p-1}(T)}{\Gamma(p)} + \frac{\delta \Gamma(\gamma-1) \Psi_0^{\gamma+p-2}(T)}{\Gamma(\gamma+p-1)}\right)\right.$$

$$+\rho\left(1-e^{-\lambda\Psi_0(\xi)}\right)\left(\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)}+\frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(\xi)}{\Gamma(\gamma+p-1)}\right)\Bigg).$$

$$\Upsilon=\frac{\Psi_0^{p+q-1}(T)}{\Gamma(p+q)}+\frac{\Psi_0^{q}(T)}{\Delta\Gamma(q+1)\Gamma(p-1)}\left(\left(1-e^{-\lambda\Psi_0(T)}\right)\Psi_0^{p-2}(T)\right.$$
$$\left.+\rho\left(1-e^{-\lambda\Psi_0(\xi)}\right)\Psi_0^{p-2}(\xi)\right).$$

3.2 Existence of solutions

In this subsection, we prove the existence of a solution for the problem (3.1)-(3.2) by applying Dhage fixed point theorem.

Theorem 3.2.1. *Suppose (H1)-(H3) holds, then the problem* (3.1)-(3.2) *has at least one solution in* $C(J,\mathbb{R})$.

Proof. First, we set: $L^*=\sup_{t\in J}|L(t)|, M^*=\sup_{t\in J}|M(t)|, \chi^*=\sup_{t\in J}|\chi(t)|, \vartheta^*=\sup_{t\in J}|\vartheta(t)|$ and $\varphi^*=\sup_{t\in J}|\varphi_i(t)|; i=1,\cdots,m$. We choose r, such that

$$r\geq\frac{\left(1-e^{-\lambda\Psi_0(T)}\right)}{\lambda}\left(\frac{\Psi_0^{p+q-1}(T)}{\Gamma(q)\Gamma(p+q)}+\frac{\Psi_0^{q}(T)}{\Gamma(q+1)\Delta}\right.$$
$$\left.\times\left(\frac{\Psi_0^{p-1}(T)}{\Gamma(p)}\left(1-e^{-\lambda\Psi_0(T)}\right)-\rho\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)}\left(1-e^{-\lambda\Psi_0(\xi)}\right)\right)\right)$$
$$\times\vartheta^*\chi^*+\sum_{i=1}^{m}\frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)}\varphi_i^*.$$

Now, we define $B_r\subset C(J,\mathbb{R})$ as

$$B_r=\{u\in E:\|u\|_\infty\leq r\}.$$

Define two operators $C:C(J,\mathbb{R})\to C(J,\mathbb{R})$ and $D:C(J,\mathbb{R})\to C(J,\mathbb{R})$ as

$$Cu(t)=\frac{1}{\Gamma(p-1)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{p-2}f(s,u(s),\delta I^{\gamma;\psi}u(s))ds,\quad t\in J,$$

and

$$Du(t)=h(t,u(t)),\quad t\in J.$$

Then, using assumptions (H1)-(H2) , we have for $u,\overline{u} \in B_r$, and each $t \in J$

$$\begin{aligned}
&|Cu(t) - C\overline{u}(t)| \\
&\le \frac{1}{\Gamma(p-1)} \int_0^t \Psi^{p-2}(t,s)|f(s,u(s),\delta I^{\gamma;\psi}u(s)) - f(s,\overline{u}(s),\delta I^{\gamma;\psi}\overline{u}(s))|ds \\
&\le \frac{1}{\Gamma(p-1)} \int_0^t \Psi^{p-2}(t,s)M(s)\Big(|u(s) - \overline{u}(s)| + \frac{\delta}{\Gamma(\gamma)}\int_0^s \Psi^{\gamma-1}(\tau,s)|u(\tau) - \overline{u}(\tau)|d\tau\Big)ds \\
&\le M^*\|u(\cdot) - \overline{u}(\cdot)\|_\infty \frac{1}{\Gamma(p-1)} \int_0^t \Psi^{p-2}(t,s)\big(1 + \frac{\delta}{\Gamma(\gamma+1)}\Psi_0^{\gamma}(s)\big)ds \\
&\le \Big(\frac{1}{\Gamma(p)}\Psi_0^{p-1}(T) + \frac{\delta\Gamma(\gamma-1)}{\Gamma(\gamma+p-1)}\Psi_0^{\gamma+p-2}(T)\Big)M^*\|u(\cdot) - \overline{u}(\cdot)\|_\infty,
\end{aligned}$$

and

$$|Cu(t)| \le \frac{\Psi_0^{p-2}(T)}{\Gamma(p-1)}\vartheta^*,$$

also

$$|Du(t) - D\overline{u}(t)| \le L^*\|u(\cdot) - \overline{u}(\cdot)\|_\infty, \qquad |Du(t)| \le \chi^*.$$

Now, we also consider two operators $A : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ and $B : B_r \to C(J,\mathbb{R})$ defined by

$$\begin{aligned}
Au(t) &= I_{0^+}^{q;\psi}\Big(Du(s)\int_0^s Cu(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau\Big)(t) \\
&\quad + \frac{1}{\Delta}I_{0^+}^{q;\psi}\Big(Du(s)\Big(1 - e^{-\lambda\Psi_0(s)}\Big)\Big)(t) \\
&\quad \times \Big(\int_0^T Cu(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau - \rho\int_0^\xi Cu(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\Big),
\end{aligned}$$

and

$$Bu(t) = \sum_{i=1}^m I_{0^+}^{\eta_i+q;\psi}g_i(s,u(s))(t).$$

We need to prove that A and B satisfy all assumptions of lemma . This can be proven in the forthcoming steps.

Step.1 A is a contraction map. Indeed, let $u(t),\overline{u}(t) \in B_r$. Then

$$\begin{aligned}
&|Au(t) - A\overline{u}(t)| \\
&\qquad \le I_{0^+}^{q;\psi}\Big(\big|Du(s)\big|\int_0^s \big|Cu(\tau) - C\overline{u}(\tau)\big|\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau\Big)(t)
\end{aligned}$$

$$+ I_{0^+}^{q;\psi}\left(\left|Du(s) - D\overline{u}(s)\right| \int_0^s \left|C\overline{u}(\tau)\right|\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau\right)(t)$$
$$+ \frac{1}{\Delta} I_{0^+}^{q;\psi}\left(\left|Du(s)\right|\left(1 - e^{-\lambda\Psi_0(s)}\right)\right)(t)\left(\int_0^T \left|Cu(\tau) - C\overline{u}(\tau)\right|\psi'(\tau)\right.$$
$$\left.\times e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau + \rho\int_0^\xi \left|Cu(\tau) - C\overline{u}(\tau)\right|\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\right)$$
$$+ \frac{1}{\Delta} I_{0^+}^{q;\psi}\left(\left|Du(s) - D\overline{u}(s)\right|\left(1 - e^{-\lambda\Psi_0(s)}\right)\right)(t)\left(\int_0^T \left|Cu(\tau)\right|\psi'(\tau)\right.$$
$$\left.\times e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau - \rho\int_0^\xi \left|Cu(\tau)\right|\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\right).$$

Using Lemma 3.1.2 and the hypotheses (H1)-(H2),we obtain

$$\begin{aligned}
&|Au(t){-}A\overline{u}(t)| \\
&\quad\le \frac{1 - e^{-\lambda\Psi_0(T)}}{\lambda}\left\{\chi^* M^*\left(\frac{\Psi_0^{p+q-1}(T)}{\Gamma(p+q)} + \frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p+q-2}(T)}{\Gamma(\gamma+p+q-1)}\right.\right.\\
&\quad + \frac{\Psi_0^{q}(T)}{\Delta\Gamma(q+1)}\left(\left(1 - e^{-\lambda\Psi_0(T)}\right)\left(\frac{\Psi_0^{p-1}(T)}{\Gamma(p)} + \frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(T)}{\Gamma(\gamma+p-1)}\right)\right.\\
&\quad \left.\left. + \rho\left(1 - e^{-\lambda\Psi_0(\xi)}\right)\left(\frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)} + \frac{\delta\Gamma(\gamma-1)\Psi_0^{\gamma+p-2}(\xi)}{\Gamma(\gamma+p-1)}\right)\right)\right)\\
&\quad + L^*\vartheta^*\left(\frac{\Psi_0^{p+q-1}(T)}{\Gamma(p+q)} + \frac{\Psi_0^{q}(T)}{\Delta\Gamma(q+1)\Gamma(p-1)}\right.\\
&\quad \times\left(\left(1 - e^{-\lambda\Psi_0(T)}\right)\Psi_0^{p-2}(T) + \rho\left(1 - e^{-\lambda\Psi_0(\xi)}\right)\right.\\
&\quad \left.\left.\left.\times \Psi_0^{p-2}(\xi)\right)\right)\right\}\|u(\cdot) - \overline{u}(\cdot)\|_\infty.
\end{aligned}$$

Moreover,

$$\|Au(t) - A\overline{u}(t)\|_\infty \le \frac{1 - e^{-\lambda\Psi_0(T)}}{\lambda}\left(\Lambda\chi^* M^* + \Upsilon L^*\vartheta^*\right)\|u(\cdot) - \overline{u}(\cdot)\|_\infty. \tag{3.13}$$

Hence by (3.12), A is a contraction map.

Step 2. B is compact and continuous on B_r.

Firstly, we prove that B is continuous on B_r. Let $u_n(t)$ be a sequence such that $u_n(t) \to u(t)$

in B_r. It follows from Lebesgue dominant convergence theorem that, for all $t \in J$,

$$\lim_{n\to\infty} Bu_n(t) = \lim_{n\to\infty} \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i+q)} \int_a^t \Psi^{\eta_i+q-1}(t,s)\, g_i(s,u_n(s))ds$$
$$= \sum_{i=1}^{m} \frac{1}{\Gamma(\eta_i+q)} \int_a^t \Psi^{\eta_i+q-1}(t,s)\, g_i(s,u(s))ds.$$

Hence $\lim_{n\to\infty} Bu_n(t) = Bu(t)$. Thus B is a continuous on B_r. Besides, we prove that B is uniformly bounded on B_r. Indeed, for any $u \in B_r$, we have

$$\|Bu(t)\|_\infty \leq \sum_{i=1}^{m} \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)} \varphi_i^* \leq r.$$

Therefore $\|Bu\| \leq r$, for all $t \in J$, which implies that B is uniformly bounded on B_r.
Now, we show that B is equicontinuous. Let $t_1, t_2 \in J$ with $t_1 > t_2$. Then for any $u(t) \in B_r$, we have

$$|Bu(t_1) - Bu(t_2)| \leq \sum_{i=1}^{m} \frac{\varphi_i^*}{\Gamma(\eta_i+q+1)} \Big(2(\psi(t_1) - \psi(t_2))^{\eta_i+q} + \big|\Psi_0^{\eta_i+q}(t_2) - \Psi_0^{\eta_i+q}(t_1)\big| \Big).$$

As $t_2 \to t_1$, $|Bu(t_1) - Bu(t_2)| \to 0$. This means that B is equicontinuous. Thus, Arzelá-Ascoli theorem shows that B is a compact operator on B_r.

Step 3. We prove that $u = Au + B\overline{u}$, for all $\overline{u} \in B_r \Rightarrow u \in B_r$.
For any $\overline{u} \in B_r$, we have

$$\begin{aligned}
\|u(\cdot)\|_\infty \leq & \frac{\left(1 - e^{-\lambda \Psi_0(T)}\right)}{\lambda} \left(\frac{\Psi_0^{p+q-1}(T)}{\Gamma(q)\Gamma(p+q)} + \frac{\Psi_0^{q}(T)}{\Gamma(q+1)\Delta} \right. \\
& \left. \times \left(\frac{\Psi_0^{p-1}(T)}{\Gamma(p)} \left(1 - e^{-\lambda \Psi_0(T)}\right) - \rho \frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)} \left(1 - e^{-\lambda \Psi_0(\xi)}\right) \right) \right) \\
& \times \vartheta^* \chi^* + \sum_{i=1}^{m} \frac{\Psi_0^{\eta_i+q}(T)}{\Gamma(\eta_i+q+1)} \varphi_i^* \\
& \leq r,
\end{aligned}$$

which implies $\|u\| \leq r$, that's mean $u \in B_r$. Hence all assumptions of lemma (3.2)are satisfied. So, the equation $u(t) = Au(t) + Bu(t)$ has at least one solution in B_r. Moreover, there exists a solution of the problem (3.1)-(3.2) in J. □

3.3 Uniqueness of solutions

Here, we prove the uniqueness theorem of (3.1)-(3.2) relying on Banach's fixed point theorem

Theorem 3.3.1. *Suppose that (H1)-(H2) and the following hypothesis holds.*

(H4) $g_i \in C(J \times \mathbb{R}, \mathbb{R})$*, and there exist positive and bounded function* $K_i(t)$*, such that*

$$|g_i(t,u) - g_i(t,\overline{u})| \leq K_i(t)|u - \overline{u}|.$$

If

$$\Xi := \frac{1 - e^{-\lambda \Psi_0(T)}}{\lambda} \left(\Lambda \chi^* M^* + \Upsilon L^* \vartheta^* \right) + \sum_{i=1}^{m} \frac{K_i^*}{\Gamma(\eta_i + q + 1)} \Psi_0^{\eta_i + q}(T) < 1, \tag{3.14}$$

then the problem (3.1)-(3.2) *has a unique solution.*

Proof. We set the operator $Q : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ as

$$Qu(t) = Au(t) + Bu(t).$$

We set $K^* = \sup_{t \in J} |K_i(t)|; i = 1, 2, \cdots, m$.
First, we show that $Q(B_r) \subset B_r$. As in the previous proof (**Step 3**) of Theorem 3.2.1, we can obtain
For $u \in B_r$ and $t \in J$

$$\begin{aligned}
\|Qu(\cdot)\|_\infty \leq & \frac{\left(1 - e^{-\lambda \Psi_0(T)}\right)}{\lambda} \left(\frac{\Psi_0^{p+q-1}(T)}{\Gamma(q)\Gamma(p+q)} + \frac{\Psi_0^q(T)}{\Gamma(q+1)\Delta} \left(\frac{\Psi_0^{p-1}(T)}{\Gamma(p)} \left(1 - e^{-\lambda \Psi_0(T)}\right) \right.\right. \\
& \left.\left. - \rho \frac{\Psi_0^{p-1}(\xi)}{\Gamma(p)} \left(1 - e^{-\lambda \Psi_0(\xi)}\right) \right) \right) \vartheta^* \chi^* + \sum_{i=1}^{m} \frac{\Psi_0^{\eta_i + q}(T)}{\Gamma(\eta_i + q + 1)} \varphi_i^* \\
& \leq R.
\end{aligned}$$

This shows that $Q(B_r) \subset B_r$. Next, we prove that Q is a contraction. For $u, \overline{u} \in B_r$

$$\|Qu(\cdot) - Q\overline{u}(\cdot)\|_\infty \leq \|Au(\cdot) - A\overline{u}(\cdot)\|_\infty + \|Bu(\cdot) - B\overline{u}(\cdot)\|_\infty$$

and

$$\begin{aligned}
&\|Bu(\cdot)-B\overline{u}(\cdot)\|_{\infty} \\
&\leq \sup_{t\in J}\left\{\sum_{i=1}^{m}\frac{1}{\Gamma(\eta_i+q)}\int_0^t \Psi^{\eta_i+q-1}(t,s)\left|g_i(s,u(s))-g_i(s,\overline{u}(s))\right|ds\right\} \\
&\leq \sum_{i=1}^{m}\frac{K_i^*}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(T)\|u(\cdot)-\overline{u}(\cdot)\|_{\infty}.
\end{aligned} \tag{3.15}$$

From (3.13),(3.14) and (3.15), we get

$$\begin{aligned}
\|Qu(\cdot)-Q\overline{u}(\cdot)\|_{\infty} \leq \Bigg(&\frac{1-e^{-\lambda\Psi_0(T)}}{\lambda}\left(\Lambda\chi^*M^*+\Upsilon L^*\vartheta^*\right) \\
&+\sum_{i=1}^{m}\frac{K_i^*}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(T)\Bigg)\|u(\cdot)-\overline{u}(\cdot)\|_{\infty}.
\end{aligned}$$

As $\Xi<1$, T is contractive map. Consequently, by Banach's fixed point theorem, we conclude that T has a unique fixed point, which is a solution of (3.1)-(3.2). □

3.4 Stability analysis

In this portion, we discuss the Ulam-Hyres and generalized Ulam-Hyres stabilities of the solution of the proposed problem. We adopt the following definitions from [43].

Let $\varepsilon>0$. Consider the subsequent inequality:

$$\left|L_{\psi}^{p}\left[\frac{{}^{c}D_{0^+}^{q;\psi}u(t)-\sum_{i=1}^{m}I_{0^+}^{\eta_i;\psi}g_i(t,u(t))}{h(t,u(t))}\right]-f\left(t,u(t),\delta I_{0^+}^{\gamma;\psi}u(t)\right)\right|\leq\varepsilon; \qquad t\in J. \tag{3.16}$$

Definition 3.4.1. The problem (3.1)-(3.2) is said to be Ulam-Hyres stable if there exists $C_f>0$ such that for each $\varepsilon>0$ and for each solution $w\in C(J,\mathbb{R})$ of (3.16), there exists a solution $u\in C(J,\mathbb{R})$ of (3.1)-(3.2) with

$$\left|w(t)-u(t)\right|\Big|\leq C_f\varepsilon, \qquad t\in J.$$

Definition 3.4.2. The problem (3.1)-(3.2) is said to be generalized Ulam-Hyres stable if there exists $\Theta\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\Theta(0)=0$ such that , for each solution $w\in C(J,\mathbb{R})$ of (3.16), there exists a solution $u \in C(J,\mathbb{R})$ of (3.1)-(3.2) with

$$\big|w(t)-u(t)\big|\big| \leq \Theta(\varepsilon), \qquad t \in J.$$

Remark 3.4.3. A function $w \in C(J,\mathbb{R})$ satisfies the problem (3.1)-(3.2), if there exists a $\phi \in C(J,\mathbb{R})$ (which depends on w) such that

(i) $\big|\phi(t)\big| \leq \varepsilon; \quad t \in J,$

(ii) for $t \in J$,

$$L_\psi^p \left[\frac{{}^cD_{0^+}^{q;\psi} w(t) - \sum_{i=1}^m I_{0^+}^{\eta_i;\psi} g_i(t,w(t))}{h(t,w(t))} \right] = f\big(t,w(t),\delta I_{0^+}^{\gamma;\psi} w(t)\big) + \phi(t).$$

Theorem 3.4.4. *Let (H1) and (H3) are fulfilled. Then the problem* (3.1)-(3.2) *is Ulam-Hyres and generalized Ulam-Hyres stable*

Proof. Let $w \in C(J,\mathbb{R})$ be a solution of the inequality (3.16) for each $\varepsilon > 0$. Then from Remark 3.4.3 and Lemma 3.1.1, we have

$$\begin{aligned}
& w(t) \\
& = I_{0^+}^{q;\psi}\left(h(s,w(s)) \int_0^s I_{0^+}^{p-1;\psi}\Big(f\big(t,w(t),\delta I_{0^+}^{\gamma;\psi} w(t)\big) + \phi(t)\Big)(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau \right)(t) \\
& + \left(\int_0^T I_{0^+}^{p-1;\psi}\Big(f\big(t,w(t),\delta I_{0^+}^{\gamma;\psi} w(t)\big) + \phi(t)\Big)(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau \right. \\
& \left. - \rho \int_0^\xi I_{0^+}^{p-1;\psi}\Big(f\big(t,w(t),\delta I_{0^+}^{\gamma;\psi} w(t)\big) + \phi(t)\Big)(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau \right) \\
& \times \frac{1}{\Delta} I_{0^+}^{q;\psi}\left(h(s,w(s))\Big(1-e^{-\lambda\Psi_0(s)}\Big)\right)(t) + \sum_{i=1}^m I_{0^+}^{\eta_i+q;\psi} g_i(s,w(s))(t).
\end{aligned}$$

Then, by Remark 3.4.3, Lemma 3.1.1 and (H1)-(H3), we obtain

$$\begin{aligned}
\big|w(t)-Qw(t)\big| = & \left| I_{0^+}^{q;\psi}\left(h(s,w(s)) \int_0^s I_{0^+}^{p-1;\psi} \phi(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau \right)(t) \right. \\
& + \left(\int_0^T I_{0^+}^{p-1;\psi} \phi(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau \right. \\
& \left. - \rho \int_0^\xi I_{0^+}^{p-1;\psi} \phi(\tau)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau \right)
\end{aligned}$$

$$\times \frac{1}{\Delta} I_{0^+}^{q;\psi}\Big(h(s,w(s))\Big(1-e^{-\lambda\Psi_0(s)}\Big)\Big)(t)\Big|$$
$$\leq \varepsilon\chi^*\Omega.$$

where,

$$\Omega = \Bigg[\Big(1-e^{-\lambda\Psi_0(\xi)}\Big)\Bigg(\frac{\Psi_0^{p+q-1}(T)}{\lambda\Gamma(p+q)\Gamma(q)} + \Bigg(\frac{\Psi_0^{p-1}(T)}{\lambda\Gamma(p)}\Big(1-e^{-\lambda\Psi_0(T)}\Big)$$
$$+\rho\frac{\Psi_0^{p-1}(\xi)}{\lambda\Gamma(p)}\Big(1-e^{-\lambda\Psi_0(\xi)}\Big)\Bigg)\frac{\Psi_0^{q}(T)}{\Delta\Gamma(q+1)}\Bigg)\Bigg].$$

Which is satisfied inequality (3.16).then for each $t \in J$,we have

$$\Big|w(t)-u(t)\Big|$$
$$= \Bigg|w(t) - I_{0^+}^{q;\psi}\Bigg(h(s,u(s))\int_0^s I_{0^+}^{p-1;\psi} f\big(\tau,u(\tau),\delta I_{0^+}^{\gamma;\psi}u(\tau)\big)\psi'(\tau)e^{-\lambda(\Psi_0(s)-\Psi_0(\tau))}d\tau\Bigg)(t)$$
$$+\Bigg(\int_0^T I_{0^+}^{p-1;\psi} f\big(\tau,u(\tau),\delta I_{0^+}^{\gamma;\psi}u(\tau)\big)\psi'(\tau)e^{-\lambda(\Psi_0(T)-\Psi_0(\tau))}d\tau$$
$$-\rho\int_0^\xi I_{0^+}^{p-1;\psi} f\big(\tau,u(\tau),\delta I_{0^+}^{\gamma;\psi}u(\tau)\big)\psi'(\tau)e^{-\lambda(\Psi_0(\xi)-\Psi_0(\tau))}d\tau\Bigg)$$
$$\times\frac{1}{\Delta} I_{0^+}^{q;\psi}\Big(h(s,u(s))\Big(1-e^{-\lambda\Psi_0(s)}\Big)\Big)(t) + \sum_{i=1}^m I_{0^+}^{\eta_i+q;\psi} g_i(s,u(s))(t)\Bigg|$$
$$\leq \Big|w(t)-Gw(t)\Big| + \Big|Gw(t)-Gu(t)\Big|$$
$$\leq \Omega\chi^*\varepsilon + \frac{1-e^{-\lambda\Psi_0(T)}}{\lambda}\Big(\Lambda\chi^*M^* + \Upsilon L^*\vartheta^*\Big)$$
$$+\sum_{i=1}^m \frac{K_i^*}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(T)\|w(\cdot)-u(\cdot)\|_\infty.$$

Then

$$\|w(\cdot)-u(\cdot)\|_\infty \leq \frac{\Omega\chi^*}{1-\Xi_1}\varepsilon.$$

where,

$$\Xi_1 = \Big(\Lambda\chi^*M^* + \Upsilon L^*\vartheta^*\Big) + \sum_{i=1}^m \frac{K_i^*}{\Gamma(\eta_i+q+1)}\Psi_0^{\eta_i+q}(T).$$

By setting $C_f = \dfrac{\Omega\chi^*}{1-\Xi_1}$, we obtain

$$\left|w(t)-u(t)\right| \leq C_f\varepsilon.$$

Therefore, the problem (3.1)-(3.2) is Ulam-Hyres stable.

Similarly, for $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that $\Theta(\varepsilon) = C_f\varepsilon$ along with $\Theta(0)=0$, the solution of the problem (3.1)-(3.2) is generalized Ulam-Hyers stable. □

3.5 Example

Consider the following ψ-Caputo HFSIDE

$$\begin{cases} \left[{}^cD_{0^+}^{\frac{3}{2};t} + 3\,{}^cD_{0^+}^{\frac{1}{2};t}\right]\left[\frac{{}^cD_{0^+}^{\frac{4}{5};t}u(t)-\sum_{i=1}^{4} I_{0^+}^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right] = f\left(t,u(t),\frac{1}{2}I_{0^+}^{\frac{5}{2};t}u(t)\right); t\in[0,1], \\ u(0)=0, \quad {}^cD_{0^+}^{\frac{4}{5};t}u(0)=0, \\ \left[\frac{{}^cD_{0^+}^{\frac{4}{5};t}u(t)-\sum_{i=1}^{4} I_{0^+}^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right]_{t=1} = \frac{7}{13}\left[\frac{{}^cD_{0^+}^{\frac{4}{5};t}u(t)-\sum_{i=1}^{4} I_{0^+}^{\eta_i;\psi} g_i(t,u(t))}{h(t,u(t))}\right]_{t=\frac{5}{6}}, \end{cases} \tag{3.17}$$

where

$$\begin{aligned} &\sum_{i=1}^{4} I_{0^+}^{\eta_i;t} g_i(t,u(t))(s) \\ &\quad = I_{0^+}^{\frac{1}{4};t}\left(\frac{1}{18}(s\sqrt{s^2+1}+\sin s+\cos u(s))\right)(t) + I_{0^+}^{\frac{9}{5};t}\left(\frac{\sin u(s)}{4\pi\sqrt{36+s^2}}\right)(t) \\ &\quad + I_{0^+}^{\frac{4}{3};t}\left(\frac{\sin^2 u(s)}{12(s+1)^2}\right)(t) + I_{0^+}^{\frac{5}{2};t}\left(\frac{1}{3\pi\sqrt{64+s^3}}\frac{|u(s)|}{2+|u(s)|}\right)(t), \end{aligned}$$

$$h(t,u(t)) = \frac{e^{-t}\sin u(t)}{t+30} + \frac{1}{60}(t^2+1),$$

and

$$f\left(t,u(t),\frac{1}{2}I_{0^+}^{\frac{5}{2};t}u(t)\right) = \frac{1}{\sqrt{t+81}}\left(\frac{|u(t)|}{1+|u(t)|} + \arctan\left(\frac{1}{2}I_{0^+}^{\frac{5}{2};t}u(t)\right)\right)$$

Here

$$p=\frac{3}{2},\qquad q=\frac{4}{5},\qquad m=4,\qquad \eta_1=\frac{7}{4},\qquad \eta_2=\frac{4}{3},\qquad \eta_3=\frac{2}{3},\qquad \eta_4=\frac{5}{6},\qquad \delta=\frac{1}{2},$$
$$\gamma=\frac{5}{2},\qquad \rho=\frac{7}{13},\qquad \xi=\frac{5}{6},$$

and

$$g_1=\frac{1}{18}(t\sqrt{t^2+1}+\sin t+\cos u(t)),\qquad g_2=\frac{\sin u(t)}{4\pi\sqrt{36+t^2}},\qquad g_3=\frac{\sin^2 u(t)}{12(t+1)^2},$$
$$g_4=\frac{1}{3\pi\sqrt{64+t^3}}\frac{|u(t)|}{2+|u(t)|}.$$

The hypothesis (H1), (H2) and (H4) are satisfied with the following positives functions:

$$L(t)=\frac{e^{-t}}{t+3},\quad \chi(t)=\frac{e^{-t}}{t+3}+\frac{t^2+1}{6},\quad M(t)=\vartheta(t)=\frac{3}{2\sqrt{t+81}},\quad K_1(t)=\frac{1}{18},$$
$$\varphi_1(t)=\frac{1}{18}(t\sqrt{t^2+1}+2),\quad K_2(t)=\varphi_2(t)=\frac{1}{4\pi\sqrt{36+t^2}},\quad K_3(t)=\varphi_3(t)=\frac{1}{12(t+1)^2}$$

and

$$K_4(t)=\varphi_4(t)=\frac{1}{3\pi\sqrt{64+t^3}},$$

which gives

$$L^*=\frac{1}{3},\quad \chi^*=\frac{1}{2},\quad M^*=\vartheta=\frac{1}{6},\quad K_1^*=\frac{1}{18},\quad \varphi_1^*=\frac{\sqrt{2}+2}{18},\quad K_2^*=\varphi_2^*=\frac{1}{24\pi},$$
$$K_3^*=\varphi_3^*=\frac{1}{12},\quad K_4^*=\varphi_4^*=\frac{1}{24\pi}.$$

With the given data, we find that

$$\Delta\simeq 0.45595101,\quad \Lambda\simeq 5.3500159,\quad \Upsilon\simeq 2.8388423,$$

and the hypothesis (H3) is satisfied by

$$\frac{1-e^{-\lambda}}{\lambda}\Big(\Lambda\chi^*M^*+\Upsilon L^*\vartheta^*\Big)+\sum_{i=1}^{4}\frac{K_i^*}{\Gamma(\eta_i+q+1)}\simeq 0.3153266651<1.$$

In the view of Theorem 3.3.1, the problem (3.17) has an unique solution. In addition, Theorem 3.4.4 ensures that (3.1)-(3.2) is Ulam-Hyres and generalized Ulam-Hyres stable. As shown in Theorem 3.4.4, for every $\epsilon > 0$ if $w \in \mathbb{R}$ satisfies

$$\left| L_{\psi}^{p}\left[\frac{{}^{c}D_{0^{+}}^{q;\psi}w(t)-\sum_{i=1}^{m}I_{0^{+}}^{\eta_{i};\psi}g_{i}(t,w(t))}{h(t,w(t))}\right]-f\left(t,w(t),\delta I_{0^{+}}^{\gamma;\psi}w(t)\right)\right| \leq \varepsilon;\ \ t\in[0,1],$$

then there exists a unique solution $u \in \mathbb{R}$ such that

$$\left|w(t)-u(t)\right|\big| \leq C_{f}\varepsilon,\ \ t\in[0,1].$$

where

$$C_{f} \simeq 0.81 < 1.$$

Hence, the problem (3.17) is Ulam-Hyres and generalized Ulam-Hyres stable.

CHAPTER 4

RANDOM HILFER-KATUGAMPOLA FRACTIONAL DIFFERENTIAL COUPLED SYSTEMS IN GENERALIZED BANACH SPACE

4.1 Introduction

In this Chapter, we study the existence and uniqueness results of Hilfer-Katugampola random nonlinear fractional differential coupled system in a generalized Banach space[1] given by:

$$\begin{cases} ({}^{\rho}D_{a^+}^{p_1,q_1}u)(t,\vartheta) = f(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \\ \\ ({}^{\rho}D_{a^+}^{p_2,q_2}v)(t,\vartheta) = g(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \end{cases} ;t \in J = [a,T], \vartheta \in \Omega, \tag{4.1}$$

[1]**F. Fredj**, H. Hammouche and M. Benchohra :Random Coupled Hilfer-Katugampola Fractional Differential Systems in Generalized Banach Spaces (submitted).

with the following initial conditions:

$$\begin{cases} ({}^\rho I_{a^+}^{1-\gamma_1} u)(a,\vartheta) = u_a(\vartheta) \\ & ;\vartheta \in \Omega, \\ ({}^\rho I_{a^+}^{1-\gamma_2} v)(a,\vartheta) = v_a(\vartheta) \end{cases} \tag{4.2}$$

where $0 \leq a < T < \infty$, $0 < p_i < 1$, $0 \leq q_i \leq 1$ and $\gamma_i = p_i + q_i(1-p_i)$; $i = 1,2$. $(\Omega, \mathcal{A})$ is measurable space, $u_a, v_a : \Omega \to \mathbb{R}^n$ are a measurable function, $f, g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ are given functions, ${}^\rho D_{a^+}^{p_i,q_i}$ is Hilfer-Katugampola fractional derivative of $p_i (0 \leq p_i \leq 1)$; $i = 1,2$ and type $q_i (0 \leq q_i \leq 1)$; $i = 1,2$ and ${}^\rho I_{a^+}^{1-\gamma_i}$ is generalized fractional integral of order $1 - \gamma_i (\gamma_i = p_i + q_i - p_i q_i)$.

The results are obtained upon some random fixed point theorems such as Perov's random fixed point theorem, the random nonlinear alternative of the Leray-Schauder type and Schauder's random fixed point theorem.

Main results

Lemma 4.1.1. *Let $f, g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}$, $i = 1,2$ such that $f(\cdot, u(\cdot,\vartheta), v(\cdot,\vartheta), \vartheta) \in C_{\gamma_1,\rho}$ and $g(\cdot, u(\cdot,\vartheta), v(\cdot,\vartheta), \vartheta) \in C_{\gamma_2,\rho}$ for all $\vartheta \in \Omega$ and any $u(\vartheta), v(\vartheta) \in \mathcal{C}$. Then the coupled systems (4.1)-(4.2) are equivalent to the problem of solution of the following system of fractional integral equations*

$$\begin{cases} u(t,\vartheta) &= \dfrac{u_a(\vartheta)}{\Gamma(\gamma_1)} \left(\dfrac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1 - 1} + \dfrac{1}{\Gamma(p_1)} \displaystyle\int_a^t \left(\dfrac{t^\rho - s^\rho}{\rho}\right)^{p_1 - 1} s^{\rho-1} f(s, u(s,\vartheta), v(s,\vartheta), \vartheta) ds, \\ v(t,\vartheta) &= \dfrac{v_a(\vartheta)}{\Gamma(\gamma_2)} \left(\dfrac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2 - 1} + \dfrac{1}{\Gamma(p_2)} \displaystyle\int_a^t \left(\dfrac{t^\rho - s^\rho}{\rho}\right)^{p_2 - 1} s^{\rho-1} g(s, u(s,\vartheta), v(s,\vartheta), \vartheta) ds. \end{cases} \tag{4.3}$$

Hypotheses:

Now, let us introduce the following assumptions.

(H1) The functions f, g are Carathéodory.

(H2) There exist measurable and bounded functions $k_i, l_i : \Omega \to (0,\infty); i = 1,2$ such that:

$$\|f(t,u_1,v_1,\vartheta) - f(t,u_2,v_2,\vartheta)\| \leq k_1(\vartheta)\|u_1 - u_2\| + l_1(\vartheta)\|v_1 - v_2\|,$$

and

$$\|g(t,u_1,v_1,\vartheta)-g(t,u_2,v_2,\vartheta)\|\leq k_2(\vartheta)\|u_1-u_2\|+l_2(\vartheta)\|v_1-v_2\|,$$

for a.e.$t\in J$, and each $u_i,v_i\in\mathbb{R}^n, i=1,2$.

(H3) There exist measurable and bounded functions $a_i,b_i:\Omega\to(0,\infty); i=1,2$ such that:

$$\|f(t,u,v,\vartheta)\|\leq a_1(\vartheta)\|u\|+b_1(\vartheta)\|v\|,$$

and

$$\|g(t,u,v,\vartheta)\|\leq a_2(\vartheta)\|u\|+b_2(\vartheta)\|v\|,$$

for a.e.$t\in J$, and each $u,v\in\mathbb{R}^n$.

4.2 Existence and Uniqueness

Theorem 4.2.1. *Assume that the hypotheses (H1) and (H2) hold. If for every $\vartheta\in\Omega$, the matrix*

$$M(\vartheta)=\begin{pmatrix}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1}\frac{\Gamma(\gamma_1)}{\Gamma(p_1+\gamma_1)}k_1(\vartheta) & \left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1+\gamma_2-\gamma_1}\frac{\Gamma(\gamma_2)}{\Gamma(p_1+\gamma_2)}l_1(\vartheta)\\ \left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_2+\gamma_1-\gamma_2}\frac{\Gamma(\gamma_1)}{\Gamma(p_2+\gamma_1)}k_2(\vartheta) & \left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_2}\frac{\Gamma(\gamma_2)}{\Gamma(p_2+\gamma_2)}l_2(\vartheta)\end{pmatrix},$$

converges to zero, then the coupled system (4.1)-(4.2) *has a unique random solution.*

Proof. From lemma 4.1.1, we define the operators $Q_1:\mathcal{C}\times\Omega\to C_{\gamma_1,\rho}$ and $Q_2:\mathcal{C}\times\Omega\to C_{\gamma_2,\rho}$ by

$$\begin{aligned}&\big(Q_1(u,v)\big)(t,\vartheta)\\ &=\frac{u_a(\vartheta)}{\Gamma(\gamma_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}+\frac{1}{\Gamma(p_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1}f(s,u(s,\vartheta),v(s,\vartheta),\vartheta)ds,\end{aligned}$$

and

$$\begin{aligned}&\big(Q_2(u,v)\big)(t,\vartheta)\\ &=\frac{v_a(\vartheta)}{\Gamma(\gamma_2)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma_2-1}+\frac{1}{\Gamma(p_2)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_2-1}s^{\rho-1}g(s,u(s,\vartheta),v(s,\vartheta),\vartheta)ds.\end{aligned}$$

Consider the operator $Q:\mathcal{C}\times\Omega\rightarrow\mathcal{C}$ defined by

$$(Q(u,v))(t,\vartheta)=((Q_1(u,v))(t,\vartheta),Q_2(u,v))(t,\vartheta)).$$

From (H1), we have f are Carathéodory functions; then $\vartheta\rightarrow f(t,u(t,\vartheta),v(t,\vartheta),\vartheta)$ are measurable functions for every $t\in J$, and the product $\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_2-1}s^{\rho-1}f(s,u(s,\vartheta),v(s,\vartheta),\vartheta)$ of continuous and a measurable function is also measurable, consequently,

$$\vartheta\rightarrow(Q_1(u,v))(t,\vartheta)\text{ and }\vartheta\rightarrow(Q_2(u,v))(t,\vartheta),$$

are measurable. As a result Q is a random operator on $\mathcal{C}\times\Omega$ into $\mathcal{C}$.

Now, we proof that the operator Q is contractive .
For all $\vartheta\in\Omega$, $(u_1,v_1),(u_2,v_2)\in\mathcal{C}$, and $t\in J$, we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}(Q_1(u_1,v_1))(t,\vartheta)-\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}(Q_1(u_2,v_2))(t,\vartheta)\right\|\\
&\le\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\frac{1}{\Gamma(p_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1}\\
&\times\left\|f\left(s,u_1(s,\vartheta),u_2(s,\vartheta),\vartheta\right)-f\left(s,v_1(s,\vartheta),v_2(s,\vartheta),\vartheta\right)\right\|ds\\
&\le\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\frac{1}{\Gamma(p_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1}\big(k_1(\vartheta)\|u_1(s,\vartheta)-u_2(s,\vartheta)\|\\
&+l_1(\vartheta)\|v_1(s,\vartheta)-v_2(s,\vartheta)\|\big)ds\\
&\le\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\frac{1}{\Gamma(p_1)}\bigg(k_1(\vartheta)\|u_1(\cdot,\vartheta)-u_2(\cdot,\vartheta)\|_{C_{\gamma_1,\rho}}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}\\
&\times s^{\rho-1}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma_1-1}ds+l_1(\vartheta)\|v_1(\cdot,\vartheta)-v_2(\cdot,\vartheta)\|_{C_{\gamma_2,\rho}}\\
&\times\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma_2-1}ds\bigg)\\
&\le\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1}\frac{\Gamma(\gamma_1)}{\Gamma(p_1+\gamma_1)}k_1(\vartheta)\|u_1(\cdot,\vartheta)-u_2(\cdot,\vartheta)\|_{C_{\gamma_1,\rho}}\\
&+\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1+\gamma_2-\gamma_1}\frac{\Gamma(\gamma_2)}{\Gamma(p_1+\gamma_2)}l_1(\vartheta)\|v_1(\cdot,\vartheta)-v_2(\cdot,\vartheta)\|_{C_{\gamma_2,\rho}}.
\end{aligned}$$

Therefore;

$$\begin{aligned}
&\Big\|\big(Q_1(u_1,v_1)\big)(.,\vartheta)-\big(Q_1(u_2,v_2)\big)(.,\vartheta)\Big\|_{C_{\gamma_1,\rho}}\\
&\le\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1}\frac{\Gamma(\gamma_1)}{\Gamma(p_1+\gamma_1)}k_1(\vartheta)\|u_1(\cdot,\vartheta)-u_2(\cdot,\vartheta)\|_{C_{\gamma_1,\rho}}\\
&+\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1+\gamma_2-\gamma_1}\frac{\Gamma(\gamma_2)}{\Gamma(p_1+\gamma_2)}l_1(\vartheta)\|v_1(\cdot,\vartheta)-v_2(\cdot,\vartheta)\|_{C_{\gamma_2,\rho}}.
\end{aligned}$$

In the same way of above inequality, we get

$$\begin{aligned}
&\Big\|\big(Q_2(u_1,v_1)\big)(\cdot,\vartheta)-\big(Q_2(u_2,v_2)\big)(\cdot,\vartheta)\Big\|_{C_{\gamma_2,\rho}}\\
&\le\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_2+\gamma_1-\gamma_2}\frac{\Gamma(\gamma_1)}{\Gamma(p_2+\gamma_1)}k_2(\vartheta)\|u_1(\cdot,\vartheta)-u_2(\cdot,\vartheta)\|_{C_{\gamma_1,\rho}}\\
&+\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_2}\frac{\Gamma(\gamma_2)}{\Gamma(p_2+\gamma_2)}l_2(\vartheta)\|v_1(\cdot,\vartheta)-v_2(\cdot,\vartheta)\|_{C_{\gamma_2,\rho}}.
\end{aligned}$$

Thus;

$$d\Big(\big(Q(u_1,v_1)\big)(\cdot,\vartheta),\big(Q(u_2,v_2)\big)(.,\vartheta)\Big)\le M(\vartheta)d\Big(\big(u_1(\cdot,\vartheta),v_1(\cdot,\vartheta)\big),\big(u_2(\cdot,\vartheta),v_2(\cdot,\vartheta)\big)\Big),$$

where,

$$d\Big(\big(u_1(\cdot,\vartheta),v_1(\cdot,\vartheta)\big),\big(u_2(\cdot,\vartheta),v_2(\cdot,\vartheta)\big)\Big)=\begin{pmatrix}\|u_1(\cdot,\vartheta)-u_2(\cdot,\vartheta)\|_{C_{\gamma_1,\rho}}\\ \|v_1(\cdot,\vartheta)-v_2(\cdot,\vartheta)\|_{C_{\gamma_2,\rho}}\end{pmatrix}$$

As for every $\vartheta\in\Omega$, the matrix $M(\vartheta)$ converges to zero, this implies that Q is a $M(\vartheta)$-contractive operator. Consequently, by theorem 1.5.4, we conclude that Q has a unique fixed point, which is a random solution of systems (4.1)-(4.2). This completes the proof. □

4.3 Existence result

Our next result is upon the fixed point theorem 1.5.5

Theorem 4.3.1. *Under the assumptions (H1) and (H3), the coupled systems* (4.1) *and* (4.2) *has at least one random solution.*

Proof. We need to proof that the operator Q satisfies all conditions of theorem 1.5.5. The proof is divided into four steps.

step 1. $Q(.,.,\vartheta)$ is continous.

Let (u_n,v_n) be a sequence such that $(u_n,v_n)\to(u,v)\in\mathcal{C}$ as $n\to\infty$. For all $\vartheta\in\Omega$, $t\in J$, we have

$$\begin{aligned}
&\left\|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}(Q_1(u_n,v_n))(t,\vartheta)-\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}(Q_1(u,v))(t,\vartheta)\right\|\\
&\leq\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\frac{1}{\Gamma(p_1)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1}\\
&\times\left\|f(s,u_n(s,\vartheta),v_n(s,\vartheta),\vartheta)-f(s,u(s,\vartheta),v(s,\vartheta),\vartheta)\right\|ds\\
&\leq\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_1}\frac{\Gamma(\gamma_1)}{\Gamma(p_1+\gamma_1)}\|f(\cdot,u_n(\cdot,\vartheta),v_n(\cdot,\vartheta),\vartheta)-f(\cdot,u(\cdot,\vartheta),v(\cdot,\vartheta),\vartheta)\|_{C_{\gamma_1,\rho}}.
\end{aligned}$$

Combining the Carathéodory property of f with the Lebesgue dominated convergence theorem, as $n\to+\infty$, we get

$$\left\|\big(Q_1(u_n,v_n)\big)(\cdot,\vartheta)-\big(Q_1(u,v)\big)(\cdot,\vartheta)\right\|_{C_{\gamma_1,\rho}}\to 0\to 0 \text{ as } n\to+\infty.$$

Likewise, we obtain

$$\begin{aligned}
&\left\|\big(Q_2(u_n,v_n)\big)(\cdot,\vartheta)-\big(Q_2(u,v)\big)(\cdot,\vartheta)\right\|_{C_{\gamma_2,\rho}}\\
&\leq\left(\frac{T^\rho-a^\rho}{\rho}\right)^{p_2}\frac{\Gamma(\gamma_2)}{\Gamma(p_2+\gamma_2)}\|g(\cdot,u_n(\cdot,\vartheta),v_n(\cdot,\vartheta),\vartheta)-g(\cdot,u(\cdot,\vartheta),v(\cdot,\vartheta),\vartheta)\|_{C_{\gamma_2,\rho}}.
\end{aligned}$$

As g is Carathéodory, we get

$$\left\|\big(Q_2(u_n,v_n)\big)(\cdot,\vartheta)-\big(Q_2(u,v)\big)(\cdot,\vartheta)\right\|_{C_{\gamma_2,\rho}}\to 0 \text{ as } n\to+\infty.$$

Moreover,

$$\left\|\big(Q(u_n,v_n)\big)(\cdot,\vartheta)-\big(Q(u,v)\big)(\cdot,\vartheta)\right\|_{\mathcal{C}}\to 0 \text{ as } n\to+\infty.$$

So the operator $Q(\cdot,\cdot,\vartheta)$ is continuous.

step 2. $Q(\cdot,\cdot,\vartheta)$ maps bounded sets into bounded sets in $\mathcal{C}$.

Let $r>0$,and

$$B_r:=\{(u,v)\in\mathcal{C}:\quad \|u\|_{C_{\gamma_1,\rho}}\leq r,\|v\|_{C_{\gamma_2,\rho}}\leq r\}.$$

For all $\vartheta \in \Omega, (u,v) \in B_r$ and $t \in J$, we have

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left(Q_1(u,v)\right)(t,\vartheta)\right\| \\
&\leq \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \frac{1}{\Gamma(p_1)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p_1-1} s^{\rho-1} \|f(s,u(s,\vartheta),v(s,\vartheta),\vartheta)\| ds \\
&\leq \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)} + \frac{1}{\Gamma(p_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \Bigg(a_1(\vartheta) \|u(\cdot,\vartheta)||_{C_{\gamma_1,\rho}} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p_1-1} \\
&\times s^{\rho-1} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{\gamma_1-1} ds + b_1(\vartheta) \|v(\cdot,\vartheta)\|_{C_{\gamma_2,\rho}} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p_1-1} \\
&\times s^{\rho-1} \left(\frac{s^\rho - a^\rho}{\rho}\right)^{\gamma_2-1} ds \Bigg) \\
&\leq \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)} + r \Bigg(\frac{\Gamma(\gamma_1)}{\Gamma(p_1+\gamma_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{p_1} a_1(\vartheta) + \frac{\Gamma(\gamma_2)}{\Gamma(p_1+\gamma_2)} \\
&\times \left(\frac{t^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_2-\gamma_1} b_1(\vartheta) \Bigg) = K_1(\vartheta),
\end{aligned}$$

therefore,

$$\left\| \left(Q_1(u,v)\right)(\cdot,\vartheta)\right\|_{C_{\gamma_1,\rho}} \leq K_1(\vartheta).$$

In similar way, we have

$$\begin{aligned}
\left\| \left(Q_2(u,v)\right)(\cdot,\vartheta)\right\|_{C_{\gamma_2,\rho}} &\leq \frac{\|v_a(\vartheta)\|}{\Gamma(\gamma_1)} + r \Bigg(\frac{\Gamma(\gamma_2)}{\Gamma(p_2+\gamma_1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_1-\gamma_2} a_2(\vartheta) + \frac{\Gamma(\gamma_2)}{\Gamma(p_2+\gamma_2)} \\
&\times \left(\frac{t^\rho - a^\rho}{\rho}\right)^{p_2} b_2(\vartheta) \Bigg) = K_2(\vartheta),
\end{aligned}$$

and thus

$$\left\| \left(Q(u,v)\right)(.,\vartheta)\right\|_{\mathcal{C}} \leq (K_1(\vartheta), K_2(\vartheta)) := K(\vartheta).$$

step 3. $Q(\cdot,\cdot,\vartheta)$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.

In this step, we proof that the map Q is completely continuous, that is Q is a map from bounded sets into equicontinuous sets of $\mathcal{C}$. For all $t_1, t_2 \in J$ with $t_1 \leq t_2$, and any $(u,v) \in B_r$, $\vartheta \in \Omega$, we have

$$\begin{aligned}
&\left\| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \big(Q_1(u,v)\big)(t_2,\vartheta) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \big(Q_1(u,v)\big)(t_1,\vartheta) \right\| \\
&\leq \frac{1}{\Gamma(p_1)} \int_a^{t_1} \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{p_1-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left(\frac{t_1^\rho - s^\rho}{\rho}\right)^{p_1-1} \right| \\
&\times s^{\rho-1} \| f(s,u(s,\vartheta),v(s,\vartheta),\vartheta) \| \, ds + \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \frac{1}{\Gamma(p_1)} \int_{t_1}^{t_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{p_1-1} \\
&\times s^{\rho-1} \| f(s,u(s,\vartheta),v(s,\vartheta),\vartheta) \| \, ds \\
&\leq \Bigg[a_1(\vartheta) r \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1 + p_1)} \Bigg(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_1-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_1-1} \right| \\
&+ \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_1} \right| \Bigg) + b_1(\vartheta) r \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 + p_1)} \Bigg(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \\
&\times \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_2-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_2-1} \right| + \Bigg| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_2-\gamma_1} \\
&- \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_1+\gamma_2-\gamma_1} \Bigg| \Bigg) \Bigg] \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

And similarly

$$\begin{aligned}
&\left\| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \big(Q_2(u,v)\big)(t_2,\vartheta) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \big(Q_2(u,v)\big)(t_1,\vartheta) \right\| \\
&\leq \Bigg[b_2(\vartheta) r \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 + p_2)} \Bigg(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_2-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_2-1} \right| \\
&+ \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_2} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_2} \right| \Bigg) + a_2(\vartheta) r \frac{\Gamma(\gamma_1)}{\Gamma(\gamma_1 + p_2)} \Bigg(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2}
\end{aligned}$$

$$\times \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_1-1} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_1-1} \right| + \left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_1-\gamma_2} - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{p_2+\gamma_1-\gamma_2} \right| \Bigg) \Bigg] \to 0 \text{ as } t_1 \to t_2.$$

Finally, from the previous **steps 1-3**, with the Arzelà-Ascoli theorem, we conclude that $Q(\cdot,\cdot,\vartheta)$ maps B_r into a precompact in $\mathcal{C}$.

step 4. Priori estimate

Let

$$E(\vartheta) = \{(u(\cdot,\vartheta), v(\cdot,\vartheta)) \in \mathcal{C} : (u(\cdot,\vartheta), v(\cdot,\vartheta)) = \sigma(\vartheta)(Q((u,v))(.,\vartheta)\},$$

for some measurable functions $\sigma : \Omega \to (0,1)$.

In this step, we need to prove that the set $E(\vartheta)$ is bounded in $\mathcal{C}$.

Let $(u(\cdot,\vartheta), v(\cdot,\vartheta)) \in E(\vartheta)$. Then, $u(\cdot,\vartheta) = \sigma(\vartheta)(Q_1((u,v))(\cdot,\vartheta)$ and $v(\cdot,\vartheta) = \sigma(\vartheta)(Q_2((u,v))(\cdot,\vartheta)$. Thus, for any $\vartheta \in \Omega$ and each $t \in J$, we have:

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} u(t,\vartheta) \right\| \\
&\le \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \frac{1}{\Gamma(p_1)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p_1-1} s^{\rho-1} \|f(s,u(s,\vartheta),v(s,\vartheta),\vartheta)\| \, ds \\
&\le \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_2} \frac{1}{\Gamma(p_1)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p_1-1} s^{\rho-1} \Bigg(a_1(\vartheta) \left\| \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} u(s,\vartheta) \right\| \\
&\qquad + b_1(\vartheta) \left\| \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} v(s,\vartheta) \right\| \Bigg) ds.
\end{aligned}$$

In same way, get:

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} v(t,\vartheta) \right\| \\
&\le \frac{\|v_a(\vartheta)\|}{\Gamma(\gamma_2)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma_1} \frac{1}{\Gamma(p_2)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{p_2-1} s^{\rho-1} \Bigg(a_2(\vartheta) \left\| \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} u(s,\vartheta) \right\| \\
&+ b_2(\vartheta) \left\| \left(\frac{s^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} v(s,\vartheta) \right\| \Bigg) ds.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} u(t,\vartheta) \right\| + \left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} v(t,\vartheta) \right\| \\
&\leq C + h(t) \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{p-1} s^{\rho-1} \left(\left\| \left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} u(s,\vartheta) \right\| \right. \\
&\left. + \left\| \left(\frac{s^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} v(s,\vartheta) \right\| \right) ds.
\end{aligned}$$

with

$$C = \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)} + \frac{\|v_a(\vartheta)\|}{\Gamma(\gamma_2)}, \quad p = max\{p_1, p_2\},$$

$$h(t) = \left(\frac{1}{\Gamma(p_1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma_2} + \frac{1}{\Gamma(p_2)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma_1} \right) max\{a_1(\vartheta) + a_2(\vartheta), b_1(\vartheta) + b_2(\vartheta)\}.$$

From lemma 1.5.8 we have

$$\begin{aligned}
&\left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_1} u(t,\vartheta) \right\| + \left\| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma_2} v(t,\vartheta) \right\| \\
&\leq C + \int_a^t \sum_{n=1}^{\infty} \frac{(h(t)\Gamma(p))^n}{\Gamma(np)} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{np-1} s^{\rho-1} C ds. \\
&\leq C \left(\sum_{n=1}^{\infty} \frac{(h(T)\Gamma(p))^n}{\Gamma(np+1)} \left(\frac{T^\rho - s^\rho}{\rho} \right)^{np} + 1 \right) < \infty.
\end{aligned}$$

Thus,

$$\left\| (u(.,\vartheta), v(.,\vartheta)) \right\|_{\mathcal{C}} < \infty.$$

Finally, as a consequence of **Steps 1-4** the operator Q is completely continuous and the set $E(\vartheta)$ is bounded. Then by theorem 1.5.5 the operator Q has a fixed point in $\mathcal{C}$, which is a random solution for the systems (4.1)-(4.2). This completes the proof. □

4.4 Example

We illustrate our results by an example. Let $\Omega = \mathbb{R}_-^* = (-\infty, 0)$ be equipped by the usual $\sigma-$algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the following random coupled Hilfer-Katugampola fractional differential system

$$\begin{cases} ({}^2D_{0^+}^{\frac{1}{2},\frac{1}{2}}u)(t,\vartheta) = f(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \\ ({}^2D_{0^+}^{\frac{1}{2},\frac{1}{2}}v)(t,\vartheta) = h(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \\ ({}^2I_{0^+}^{\frac{1}{4}}u)(0,\vartheta) = \cos(\vartheta) \\ ({}^2I_{0^+}^{\frac{1}{4}}v)(0,\vartheta) = \sin(\vartheta) \end{cases} ;t \in [0,1], \vartheta \in \Omega, \tag{4.4}$$

where:

$$f(t,u(t,\vartheta),v(t,\vartheta),\vartheta) = \frac{t^{-\frac{1}{4}}\vartheta^2(u(t)+v(t))\sin t}{64(1+\vartheta^2+\sqrt{t})(1+|u|+|v|)}; \qquad t \in [0,1], \vartheta \in \Omega,$$

and

$$h(t,u(t,\vartheta),v(t,\vartheta),\vartheta) = \frac{\vartheta^2(u(t)+v(t))\cos t}{64(1+|u|+|v|)}; \qquad t \in [0,1], \vartheta \in \Omega.$$

Clearly, the functions f and h are Carathéodory. The hypothesis (H2) is satisfied with the following measurable functions:

$$a_1(\vartheta) = a_2(\vartheta) = b_1(\vartheta) = b_2(\vartheta) = \frac{\vartheta^2}{64(1+\vartheta^2)}; \qquad \vartheta \in \Omega.$$

For all $\vartheta \in \Omega$, the matrix A is defined as follows

$$M(\vartheta) = \frac{\vartheta^2}{64\sqrt{2}(1+\vartheta^2)\Gamma(\frac{3}{2})} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Where the eigenvalues of the matrix M are:

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{\vartheta^2}{32\sqrt{2}(1+\vartheta^2)\Gamma(\frac{3}{2})} \simeq 0.0249\frac{\vartheta^2}{1+\vartheta^2} < 1.$$

Thus, $M(\vartheta)$ is converges to zero, then by theorem 4.2.1 the coupled system (4.4) has unique random solution on $[0,1]$.

4.5 Random Coupled Fractional Differential Systems in Banach space

In this section, we will investigate the same previous system (4.1)-(4.2), but in the case $n = 1$ and that is due to the nature of the hypotheses that we will use to prove the next result. Now the system's data in this case defined as follows

$$\begin{cases} ({}^{\rho}D_{a^+}^{p_1,q_1}u)(t,\vartheta) = f(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \\ \\ ({}^{\rho}D_{a^+}^{p_2,q_2}v)(t,\vartheta) = g(t,u(t,\vartheta),v(t,\vartheta),\vartheta) \end{cases} ;t \in J = [a,T], \vartheta \in \Omega, \tag{4.5}$$

with the following initial conditions:

$$\begin{cases} ({}^{\rho}I_{a^+}^{1-\gamma_1}u)(a,\vartheta) = u_a(\vartheta) \\ \\ ({}^{\rho}I_{a^+}^{1-\gamma_2}v)(a,\vartheta) = v_a(\vartheta) \end{cases} ;\vartheta \in \Omega, \tag{4.6}$$

where $\Omega \in \mathbb{R}$, $u_a, v_a : \Omega \to \mathbb{R}$ are a measurable function, $f, g : J \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$.

4.6 Existence result

This result is based on Itoh's random fixed point theorem.

Theorem 4.6.1. *Assume that the hypotheses (H1) and the following hypotheses hold*

(H′1) There exit measurable and bounded functions $c_i, d_i : \Omega \to L^{\infty}(J, [0,\infty))$; $i = 1, 2$, such that

$$|f(t,u,v,\vartheta)| \leq \frac{c_1(t,\vartheta)|u| + d_1(t,\vartheta)|v|}{1 + |u| + |v|},$$

and

$$|g(t,u,v,\vartheta)| \leq \frac{c_2(t,\vartheta)|u| + d_2(t,\vartheta)|v|}{1 + |u| + |v|},$$

for a.e $t \in J$ and each $u, v \in \mathbb{R}$, $\vartheta \in \Omega$.

Then the systems (4.5) and (4.6) has at least one random solution defined on $J \times \Omega$.

Proof. We set

$$\begin{aligned}
&\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\big(Q_1(u,v)\big)(t,\vartheta)\right| \\
&\le \frac{\|u_a(\vartheta)\|}{\Gamma(\gamma_1)}+\frac{1}{\Gamma(p_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1} \\
&\times|f(s,u(s,\vartheta),v(s,\vartheta),\vartheta)|ds \\
&\le \frac{|u_a(\vartheta)|}{\Gamma(\gamma_1)}+\frac{1}{\Gamma(p_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1} \\
&\times\frac{|c_1(s,\vartheta)||u(s,\vartheta)|+|d_1(s,\vartheta)||v(s,\vartheta)|}{1+|u(s,\vartheta)|+|v(s,\vartheta)|}ds \\
&\le \frac{|u_a(\vartheta)|}{\Gamma(\gamma_1)}+\frac{1}{\Gamma(p_1)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma_1}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{p_1-1}s^{\rho-1} \\
&\times\big(\|c_1(\cdot,\vartheta)\|_\infty+\|d_1(\cdot,\vartheta)\|_\infty\big)ds \\
&\le \frac{|u_a(\vartheta)|}{\Gamma(\gamma_1)}+\frac{\|c_1(\cdot,\vartheta)\|_\infty+\|d_1(\cdot,\vartheta)\|_\infty}{\Gamma(p_1+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma_1+p_1}.
\end{aligned}$$

Similarly, we get

$$\left\|\big(Q_2(u,v)\big)(\cdot,\vartheta)\right\|_{C_{\gamma_2}}\le\frac{|v_a(\vartheta)|}{\Gamma(\gamma_2)}+\frac{\|c_2(\cdot,\vartheta)\|_\infty+\|d_2(\cdot,\vartheta)\|_\infty}{\Gamma(p_2+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma_2+p_2}.$$

Therefore

$$\left\|\big(Q(u,v)\big)(\cdot,\vartheta)\right\|_{\mathcal{C}}\le\frac{|u_a(\vartheta)|}{\Gamma(\gamma_1)}+\frac{|v_a(\vartheta)|}{\Gamma(\gamma_2)}+\sum_{i=1}^{2}\frac{\|c_i(\cdot,\vartheta)\|_\infty+\|d_i(\cdot,\vartheta)\|_\infty}{\Gamma(p_i+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma_i+p_i}:=d(\vartheta),$$

We set

$$B_d=B(0,d)=\{(u,v)\in\mathcal{C}:\|(u,v)\|_{\mathcal{C}}\le d(\vartheta)\},$$

where B_d is a closed, bounded and convex subset of $\mathcal{C}$.

We need to prove that the operator $Q:B_d\times\Omega\to B_d$ satisfies the conditions of theorem 1.5.6.

step 1. Clearly, $Q:B_d\times\Omega\to B_d$ is a random continuous operator.

step 2. $Q(B_d)$ is uniformly bounded.

Since, $Q(u,v,\vartheta)\subset B_d$, for all $(u,v)\in B_d$, and B_d is bounded.

step 3. $Q(B_d)$ is relatively compact.

For all $t_1, t_2 \in J, t_1 \le t_2$, and any $(u,v) \in B_d$, $\vartheta \in \Omega$, we have

$$\begin{aligned}
&\left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} (Q_1(u,v))(t_2,\vartheta) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} (Q_1(u,v))(t_1,\vartheta) \right| \\
&\le \left[\frac{\|c_1(\cdot,\vartheta)\|_\infty + \|d_1(\cdot,\vartheta)\|_\infty}{\Gamma(p_1+1)} \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{p_1} \right. \\
&+ \frac{\|c_1(\cdot,\vartheta)\|_\infty + \|d_1(\cdot,\vartheta)\|_\infty}{\Gamma(p_1)} \int_a^{t_1} \left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{p_1-1} \right. \\
&- \left. \left. \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_1} \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{p_1-1} \right) s^{\rho-1} ds \right] \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

and

$$\begin{aligned}
&\left| \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} (Q_2(u,v))(t_2,\vartheta) - \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} (Q_2(u,v))(t_1,\vartheta) \right| \\
&\le \left[\frac{\|c_2(\cdot,\vartheta)\|_\infty + \|d_2(\cdot,\vartheta)\|_\infty}{\Gamma(p_2+1)} \left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{p_2} \right. \\
&+ \frac{\|c_2(\cdot,\vartheta)\|_\infty + \|d_2(\cdot,\vartheta)\|_\infty}{\Gamma(p_2)} \int_a^{t_1} \left(\left(\frac{t_2^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \left(\frac{t_2^\rho - s^\rho}{\rho}\right)^{p_2-1} \right. \\
&- \left. \left. \left(\frac{t_1^\rho - a^\rho}{\rho}\right)^{1-\gamma_2} \left(\frac{t_2^\rho - t_1^\rho}{\rho}\right)^{p_2-1} \right) s^{\rho-1} ds \right] \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

As a consequence of **steps 1-3** and using the Ascoli-Arzela theorem, we deduce $Q : B_d \times \Omega \to B_d$ is continuous, compact and satisfies the assumption of theorem 1.5.6. Then the operator Q has a fixed point which is a random solution of the systems (4.5)-(4.6) on J. □

CHAPTER 5

RANDOM MULTI-FRACTIONAL DIFFERENTIAL COUPLED SYSTEM IN GENERALIZED BANACH SPACE

5.1 Introduction

This Chapter, aims to investigate a random coupled system with multiple fractional derivatives of $\psi-$Caputo of different orders subject to non-local integral and boundary condition and proves the uniqueness of random solution by applying random versions of the Pervo fixed point theorem, while the existence of solutions is derived by a random version of a Krasnoselskii-type fixed point theorem. Also, we study the Ulam-Hyres stability of the proposed problem. The stability analysis of functional and differential equations is very useful in various applications. Considerable attention has been paid to the study of different kinds of Ulam stability. For details, see [12, 35, 27, 52, 55]. To the best of our knowledge, the Ulam–Hyers stability has been very rarely studied for a random coupled system of fractional differential equations in generalized Banach space. Therefore, in this chapter, we study the existence, uniqueness and Ulam–Hyers stability results to the following nonlinear random multi-fractional equations. [1]

[1]**F. Fredj**, H. Hammouche, :Existence and Ulam-Hyres stability of random coupled system of multi-fractional differential equations in generalized Banach space (submitted).

$$\begin{cases} {}^cD_{0^+}^{p_1;\psi}[{}^cD^{q_1;\psi}u(t,\vartheta) - h(t,u_t(\vartheta),v_t(\vartheta),\vartheta)] = f(t,u_t(\vartheta),v_t(\vartheta),{}^cD_{0^+}^{\delta_1;\psi}v(t,\vartheta),\vartheta), \\ {}^cD_{0^+}^{p_2;\psi}[{}^cD^{q_2;\psi}v(t,\vartheta) - k(t,u_t(\vartheta),v_t(\vartheta),\vartheta)] = g(t,u_t(\vartheta),{}^cD_{0^+}^{\delta_2;\psi}u(t,\vartheta),v_t(\vartheta),\vartheta), \end{cases} \tag{5.1}$$

subject to the following coupled non-local integral and boundary condition

$$\begin{cases} u(0,\vartheta) = \chi(v(\vartheta)), \quad D_\psi u(0,\vartheta) = 0, \quad \int_0^T v(\tau,\vartheta)d\tau = \kappa_2 u(\xi,\vartheta) \\ v(0,\vartheta) = \varphi(u(\vartheta)), \quad D_\psi v(0,\vartheta) = 0, \quad \int_0^T u(\tau,\vartheta)d\tau = \kappa_1 v(\rho,\vartheta) \end{cases} ;\xi,\rho \in J, \tag{5.2}$$

where $t \in J = [0,T], \vartheta \in \Omega$ and $u_t(\vartheta) = u(t,\vartheta)$. ${}^cD_{0^+}^{q_i;\psi}$ and ${}^cD_{0^+}^{\sigma_i;\psi}$ are the $\psi-$Caputo derivative of order $1 < q_i \leq 2$ and $0 < \sigma_i \leq 1(\sigma_i \in \{q_i,\delta\}\ ;i = 1,2)$ respectively. $(\Omega,\mathcal{A})$ is measurable space. $f,g : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $h,k : J \times \mathbb{R}^n \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ are given functions. $\varphi,\chi : \mathbb{R}^n \to \mathbb{R}^n$ are given continuous function, and κ_i are real constants ; $i = 1,2$. The differential operator D_ψ is defined by

$$D_\psi = \frac{1}{\psi'(t)}\frac{d}{dt}.$$

Lemma 5.1.1. *Let* $0 < p_1,p_2 < 1$, $1 < q_1,q_2 < 2$. *For any functions* $H,K,F,G \in C(J,\mathbb{R})$, *the following linear fractional boundary value problem*

$$\begin{cases} {}^cD_{0^+}^{p_1;\psi}[{}^cD_{0^+}^{q_1;\psi}u(t) - H(t)] = F(t), \quad t \in J, \\ {}^cD_{0^+}^{p_2;\psi}[{}^cD_{0^+}^{q_2;\psi}v(t) - F(t)] = G(t), \quad t \in J, \\ u(0) = \chi(v), \quad D_\psi u(0) = 0, \quad \int_0^T v(\tau)d\tau = \kappa_2 u(\xi), \qquad ;\xi,\rho \in J, \\ v(0) = \varphi(u), \quad D_\psi v(0) = 0, \quad \int_0^T u(\tau)d\tau = \kappa_1 v(\rho), \end{cases} \tag{5.3}$$

has a unique solution, which is given by

$$\begin{aligned} u(t) &= I_{0^+}^{q_1;\psi} H(t) + I_{0^+}^{p_2+q_2;\psi} F(t) + \frac{\Psi_0^{q_1}(t)}{\Delta_1 \Gamma(q_1+1)} \left[\frac{\int_0^T \Psi_0^{q_2}(\tau) d\tau}{\Psi_0^{q_2}(\xi)} \left(I_{0^+}^{q_2;\psi} K(\xi) \right.\right. \\ &+ I_{0^+}^{p_2+q_2;\psi} G(\xi) - \frac{1}{\kappa_2} \left(\left. \int_0^T I_{0^+}^{q_1;\psi} H(\tau) d\tau + \int_0^T I_{0^+}^{p_2+q_2;\psi} F(\tau) d\tau \right) \right) \\ &+ \kappa_1 \left(I_{0^+}^{q_1;\psi} H(\rho) + I_{0^+}^{p_2+q_2;\psi} F(\rho) \right) - \int_0^T I_{0^+}^{q_2;\psi} K(\tau) d\tau \\ &\left. - \int_0^T I_{0^+}^{p_2+q_2;\psi} G(\tau) d\tau \right] + (\Psi_0^{q_1}(t)\lambda_1 + 1)\chi(v) + \lambda_2 \Psi_0^{q_1}(t) \varphi(u). \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} v(t) &= I_{0^+}^{q_2;\psi} K(t) + I_{0^+}^{p_2+q_2;\psi} G(t) + \frac{\Psi_0^{q_2}(t)}{\Delta_2 \Gamma(q_2+1)} \left[\frac{\int_0^T \Psi_0^{q_1}(\tau) d\tau}{\Psi_0^{q_2}(\rho)} \left(I_{0^+}^{q_1;\psi} H(\rho) \right.\right. \\ &+ I_{0^+}^{p_2+q_2;\psi} F(\rho) - \frac{1}{\kappa_1} \left(\left. \int_0^T I_{0^+}^{q_2;\psi} K(\tau) d\tau + \int_0^T I_{0^+}^{p_2+q_2;\psi} G(\tau) d\tau \right) \right) \\ &+ \kappa_2 \left(I_{0^+}^{q_2;\psi} K(\xi) + I_{0^+}^{p_2+q_2;\psi} G(\xi) \right) - \int_0^T I_{0^+}^{q_1;\psi} H(\tau) d\tau \\ &\left. - \int_0^T I_{0^+}^{p_2+q_2;\psi} F(\tau) d\tau \right] + (\Psi_0^{q_2}(t)\lambda_3 + 1)\varphi(u) + \lambda_4 \Psi_0^{q_2}(t) \chi(v). \end{aligned} \tag{5.5}$$

where

$$\Delta_1 = \frac{\int_0^T \Psi_0^{q_1}(\tau) d\tau \int_0^T \Psi_0^{q_2}(\tau) d\tau - \kappa_1 \kappa_2 \Psi_0^{q_1}(\rho) \Psi_0^{q_2}(\xi)}{\kappa_2 \Psi_0^{q_2}(\xi) \Gamma(q_1+1)},$$

$$\lambda_1 = \frac{\kappa_1 \kappa_2 \Psi_0^{q_2}(\xi) - T \int_0^T \Psi_0^{q_2}(\tau) d\tau}{\kappa_2 \Psi_0^{q_2}(\xi) \Gamma(q_1+1) \Delta_1}, \quad \lambda_2 = \frac{\int_0^T \Psi_0^{q_2}(\tau) d\tau - T \Psi_0^{q_2}(\xi)}{\Psi_0^{q_2}(\xi) \Gamma(q_1+1) \Delta_1},$$

$$\Delta_2 = \frac{\int_0^T \Psi_0^{q_1}(\tau) d\tau \int_0^T \Psi_0^{q_2}(\tau) d\tau - \kappa_1 \kappa_2 \Psi_0^{q_1}(\rho) \Psi_0^{q_2}(\xi)}{\kappa_1 \Psi_0^{q_1}(\rho) \Gamma(q_2+1)},$$

$$\lambda_3=\frac{\kappa_1\kappa_2\Psi_0^{q_1}(\rho)-T\displaystyle\int_0^T\Psi_0^{q_1}(\tau)d\tau}{\kappa_1\Psi_0^{q_1}(\rho)\Gamma(q_2+1)\Delta_2},\quad \lambda_4=\frac{\displaystyle\int_0^T\Psi_0^{q_1}(\tau)d\tau-T\Psi_0^{q_1}(\rho)}{\Psi_0^{q_1}(\rho)\Gamma(q_2+1)\Delta_2}.$$

Proof. According to lemma 1.2.10, the general solutions of linear boundary value problem (5.3) can expressed by

$$u(t)=I_{0^+}^{q_1;\psi}H(t)+I_{0^+}^{p_1+q_1;\psi}F(t)+\frac{\Psi_0^{q_1}(t)}{\Gamma(q_1+1)}c_0+c_1\Psi_0(t)+c_2, \tag{5.6}$$

and

$$v(t)=I_{0^+}^{q_2;\psi}K(t)+I_{0^+}^{p_2+q_2;\psi}G(t)+\frac{\Psi_0^{q_2}(t)}{\Gamma(q_2+1)}d_0+d_1\Psi_0(t)+d_2, \tag{5.7}$$

where $c_i,d_i\in\mathbb{R}(i=1,2)$ are arbitrary constants.
With the help of conditions $u(0)=\chi(v)$, $v(0)=\varphi(u)$ and $D_\psi u(0)=0$, $D_\psi v(0)=0$, we find $c_2=\chi(v)$, $d_2=\varphi(u)$ and $c_1=0$, $d_1=0$ respectively. Applying the boundary conditions $\int_0^T v(\tau)d\tau=\kappa_1 u(\xi)$ and $\int_0^T u(\tau)d\tau=\kappa_2 u(\rho)$, from (5.6) and (5.7), we have

$$\begin{aligned}\frac{\displaystyle\int_0^T\Psi_0^{q_2}(\tau)d\tau}{\Gamma(q_2+1)}d_0-\frac{\kappa_1\Psi_0^{q_1}(\rho)}{\Gamma(q_1+1)}c_0&=\kappa_1\big(I_{0^+}^{q_1;\psi}H(\rho)+I_{0^+}^{p_2+q_2;\psi}F(\rho)+\chi(v)\big)\\&\quad-\int_0^T I_{0^+}^{q_2;\psi}K(\tau)d\tau-\int_0^T I_{0^+}^{p_2+q_2;\psi}G(\tau)d\tau-T\varphi(u),\end{aligned}$$

and

$$\begin{aligned}\frac{\displaystyle\int_0^T\Psi_0^{q_1}(\tau)d\tau}{\Gamma(q_1+1)}c_0-\frac{\kappa_2\Psi_0^{q_2}(\xi)}{\Gamma(q_2+1)}d_0&=\kappa_2\big(I_{0^+}^{q_2;\psi}K(\xi)+I_{0^+}^{p_2+q_2;\psi}G(\xi)+\varphi(u)\big)\\&\quad-\int_0^T I_{0^+}^{q_1;\psi}H(\tau)d\tau-\int_0^T I_{0^+}^{p_2+q_2;\psi}F(\tau)d\tau-T\chi(v).\end{aligned}$$

Solving the resulting equations for c_0 and d_0, we find that

$$\begin{aligned}c_0=\frac{\displaystyle\int_0^T\Psi_0^{q_2}(\tau)d\tau}{\Delta_1\Psi_0^{q_2}(\xi)}\Bigg[&I_{0^+}^{q_2;\psi}K(\xi)+I_{0^+}^{p_2+q_2;\psi}G(\xi)+\varphi(u(\vartheta))-\frac{1}{\kappa_2}\bigg(\int_0^T I_{0^+}^{q_1;\psi}H(\tau)d\tau\\&+\int_0^T I_{0^+}^{p_2+q_2;\psi}F(\tau)d\tau+T\chi(v(\vartheta))\bigg)\Bigg]+\frac{\kappa_1}{\Delta_1}\bigg(I_{0^+}^{q_1;\psi}H(\rho)+I_{0^+}^{p_2+q_2;\psi}F(\rho)\end{aligned}$$

$$+\chi(v(\vartheta))\Big)-\frac{1}{\Delta_1}\Big(\int_0^T I_{0^+}^{q_2;\psi}K(\tau)d\tau+\int_0^T I_{0^+}^{p_2+q_2;\psi}G(\tau)d\tau+T\varphi(u(\vartheta))\Big),$$

$$\begin{aligned}d_0=&\frac{\int_0^T\Psi_0^{q_1}(\tau)d\tau}{\Delta_2\Psi_0^{q_1}(\rho)}\Big[I_{0^+}^{q_1;\psi}H(\rho)+I_{0^+}^{p_2+q_2;\psi}F(\rho)+\chi(v(\vartheta))-\frac{1}{\kappa_1}\Big(\int_0^T I_{0^+}^{q_2;\psi}K(\tau)d\tau\\&+\int_0^T I_{0^+}^{p_2+q_2;\psi}G(\tau)d\tau+T\varphi(u(\vartheta))\Big)\Big]+\frac{\kappa_2}{\Delta_2}\Big(I_{0^+}^{q_2;\psi}K(\xi)+I_{0^+}^{p_2+q_2;\psi}G(\xi)\\&+\varphi(u(\vartheta))\Big)-\frac{1}{\Delta_2}\Big(\int_0^T I_{0^+}^{q_1;\psi}H(\tau)d\tau+\int_0^T I_{0^+}^{p_2+q_2;\psi}F(\tau)d\tau+T\chi(u)\Big).\end{aligned}$$

Inserting c_0, c_1, c_2, d_0, d_1 and d_2 in (5.6) and (5.7), which leads to the solution system (5.4) and (5.5). □

Main results

Lemma 5.1.2. *For given functions $f,h,g,k\in C(J,\mathbb{R}^n)$, $i=1,2$. A functions $u,v\in C^2$ is a random solution of systems (5.1)-(5.2) if and only if u,v satisfies the following random coupled system integral equations*

$$\begin{aligned}&u(t,\vartheta)\\&=I_{0^+}^{q_1;\psi}h(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(t,\vartheta)+I_{0^+}^{p_1+q_1;\psi}f(s,u_s(\vartheta),v_s(\vartheta),{}^cD_{0^+}^{\delta_1;\psi}v(s,\vartheta),\vartheta)(t,\vartheta)\\&+\frac{\Psi_0^{q_1}(t)}{\Delta_1\Gamma(q_1+1)}\Bigg[\frac{\int_0^T\Psi_0^{q_2}(\tau)d\tau}{\Psi_0^{q_2}(\xi)}\Bigg(I_{0^+}^{q_2;\psi}k(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\xi,\vartheta)\\&+I_{0^+}^{p_2+q_2;\psi}g(s,u_s(\vartheta),{}^cD_{0^+}^{\delta_2;\psi}u(s,\vartheta),v_s(\vartheta),\vartheta)(\xi,\vartheta)-\frac{1}{\kappa_2}\Big(\int_0^T I_{0^+}^{q_1;\psi}h(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\tau,\vartheta)d\tau\\&+\int_0^T I_{0^+}^{p_1+q_1;\psi}f(s,u_s(\vartheta),v_s(\vartheta),{}^cD_{0^+}^{\delta_1;\psi}v(s,\vartheta),\vartheta)(\tau,\vartheta)d\tau\Big)\Bigg)+\kappa_1\Big(I_{0^+}^{q_1;\psi}h(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\rho,\vartheta)\\&+I_{0^+}^{p_1+q_1;\psi}f(s,u_s(\vartheta),v_s(\vartheta),{}^cD_{0^+}^{\delta_1;\psi}v(s,\vartheta),\vartheta)(\rho,\vartheta)\Big)-\int_0^T I_{0^+}^{q_2;\psi}k(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\tau,\vartheta)d\tau\\&-\int_0^T I_{0^+}^{p_2+q_2;\psi}g(s,u_s(\vartheta),{}^cD_{0^+}^{\delta_2;\psi}u(s,\vartheta),v_s(\vartheta),\vartheta)(\tau,\vartheta)d\tau\Bigg]+(\Psi_0^{q_1}(t)\lambda_1+1)\chi(v(\vartheta))\\&+\lambda_2\Psi_0^{q_1}(t)\varphi(u(\vartheta)),\end{aligned}\tag{5.8}$$

and

$$
\begin{aligned}
& v(t,\vartheta) \\
& = I_{0^+}^{q_2;\psi} k(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(t,\vartheta) + I_{0^+}^{p_2+q_2;\psi} g(s,u_s(\vartheta),{}^c D_{0^+}^{\delta_2;\psi} u(s,\vartheta),v_s(\vartheta),\vartheta)(t,\vartheta) \\
& + \frac{\Psi_0^{q_2}(t)}{\Delta_2 \Gamma(q_2+1)} \Bigg[\frac{\int_0^T \Psi_0^{q_1}(\tau) d\tau}{\Psi_0^{q_1}(\rho)} \Bigg(I_{0^+}^{q_1;\psi} h(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\rho,\vartheta) \\
& + I_{0^+}^{p_1+q_1;\psi} f(s,u_s(\vartheta),v_s(\vartheta),{}^c D_{0^+}^{\delta_1;\psi} v(s,\vartheta),\vartheta)(\rho,\vartheta) - \frac{1}{\kappa_1} \bigg(\int_0^T I_{0^+}^{q_2;\psi} k(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\tau,\vartheta) d\tau \\
& + \int_0^T I_{0^+}^{p_2+q_2;\psi} g(s,u_s(\vartheta),{}^c D_{0^+}^{\delta_2;\psi} u(s,\vartheta),v_s(\vartheta),\vartheta)(\tau,\vartheta) d\tau \bigg) \Bigg) + \kappa_2 \Big(I_{0^+}^{q_2;\psi} k(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\xi,\vartheta) \\
& + I_{0^+}^{p_2+q_2;\psi} g(s,u_s(\vartheta),{}^c D_{0^+}^{\delta_2;\psi} u(s,\vartheta),v_s(\vartheta),\vartheta)(\xi,\vartheta) \Big) - \int_0^T I_{0^+}^{q_1;\psi} h(s,u_s(\vartheta),v_s(\vartheta),\vartheta)(\tau,\vartheta) d\tau \\
& - \int_0^T I_{0^+}^{p_1+q_1;\psi} f(s,u_s(\vartheta),v_s(\vartheta),{}^c D_{0^+}^{\delta_1;\psi} v(s,\vartheta),\vartheta)(\tau,\vartheta) d\tau \Bigg] + (\Psi_0^{q_2}(t)\lambda_3 + 1)\varphi(u(\vartheta)) \\
& + \lambda_4 \Psi_0^{q_2}(t) \chi(v(\vartheta)),
\end{aligned}
\tag{5.9}
$$

For $0 < \delta_1, \delta_2 < 1$. Let us introduce the spaces

$$E = \{u(t,\vartheta) : u(t,\vartheta) \in C(J,\mathbb{R}^n) \text{ and } {}^c D_{0^+}^{\delta_2;\psi} u(t,\vartheta) \in C(J,\mathbb{R}^n)\},$$

$$F = \{v(t,\vartheta) : v(t,\vartheta) \in C(J,\mathbb{R}^n) \text{ and } {}^c D_{0^+}^{\delta_1;\psi} v(t,\vartheta) \in C(J,\mathbb{R}^n)\},$$

endowed with the norm

$$\|u(\cdot,\vartheta)\|_E = \|u(\cdot,\vartheta)\|_\infty + \|{}^c D_{0^+}^{\delta_2;\psi} u(\cdot,\vartheta)\|_\infty = \sup_{t\in J} \|u(t,\vartheta)\| + \sup_{t\in J} \|{}^c D_{0^+}^{\delta_2;\psi} u(t,\vartheta)\|.$$

$$\|v(\cdot,\vartheta)\|_F = \|(\cdot,\vartheta)\|_\infty + \|{}^c D_{0^+}^{\delta_1;\psi} v(\cdot,\vartheta)\|_\infty = \sup_{t\in J} \|v(t,\vartheta)\| + \sup_{t\in J} \|{}^c D_{0^+}^{\delta_1;\psi} v(t,\vartheta)\|.$$

It is clear that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are Banach spaces. It follows that the product space $(E \times F, \|\cdot\|_{E\times F})$ is a Banach space with norm

$$\|(u(\cdot,\vartheta), v(\cdot,\vartheta))\|_{E\times F} = \|u(\cdot,\vartheta)\|_E + \|v(\cdot,\vartheta)\|_F, \qquad u,v \in E \times F.$$

To define a fixed point problem equivalent to system (3.3)-(3.4), we introduce the operators

$$Q : J \times E \times F \times \Omega \to E \times F,$$

defined by

$$Q(u,v)(t,\vartheta)=\begin{pmatrix} Q_1(u,v)(t,\vartheta) \\ \\ Q_2(u,v)(t,\vartheta) \end{pmatrix},$$

where

$$\begin{aligned} Q_1(u,v)(t,\vartheta) &= I_{0^+}^{q_1;\psi}h_{u,v}(t,\vartheta)+I_{0^+}^{p_1+q_1;\psi}f_{u,v}(t,\vartheta)+\frac{\Psi_0^{q_1}(t)}{\Delta_1\Gamma(q_1+1)}\left[\frac{\int_0^T\Psi_0^{q_2}(\tau)d\tau}{\Psi_0^{q_2}(\xi)}\right. \\ &\times\left(I_{0^+}^{q_2;\psi}k_{u,v}(\xi,\vartheta)+I_{0^+}^{p_2+q_2;\psi}g_{u,v}(\xi,\vartheta)-\frac{1}{\kappa_2}\left(\int_0^T I_{0^+}^{q_1;\psi}h_{u,v}(\tau,\vartheta)d\tau\right.\right. \\ &+\left.\left.\int_0^T I_{0^+}^{p_1+q_1;\psi}f_{u,v}(\tau,\vartheta)d\tau\right)\right)+\kappa_1\left(I_{0^+}^{q_1;\psi}h_{u,v}(\rho,\vartheta)+I_{0^+}^{p_1+q_1;\psi}f_{u,v}(\rho,\vartheta)\right) \\ &-\left.\int_0^T I_{0^+}^{q_2;\psi}k_{u,v}(\tau,\vartheta)d\tau-\int_0^T I_{0^+}^{p_2+q_2;\psi}g_{u,v}(\tau,\vartheta)d\tau\right] \\ &+(\Psi_0^{q_1}(t)\lambda_1+1)\chi(v(\vartheta))+\lambda_2\Psi_0^{q_1}(t)\varphi(u(\vartheta)), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} Q_2(u,v)(t,\vartheta) &= I_{0^+}^{q_2;\psi}k_{u,v}(t,\vartheta)+I_{0^+}^{p_2+q_2;\psi}g_{u,v}(t,\vartheta)+\frac{\Psi_0^{q_2}(t)}{\Delta_2\Gamma(q_2+1)}\left[\frac{\int_0^T\Psi_0^{q_1}(\tau)d\tau}{\Psi_0^{q_1}(\rho)}\right. \\ &\times\left(I_{0^+}^{q_1;\psi}h_{u,v}(\rho,\vartheta)+I_{0^+}^{p_1+q_1;\psi}f_{u,v}(\rho,\vartheta)-\frac{1}{\kappa_1}\left(\int_0^T I_{0^+}^{q_2;\psi}k_{u,v}(\tau,\vartheta)d\tau\right.\right. \\ &+\left.\left.\int_0^T I_{0^+}^{p_2+q_2;\psi}g_{u,v}(\tau,\vartheta)d\tau\right)\right)+\kappa_2\left(I_{0^+}^{q_2;\psi}k_{u,v}(\xi,\vartheta)+I_{0^+}^{p_2+q_2;\psi}g_{u,v}(\xi,\vartheta)\right) \\ &-\left.\int_0^T I_{0^+}^{q_1;\psi}h_{u,v}(\tau,\vartheta)d\tau-\int_0^T I_{0^+}^{p_1+q_1;\psi}f_{u,v}(\tau,\vartheta)d\tau\right] \\ &+(\Psi_0^{q_2}(t)\lambda_3+1)\varphi(u(\vartheta))+\lambda_4\Psi_0^{q_2}(t)\chi(v(\vartheta)), \end{aligned} \tag{5.11}$$

with $f_{u,v}(t,\vartheta)=f(t,u_t(\vartheta),v_t(\vartheta),{}^cD_{0^+}^{\delta_1;\psi}v(t,\vartheta),\vartheta)$, $h_{u,v}(t,\vartheta)=h(t,u_t(\vartheta),v_t(\vartheta),\vartheta)$ $g_{u,v}(t,\vartheta)=g(t,u_t(\vartheta),{}^cD_{0^+}^{\delta_2;\psi}u(t,\vartheta),v_t(\vartheta),\vartheta)$ and $k_{u,v}(t,\vartheta)=k(t,u_t(\vartheta),v_t(\vartheta),\vartheta)$; $i=1,2$.

The maps $\chi(v)$ and $\varphi(u)$ are measurable for all $\vartheta\in\Omega$. In view of Lemma 5.1.2 and (5.10), (5.11), we obtain a fixed point problem $Q(u,v) = (u,v)$. Thus, the system (5.1)-(5.2) has a random solution if and only if we show that the operator Q has a random fixed point.

For the sake of computational convenience, we introduce the following notation.

$$K_1 = \left(1 + \frac{T^2\Psi_0^{q_1+q_2}(T)}{|\kappa_2\Delta_1|\Psi_0^{q_2}(\xi)\Gamma(q_1+1)} + \frac{|\kappa_1|\Psi_0^{q_1}(\rho)}{|\Delta_1|\Gamma(q_1+1)}\right)\frac{\Psi_0^{q_1}(T)}{\Gamma(q_1+1)},$$

$$K_2 = \left(\frac{2T\Psi_0^{q_1+q_2}(T)}{|\Delta_2|\Gamma(q_1+1)\Gamma(q_2+1)}\right), \quad \widetilde{K}_1 = \left(\frac{2T\Psi_0^{q_1+q_2}(T)}{|\Delta_1|\Gamma(q_1+1)\Gamma(q_2+1)}\right),$$

$$\widetilde{K}_2 = \left(1 + \frac{T^2\Psi_0^{q_1+q_2}(T)}{|\kappa_2\Delta_2|\Psi_0^{q_1}(\rho)\Gamma(q_2+1)} + \frac{|\kappa_2|\Psi_0^{q_2}(\xi)}{|\Delta_2|\Gamma(q_2+1)}\right)\frac{\Psi_0^{q_2}(T)}{\Gamma(q_2+1)},$$

$$L_1 = \left(1 + \frac{T^2\Psi_0^{q_1+q_2}(T)}{|\kappa_2\Delta_1|\Psi_0^{q_2}(\xi)} + \frac{|\kappa_1|\Psi_0^{p_1+q_1}(\rho)\Psi_0^{-p_1}(T)}{|\Delta_1|}\right)\frac{\Psi_0^{p_1+q_1}(T)}{\Gamma(q_1+1)\Gamma(p_1+q_1+1)},$$

$$L_2 = \left(\Psi_0^{q_1}(\rho) + \Psi_0^{p_1}(T)\right)\frac{T\Psi_0^{q_1+q_2}(T)}{|\Delta_2|\Gamma(q_2+1)\Gamma(p_1+q_1+1)},$$

$$\widetilde{L}_1 = \left(\Psi_0^{q_2}(\xi) + \Psi_0^{p_2}(T)\right)\frac{T\Psi_0^{q_1+q_2}(T)}{|\Delta_1|\Gamma(q_1+1)\Gamma(p_2+q_2+1)}, \text{ and}$$

$$\widetilde{L}_2 = \left(1 + \frac{T^2\Psi_0^{q_1+q_2}(T)}{|\kappa_1\Delta_2|\Psi_0^{q_1}(\rho)} + \frac{|\kappa_2|\Psi_0^{p_2+q_2}(\xi)\Psi_0^{-p_2}(T)}{|\Delta_2|}\right)\frac{\Psi_0^{p_2+q_2}(T)}{\Gamma(q_2+1)\Gamma(p_2+q_2+1)}.$$

5.2 Existence and Uniqueness

The first result is concerned with the existence and uniqueness of random solution for the system (3.3)-(3.4) and it is based on random versions of the Pervo fixed point theorem.

Theorem 5.2.1. *Assume that the following hypotheses holds.*

(H1) The functions f, g, h and k are Carathéodory.

(H2) There exist measurable functions $a_i, b_i, n_i, l_i, k_i : J \to L^\infty(\Omega, \mathbb{R}_+); i = 1,2$ such that:

$$\|f(t,u_1,u_2,u_3,\vartheta) - f(t,\overline{u}_1,\overline{u}_2,\overline{u}_3,\vartheta)\| \le k_1(t,\vartheta)\|u_1-\overline{u}_1\| + l_1(t,\vartheta)\|u_2-\overline{u}_2\| + n_1(t,\vartheta)\|u_3-\overline{u}_3\|,$$

$$\|g(t,u_1,u_2,u_3,\vartheta)-g(t,\overline{u}_1,\overline{u}_2,\overline{u}_3,\vartheta)\| \leq k_2(t,\vartheta)\|u_1-\overline{u}_1\| + l_2(t,\vartheta)\|u_2-\overline{u}_2\| \\ + n_2(t,\vartheta)\|u_3-\overline{u}_3\|,$$

and

$$\|h(t,u_1,u_2,\vartheta)-h(t,\overline{u}_1,\overline{u}_2,\vartheta)\| \leq a_1(t,\vartheta)\|u_1-\overline{u}_1\| + b_1(t,\vartheta)\|u_2-\overline{u}_2\|,$$

$$\|k(t,u_1,u_2,\vartheta)-k(t,\overline{u}_1,\overline{u}_2,\vartheta)\| \leq a_2(t,\vartheta)\|u_1-\overline{u}_1\| + b_2(t,\vartheta)\|u_2-\overline{u}_2\|,$$

for a.e.t $\in J$, *and each* $u_i, \overline{u}_i \in \mathbb{R}^n, i=1,2,3.$

(H3) *The functions* χ, φ *are continuous functions with* $\chi(0)=\varphi(0)=0$. *There exist bounded measurable functions* $l_\chi, l_\varphi : \Omega \to (0,\infty)$, *such that:*

$$\|\chi(u)-\chi(\overline{u})\| \leq l_\chi(\vartheta)\|u-\overline{u}\|;$$

$$\|\varphi(u)-\varphi(\overline{u})\| \leq l_\varphi(\vartheta)\|u-\overline{u}\|;$$

for each $u, \overline{u} \in \mathbb{R}^n$.

(H4) $M(\vartheta) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *is random variable matrix, such that for every* $\vartheta \in \Omega$, *the matrix*

$$M(\vartheta) = \begin{pmatrix} \dfrac{\|\Psi_0^{1-\delta_2}(T)\|}{\Gamma(2-\delta_2)} \Upsilon_1'(\vartheta) + \Upsilon_1(\vartheta) & \dfrac{\|\Psi_0^{1-\delta_2}(T)\|}{\Gamma(2-\delta_2)} \Upsilon_2'(\vartheta) + \Upsilon_2(\vartheta) \\ \\ \dfrac{\|\Psi_0^{1-\delta_1}(T)\|}{\Gamma(2-\delta_1)} \Upsilon_3'(\vartheta) + \Upsilon_3(\vartheta) & \dfrac{\|\Psi_0^{1-\delta_1}(T)\|}{\Gamma(2-\delta_1)} \Upsilon_4'(\vartheta) + \Upsilon_4(\vartheta) \end{pmatrix}.$$

converges to zero.
Then the coupled system (3.3)-(3.4) *has a unique random solution.*
Where,

$$\Upsilon_1(\vartheta) = \max\{K_1\|a_1(\cdot,\vartheta)\|_\infty + L_1\|k_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|a_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|k_2(\cdot,\vartheta)\|_\infty \\ + |\lambda_2| l_\varphi(\vartheta)\Psi_0^{q_1}(T), \widetilde{L}_1\|l_2(\cdot,\vartheta)\|_\infty\},$$

$$\Upsilon_2(\vartheta) = \max\{K_1\|b_1(\cdot,\vartheta)\|_\infty + L_1\|l_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|b_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|n_2(\cdot,\vartheta)\|_\infty \\ + l_\chi(\vartheta)(|\lambda_1|\Psi_0^{q_1}(T)+1), L_1\|n_1(\cdot,\vartheta)\|_\infty\},$$

$$\begin{aligned}\Upsilon_3(\vartheta) = \max\big\{&K_2\|a_2(\cdot,\vartheta)\|_\infty + L_2\|k_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|a_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|k_1(\cdot,\vartheta)\|_\infty \\ &+ l_\chi(\vartheta)(|\lambda_4|\Psi_0^{q_2}(T)+1), \widetilde{L}_2\|l_1(\cdot,\vartheta)\|_\infty\big\},\end{aligned}$$

$$\begin{aligned}\Upsilon_4(\vartheta) = \max\big\{&K_2\|b_2(\cdot,\vartheta)\|_\infty + L_2\|l_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|b_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|n_1(\cdot,\vartheta)\|_\infty \\ &+ |\lambda_3|l_\varphi(\vartheta)\Psi_0^{q_2}(T), L_2\|n_2(\cdot,\vartheta)\|_\infty\big\},\end{aligned}$$

$$\begin{aligned}\Upsilon_1'(\vartheta) = \max\Big\{&\big(K_1\|a_1(\cdot,\vartheta)\|_\infty + L_1\|k_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|a_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|k_2(\cdot,\vartheta)\|_\infty\big)\frac{q_1}{\Psi_0(T)} \\ &+ |\lambda_1|q_1 l_\varphi(\vartheta)\Psi_0^{q_1-1}(T), \frac{q_1\widetilde{L}_1}{\Psi_0(T)}\|l_2(\cdot,\vartheta)\|_\infty\Big\},\end{aligned}$$

$$\begin{aligned}\Upsilon_2'(\vartheta) = \max\Big\{&\big(K_1\|b_1(\cdot,\vartheta)\|_\infty + L_1\|l_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|b_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|n_2(\cdot,\vartheta)\|_\infty\big)\frac{q_1}{\Psi_0(T)} \\ &+ |\lambda_2|q_1 l_\chi(\vartheta)\Psi_0^{q_1-1}(T), \frac{q_1 L_1}{\Psi_0(T)}\|n_1(\cdot,\vartheta)\|_\infty\Big\},\end{aligned}$$

$$\begin{aligned}\Upsilon_3'(\vartheta) = \max\Big\{&\big(K_2\|a_2(\cdot,\vartheta)\|_\infty + L_2\|k_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|a_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|k_1(\cdot,\vartheta)\|_\infty\big)\frac{q_2}{\Psi_0(T)} \\ &+ |\lambda_3|q_2 l_\chi(\vartheta)\Psi_0^{q_2-1}(T), \frac{q_2\widetilde{L}_2}{\Psi_0(T)}\|l_1(\cdot,\vartheta)\|_\infty\Big\},\end{aligned}$$

$$\begin{aligned}\Upsilon_4'(\vartheta) = \max\Big\{&\big(K_2\|b_2(\cdot,\vartheta)\|_\infty + L_2\|l_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|b_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|n_1(\cdot,\vartheta)\|_\infty\big)\frac{q_2}{\Psi_0(T)} \\ &+ |\lambda_4|q_2 l_\varphi(\vartheta)\Psi_0^{q_2-1}(T), \frac{q_2 L_2}{\Psi_0(T)}\|n_2(\cdot,\vartheta)\|_\infty\Big\}.\end{aligned}$$

Proof. First, we need to show that the operator Q is a random operator on $E \times F$.
From (H1) and Definition 1.3.3 the maps f, g, k and h are measurable with respect to the variable ϑ. In the view of the Definition 1.3.6, we conclude that the maps

$$\vartheta \to Q_1(u,v)(t,\vartheta) \text{ and } \vartheta \to Q_2(u,v)(t,\vartheta)$$

are measurable. As a result, the operator Q is a random operator on $E \times F \times \Omega$ into $E \times F$.

Next, we prove that the operator Q is contractive.

For all $\vartheta \in \Omega$, $(u,v),(\overline{u},\overline{v}) \in E \times F$, and $t \in J$, we have

$$
\begin{aligned}
&\left\| Q_1(u,v)(t,\vartheta) - Q_1(\overline{u},\overline{v})(t,\vartheta) \right\| \\
&= \| I_{0^+}^{q_1;\psi} h_{u,v}(t,\vartheta) - I_{0^+}^{q_1;\psi} h_{\overline{u},\overline{v}}(t,\vartheta) \| + \| I_{0^+}^{p_1+q_1;\psi} f_{u,v}(t,\vartheta) - I_{0^+}^{p_1+q_1;\psi} f_{\overline{u},\overline{v}}(t,\vartheta) \| \\
&+ \frac{\Psi_0^{q_1}(t)}{|\Delta_1| \Gamma(q_1+1)} \left[\frac{\int_0^T \Psi_0^{q_2}(\tau) d\tau}{\Psi_0^{q_2}(\xi)} \Bigg(\| I_{0^+}^{q_2;\psi} k_{u,v}(\xi,\vartheta) - I_{0^+}^{q_2;\psi} k_{\overline{u},\overline{v}}(\xi,\vartheta) \| \right. \\
&+ \| I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\xi,\vartheta) - I_{0^+}^{p_2+q_2;\psi} g_{\overline{u},\overline{v}}(\xi,\vartheta) \| - \frac{1}{|\kappa_2|} \Bigg(\int_0^T \| I_{0^+}^{q_1;\psi} h_{u,v}(\tau,\vartheta) \\
&- I_{0^+}^{q_1;\psi} h_{\overline{u},\overline{v}}(\tau,\vartheta) \| d\tau + \int_0^T \| I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\tau,\vartheta) - I_{0^+}^{p_1+q_1;\psi} f_{\overline{u},\overline{v}}(\tau,\vartheta) \| d\tau \Bigg) \Bigg) \\
&+ |\kappa_1| \Bigg(\| I_{0^+}^{q_1;\psi} h_{u,v}(\rho,\vartheta) - I_{0^+}^{q_1;\psi} h_{\overline{u},\overline{v}}(\rho,\vartheta) \| + \| I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\rho,\vartheta) - I_{0^+}^{p_1+q_1;\psi} f_{\overline{u},\overline{v}}(\rho,\vartheta) \| \Bigg) \\
&+ \int_0^T \| I_{0^+}^{q_2;\psi} k_{u,v}(\tau,\vartheta) - I_{0^+}^{q_2;\psi} k_{\overline{u},\overline{v}}(\tau,\vartheta) \| d\tau + \int_0^T \| I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\tau,\vartheta) \\
&\left. - I_{0^+}^{p_2+q_2;\psi} g_{\overline{u},\overline{v}}(\tau,\vartheta) \| d\tau \right] + (|\lambda_1| \Psi_0^{q_1}(t) + 1) \| \chi(v_1) - \chi(v_2) \| + |\lambda_2| \Psi_0^{q_1}(t) \| \varphi(u_1) - \varphi(u_1) \| \\
&\leq \Big(K_1 \| a_1(\cdot,\vartheta) \|_\infty + L_1 \| k_1(\cdot,\vartheta) \|_\infty + \widetilde{K}_1 \| a_2(\cdot,\vartheta) \|_\infty + \widetilde{L}_1 \| k_2(\cdot,\vartheta) \|_\infty + |\lambda_1| l_\varphi(\vartheta) \Psi_0^{q_2}(T) \Big) \\
&\times \| u(\cdot,\vartheta) - \overline{u}(\cdot,\vartheta) \|_\infty + \widetilde{L}_1 \| l_2(\cdot,\vartheta) \|_\infty \| ^c D_{0^+}^{\delta_2;\psi} u(\cdot,\vartheta) - ^c D_{0^+}^{\delta_2;\psi} \overline{u}(\cdot,\vartheta) \|_\infty + \Big(K_1 \| b_1(\cdot,\vartheta) \|_\infty \\
&+ L_1 \| l_1(\cdot,\vartheta) \|_\infty + \widetilde{K}_1 \| b_2(\cdot,\vartheta) \|_\infty + \widetilde{L}_1 \| n_2(\cdot,\vartheta) \|_\infty + l_\chi(\vartheta) (|\lambda_2| \Psi_0^{q_2}(T) + 1) \Big) \\
&\times \| v(\cdot,\vartheta) - \overline{v}(\cdot,\vartheta) \|_\infty + L_1 \| n_1(\cdot,\vartheta) \|_\infty \| ^c D_{0^+}^{\delta_1;\psi} v(\cdot,\vartheta) - ^c D_{0^+}^{\delta_1;\psi} \overline{v}(\cdot,\vartheta) \|_\infty \\
&\leq \Upsilon_1(\vartheta) \| u(\cdot,\vartheta) - \overline{u}(\cdot,\vartheta) \|_E + \Upsilon_2(\vartheta) \| v(\cdot,\vartheta) - \overline{v}(\cdot,\vartheta) \|_F
\end{aligned}
$$

Also we have

$$\begin{aligned}
&\left\|D_\psi Q_1(u,v)(t,\vartheta) - D_\psi Q_1(\overline{u},\overline{v})(t,\vartheta)\right\| \\
&= \|I_{0^+}^{q_1-1;\psi} h_{u,v}(t,\vartheta) - I_{0^+}^{q_1-1;\psi} h_{\overline{u},\overline{v}}(t,\vartheta)\| + \|I_{0^+}^{p_1+q_1-1;\psi} f_{u,v}(t,\vartheta) - I_{0^+}^{p_1+q_1-1;\psi} f_{\overline{u},\overline{v}}(t,\vartheta)\| \\
&+ \frac{\Psi_0^{q_1-1}(t)}{|\Delta_1|\Gamma(q_1)}\left[\frac{\int_0^T \Psi_0^{q_2}(\tau)d\tau}{\Psi_0^{q_2}(\xi)}\Bigg(\|I_{0^+}^{q_2;\psi} k_{u,v}(\xi,\vartheta) - I_{0^+}^{q_2;\psi} k_{\overline{u},\overline{v}}(\xi,\vartheta)\|\right. \\
&+ \|I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\xi,\vartheta) - I_{0^+}^{p_2+q_2;\psi} g_{\overline{u},\overline{v}}(\xi,\vartheta)\| - \frac{1}{|\kappa_2|}\Bigg(\int_0^T \|I_{0^+}^{q_1;\psi} h_{u,v}(\tau,\vartheta) - I_{0^+}^{q_1;\psi} h_{\overline{u},\overline{v}}(\tau,\vartheta)\|d\tau \\
&+ \int_0^T \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\tau,\vartheta) - I_{0^+}^{p_1+q_1;\psi} f_{\overline{u},\overline{v}}(\tau,\vartheta)\|d\tau\Bigg)\Bigg) + |\kappa_1|\Big(\|I_{0^+}^{q_1;\psi} h_{u,v}(\rho,\vartheta) - I_{0^+}^{q_1;\psi} h_{\overline{u},\overline{v}}(\rho,\vartheta)\| \\
&+ \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\rho,\vartheta) - I_{0^+}^{p_1+q_1;\psi} f_{\overline{u},\overline{v}}(\rho,\vartheta)\|\Big) + \int_0^T \|I_{0^+}^{q_2;\psi} k_{u,v}(\tau,\vartheta) - I_{0^+}^{q_2;\psi} k_{\overline{u},\overline{v}}(\tau,\vartheta)\|d\tau \\
&\left.+ \int_0^T \|I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\tau,\vartheta) - I_{0^+}^{p_2+q_2;\psi} g_{\overline{u},\overline{v}}(\tau,\vartheta)\|d\tau\right] + |\lambda_1| q_1 \Psi_0^{q_1-1}(t)\|\chi(v_1) - \chi(v_2)\| \\
&+ |\lambda_2| q_1 \Psi_0^{q_1-1}(t)\|\varphi(u_1) - \varphi(u_1)\| \\
&\leq \Upsilon_1'(\vartheta)\|u(\cdot,\vartheta) - \overline{u}(\cdot,\vartheta)\|_E + \Upsilon_2'(\vartheta)\|v(\cdot,\vartheta) - \overline{v}(\cdot,\vartheta)\|_F.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\|{}^cD_{0^+}^{\delta_2;\psi} Q_1(u,v)(t,\vartheta) - {}^cD_{0^+}^{\delta_2;\psi} Q_1(\overline{u},\overline{v})(t,\vartheta)\right\| \\
&\leq \int_0^t \frac{\Psi^{-\delta_2}(t,s)}{\Gamma(1-\delta_2)}\left\|D_\psi T_1(u,v)(t,\vartheta) - D_\psi T_1(\overline{u},\overline{v})(t,\vartheta)\right\|ds \\
&\leq \frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\left(\Upsilon_1'(\vartheta)\|u(\cdot,\vartheta) - \overline{u}(\cdot,\vartheta)\|_E + \Upsilon_2'(\vartheta)\|v(\cdot,\vartheta) - \overline{v}(\cdot,\vartheta)\|_F\right).
\end{aligned}$$

From the above inequalities, we obtain

$$\begin{aligned}
&\left\|Q_1(u,v)(t,\vartheta) - Q_1(\overline{u},\overline{v})(t,\vartheta)\right\|_E \\
&= \left\|Q_1(u,v)(t,\vartheta) - Q_1(\overline{u},\overline{v})(t,\vartheta)\right\|_\infty + \left\|{}^cD_{0^+}^{\delta_2;\psi} Q_1(u,v)(t,\vartheta) -{}^c D_{0^+}^{\delta_2;\psi} Q_1(\overline{u},\overline{v})(t,\vartheta)\right\|_\infty \\
&\leq \left(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Upsilon_1'(\vartheta) + \Upsilon_1(\vartheta)\right)\|u(\cdot,\vartheta) - \overline{u}(\cdot,\vartheta)\|_E + \left(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Upsilon_2'(\vartheta) + \Upsilon_2(\vartheta)\right) \\
&\times \|v(\cdot,\vartheta) - \overline{v}(\cdot,\vartheta)\|_F.
\end{aligned}$$

In a similar way, we can find that

$$\begin{aligned}&\left\|Q_2(u,v)(t,\vartheta)-Q_2(\overline{u},\overline{v})(t,\vartheta)\right\|_F\\&\leq\left(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Upsilon_3'(\vartheta)+\Upsilon_3(\vartheta)\right)\|u(\cdot,\vartheta)-\overline{u}(\cdot,\vartheta)\|_E+\left(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Upsilon_4'(\vartheta)+\Upsilon_4(\vartheta)\right)\\&\times\|v(\cdot,\vartheta)-\overline{v}(\cdot,\vartheta)\|_F.\end{aligned}$$

Thus,

$$d\Big(Q(u,v)(.,\vartheta),Q(\overline{u},\overline{v}_2)(\cdot,\vartheta)\Big)\leq M(\vartheta)d\Big((u(\cdot,\vartheta),v(\cdot,\vartheta)),(\overline{u}(\cdot,\vartheta),\overline{v}(\cdot,\vartheta))\Big)$$

where

$$d\Big((u(\cdot,\vartheta),v(\cdot,\vartheta)),(\overline{u}(\cdot,\vartheta),\overline{v}(\cdot,\vartheta))\Big)=\begin{pmatrix}\|u(\cdot,\vartheta)-\overline{u}(\cdot,\vartheta)\|_E\\ \\ \|v(\cdot,\vartheta)-\overline{v}(\cdot,\vartheta)\|_F.\end{pmatrix}$$

As for every $\vartheta\in\Omega$, the matrix $M(\vartheta)$ converges to zero, this implies that the operator Q is a $M(\vartheta)-$contractive operator. Consequently, by theorem 1.5.4, we conclude that Q has a unique fixed point, which is a random solution of systems (5.1)-(5.2). This completes the proof. □

5.3 Existence result

In the next result, we prove the existence of solution for the system (3.3)-(3.4) by applying a random version of a Krasnoselskii-type fixed point theorem.

Theorem 5.3.1. *Assume that (H1)-(H2) and the following hypotheses holds.*

(H5) There exist measurable functions $\widetilde{\phi}_i,\widetilde{\theta}_i,\widetilde{a}_i,\widetilde{b}_i,\widetilde{n}_i,\widetilde{l}_i,\widetilde{k}_i:J\to L^\infty(\Omega,\mathbb{R}_+);i=1,2$ *such that:*

$$\|f(t,u_1,u_2,u_3,\vartheta)\|\leq\widetilde{\phi}_1(t,\vartheta)+\widetilde{k}_1(t,\vartheta)\|u_1\|+\widetilde{l}_1(t,\vartheta)\|u_2\|+\widetilde{n}_1(t,\vartheta)\|u_3\|,$$

$$\|g(t,u_1,u_2,u_3,\vartheta)\|\leq\widetilde{\phi}_2(t,\vartheta)+\widetilde{k}_2(t,\vartheta)\|u_1\|+\widetilde{l}_2(t,\vartheta)\|u_2\|+\widetilde{n}_2(t,\vartheta)\|u_3\|,$$

and

$$\|h(t,u_1,u_2,\vartheta)\|\leq\widetilde{\theta}_1(t,\vartheta)+\widetilde{a}_1(t,\vartheta)\|u_1\|+\widetilde{b}_1(t,\vartheta)\|u_2\|,$$

$$\|k(t,u_1,u_2,\vartheta)\|\leq\widetilde{\theta}_2(t,\vartheta)+\widetilde{a}_2(t,\vartheta)\|u_1\|+\widetilde{b}_2(t,\vartheta)\|u_2\|,$$

for a.e. $t \in J$, *and each* $u_i \in \mathbb{R}^n, i = 1,2,3.$

(H6) there exist positive constants N_{q_1}, N_{q_2}, *such that*

$$\max\{N_{q_1} l_\varphi(\vartheta), N_{q_2} l_\chi(\vartheta)\} < 1,$$

where,

$$N_{q_1} = |\lambda_2| \left(\Psi_0^{q_1}(T) + \frac{\Gamma(q_1+1)}{\Gamma(q_1+1-\delta_2)} \Psi_0^{q_1-\delta_2}(T) \right),$$

$$N_{q_2} = |\lambda_4| \left(\Psi_0^{q_2}(T) + \frac{\Gamma(q_2+1)}{\Gamma(q_2+1-\delta_1)} \Psi_0^{q_2-\delta_1}(T) \right).$$

Let

$$\widetilde{M}(\vartheta) = \begin{pmatrix} 1 - N_{\delta_2} Y_1'(\vartheta) - Y_1(\vartheta) & - N_{\delta_2} Y_2'(\vartheta) - Y_2(\vartheta) \\ \\ -N_{\delta_1} Y_3'(\vartheta) - Y_3(\vartheta) & 1 - N_{\delta_1} Y_4'(\vartheta) - Y_4(\vartheta) \end{pmatrix}$$

if $\det \widetilde{M} > 0$. *Then the coupled system* (3.3)-(3.4) *has at least a random solution. Where*

$$N_{\delta_i} = \frac{\Psi_0^{1-\delta_i}(T)}{\Gamma(2-\delta_i)}; i = 1,2.$$

In what follows we use the following notations.

$$\Theta_1(\vartheta) = \max\big\{K_1\|a_1(\cdot,\vartheta)\|_\infty + L_1\|k_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|a_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|k_2(\cdot,\vartheta)\|_\infty \\ ,\widetilde{L}_1\|l_2(\cdot,\vartheta)\|_\infty\big\},$$

$$\Theta_2(\vartheta) = \max\big\{K_1\|b_1(\cdot,\vartheta)\|_\infty + L_1\|l_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|b_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|n_2(\cdot,\vartheta)\|_\infty \\ + (|\lambda_1|\Psi_0^{q_1}(T)+1)l_\chi(\vartheta), L_1\|n_1(\cdot,\vartheta)\|_\infty\big\},$$

$$\Theta_3(\vartheta) = \max\big\{K_2\|a_2(\cdot,\vartheta)\|_\infty + L_2\|k_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|a_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|k_1(\cdot,\vartheta)\|_\infty \\ ,\widetilde{L}_2\|l_1(\cdot,\vartheta)\|_\infty\big\},$$

$$\Theta_4(\vartheta) = \max\big\{K_2\|b_2(\cdot,\vartheta)\|_\infty + L_2\|l_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|b_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|n_1(\cdot,\vartheta)\|_\infty \\ + |\lambda_3|\Psi_0^{q_2}(T)l_\varphi(\vartheta), L_2\|n_2(\cdot,\vartheta)\|_\infty\big\},$$

$$\Theta_1'(\vartheta) = \max\Big\{ \big(K_1\|a_1(\cdot,\vartheta)\|_\infty + L_1\|k_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|a_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|k_2(\cdot,\vartheta)\|_\infty\big)\frac{q_1}{\Psi_0(T)} + |\lambda_1|q_1\Psi_0^{q_1-1}(T)l_\varphi(\vartheta), \frac{q_1\widetilde{L}_1}{\Psi_0(T)}\|l_2(\cdot,\vartheta)\|_\infty\Big\},$$

$$\Theta_2'(\vartheta) = \max\Big\{ \big(K_1\|b_1(\cdot,\vartheta)\|_\infty + L_1\|l_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|b_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|n_2(\cdot,\vartheta)\|_\infty\big)\frac{q_1}{\Psi_0(T)}, \frac{q_1L_1}{\Psi_0(T)}\|n_1(\cdot,\vartheta)\|_\infty\Big\},$$

$$\Theta_3'(\vartheta) = \max\Big\{ \big(K_2\|a_2(\cdot,\vartheta)\|_\infty + L_2\|k_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|a_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|k_1(\cdot,\vartheta)\|_\infty\big)\frac{q_2}{\Psi_0(T)} + |\lambda_3|q_2\Psi_0^{q_2-1}(T)l_\chi(\vartheta), \frac{q_2\widetilde{L}_2}{\Psi_0(T)}\|l_1(\cdot,\vartheta)\|_\infty\Big\},$$

$$\Theta_4'(\vartheta) = \max\Big\{ \big(K_2\|b_2(\cdot,\vartheta)\|_\infty + L_2\|l_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|b_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|n_1(\cdot,\vartheta)\|_\infty\big)\frac{q_2}{\Psi_0(T)}, \frac{q_2L_2}{\Psi_0(T)}\|n_2(\cdot,\vartheta)\|_\infty\Big\},$$

$$\widetilde{\Theta}_1(\vartheta) = \max\big\{K_1\|\widetilde{a}_1(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{k}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{a}_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{k}_2(\cdot,\vartheta)\|_\infty, \widetilde{L}_1\|\widetilde{l}_2(\cdot,\vartheta)\|_\infty\big\},$$

$$\widetilde{\Theta}_2(\vartheta) = \max\big\{K_1\|\widetilde{b}_1(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{l}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{b}_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{n}_2(\cdot,\vartheta)\|_\infty + (|\lambda_1|\Psi_0^{q_1}(T)+1)l_\chi(\vartheta), L_1\|\widetilde{n}_1(\cdot,\vartheta)\|_\infty\big\},$$

$$\widetilde{\Theta}_3(\vartheta) = \max\big\{K_2\|\widetilde{a}_2(\cdot,\vartheta)\|_\infty + L_2\|\widetilde{k}_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|\widetilde{a}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|\widetilde{k}_1(\cdot,\vartheta)\|_\infty, \widetilde{L}_2\|\widetilde{l}_1(\cdot,\vartheta)\|_\infty\big\},$$

$$\widetilde{\Theta}_4(\vartheta) = \max\big\{K_2\|\widetilde{b}_2(\cdot,\vartheta)\|_\infty + L_2\|\widetilde{l}_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|\widetilde{b}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|\widetilde{n}_1(\cdot,\vartheta)\|_\infty + |\lambda_3|\Psi_0^{q_2}(T)l_\varphi(\vartheta), L_2\|\widetilde{n}_2(\cdot,\vartheta)\|_\infty\big\},$$

$$\widetilde{\Theta}'_1(\vartheta) = \max\Big\{ \big(K_1\|\widetilde{a}_1(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{k}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{a}_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{k}_2(\cdot,\vartheta)\|_\infty\big)\frac{q_1}{\Psi_0(T)} + |\lambda_1|q_1\Psi^{q_1-1}(T)l_\varphi(\vartheta), \frac{q_1\widetilde{L}_1}{\|\Psi_0^1(T)\|}\|\widetilde{l}_2(\cdot,\vartheta)\|_\infty\Big\},$$

$$\widetilde{\Theta}'_2(\vartheta) = \max\Big\{ \big(K_1\|\widetilde{b}_1(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{l}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{b}_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{n}_2(\cdot,\vartheta)\|_\infty\big)\frac{q_1}{\Psi_0(T)}, \frac{q_1L_1}{\Psi_0(T)}\|\widetilde{n}_1(\cdot,\vartheta)\|_\infty\Big\},$$

$$\widetilde{\Theta}'_3(\vartheta) = \max\Big\{ \big(K_2\|\widetilde{a}_2(\cdot,\vartheta)\|_\infty + L_2\|\widetilde{k}_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|\widetilde{a}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|\widetilde{k}_1(\cdot,\vartheta)\|_\infty\big)\frac{q_2}{\Psi_0(T)} + q_2\|\lambda_3\Psi_0^{q_2-1}(T)\|l_\chi(\vartheta), \frac{q_2\widetilde{L}_2}{\Psi_0(T)}\|\widetilde{l}_1(\cdot,\vartheta)\|_\infty\Big\},$$

$$\widetilde{\Theta}'_4(\vartheta) = \max\Big\{ \big(K_2\|\widetilde{b}_2(\cdot,\vartheta)\|_\infty + L_2\|\widetilde{l}_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|\widetilde{b}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|\widetilde{n}_1(\cdot,\vartheta)\|_\infty\big)\frac{q_2}{\Psi_0(T)}, \frac{q_2L_2}{\Psi_0(T)}\|\widetilde{n}_2(\cdot,\vartheta)\|_\infty\Big\},$$

$$\Phi_1(\vartheta) = K_1\|\theta_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\theta_2(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{\phi}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{\phi}_2(\cdot,\vartheta)\|_\infty,$$

and

$$\Phi_2(\vartheta) = K_2\|\theta_2(\cdot,\vartheta)\|_\infty + \widetilde{K}_2\|\theta_1(\cdot,\vartheta)\|_\infty + L_2\|\widetilde{\phi}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_2\|\widetilde{\phi}_1(\cdot,\vartheta)\|_\infty.$$

Proof. Let us subdivide the operator Q into two operators $A, B : E \times F \times \Omega \to E \times F$ as follows:

$$Q(u,v)(t,\vartheta) = A(u,v)(t,\vartheta) + B(u,v)(t,\vartheta) \qquad (u,v) \in E \times F, (t,\vartheta) \in J \times \Omega,$$

where

$$A(u,v)(t,\vartheta) = (A_1(u,v)(t,\vartheta), A_2(u,v)(t,\vartheta)),$$

and

$$B(u,v)(t,\vartheta) = (B_1(u,v)(t,\vartheta), B_2(u,v)(t,\vartheta)),$$

with

$$A_1(u,v)(t,\vartheta) = \lambda_2 \Psi_0^{q_1}(t)\varphi(u(\vartheta)), \tag{5.12}$$

$$A_2(u,v)(t,\vartheta) = \lambda_4 \Psi_0^{q_2}(t)\chi(v(\vartheta)), \tag{5.13}$$

and

$$\begin{aligned}
B_1(u,v)(t,\vartheta) &= I_{0^+}^{q_1;\psi} h_{u,v}(t,\vartheta) + I_{0^+}^{p_1+q_1;\psi} f_{u,v}(t,\vartheta) + \frac{\Psi_0^{q_1}(t)}{\Delta_1 \Gamma(q_1+1)} \left[\frac{\int_0^T \Psi_0^{q_2}(\tau)d\tau}{\Psi_0^{q_2}(\xi)} \right. \\
&\times \left(I_{0^+}^{q_2;\psi} k_{u,v}(\xi,\vartheta) + I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\xi,\vartheta) - \frac{1}{\kappa_2} \left(\int_0^T I_{0^+}^{q_1;\psi} h_{u,v}(\tau,\vartheta)d\tau \right.\right. \\
&+ \left.\left. \int_0^T I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\tau,\vartheta)d\tau \right) \right) + \kappa_1 \left(I_{0^+}^{q_1;\psi} h_{u,v}(\rho,\vartheta) + I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\rho,\vartheta) \right) \\
&- \left. \int_0^T I_{0^+}^{q_2;\psi} k_{u,v}(\tau,\vartheta)d\tau - \int_0^T I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\tau,\vartheta)d\tau \right] \\
&+ (\Psi_0^{q_1}(t)\lambda_1 + 1)\chi(v(\vartheta)),
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
B_2(u,v)(t,\vartheta) &= I_{0^+}^{q_2;\psi} k_{u,v}(t,\vartheta) + I_{0^+}^{p_2+q_2;\psi} g_{u,v}(t,\vartheta) + \frac{\Psi_0^{q_2}(t)}{\Delta_2 \Gamma(q_2+1)} \left[\frac{\int_0^T \Psi_0^{q_1}(\tau)d\tau}{\Psi_0^{q_1}(\rho)} \right. \\
&\times \left(I_{0^+}^{q_1;\psi} h_{u,v}(\rho,\vartheta) + I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\rho,\vartheta) - \frac{1}{\kappa_1} \left(\int_0^T I_{0^+}^{q_2;\psi} k_{u,v}(\tau,\vartheta)d\tau \right.\right. \\
&+ \left.\left. \int_0^T I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\tau,\vartheta)d\tau \right) \right) + \kappa_2 \left(I_{0^+}^{q_2;\psi} k_{u,v}(\xi,\vartheta) + I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\xi,\vartheta) \right) \\
&- \left. \int_0^T I_{0^+}^{q_1;\psi} h_{u,v}(\tau,\vartheta)d\tau - \int_0^T I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\tau,\vartheta)d\tau \right] \\
&+ (\Psi_0^{q_2}(t)\lambda_3 + 1)\varphi(u(\vartheta)).
\end{aligned} \tag{5.15}$$

We need to prove that the operators A and B satisfies all conditions of the Theorem 1.5.7. The proof is divided into several steps.

step 1. A is $\widetilde{M}(\vartheta)-$contraction operator:
As in the previous proof of Theorem 5.3.1, we can obtain
For all $\vartheta \in \Omega$, $(u,v),(\overline{u},\overline{v}) \in E \times F$, and $t \in J$

$$\begin{aligned}&\Big\|\big(A_1(u,v)\big)(\cdot,\vartheta) - \big(A_1(\overline{u},\overline{v})\big)(\cdot,\vartheta)\Big\|_E \\ &\leq |\lambda_2| \left(\Psi_0^{q_1}(T) + \frac{\Gamma(q_1+1)}{\Gamma(q_1+1-\delta_2)}\Psi_0^{q_1-\delta_2}(T)\right) l_\varphi(\vartheta)\|u(\cdot,\vartheta) - \overline{u}(\cdot,\vartheta)\|_E,\end{aligned}$$

And

$$\begin{aligned}&\Big\|\big(A_2(u,v)\big)(\cdot,\vartheta) - \big(A_2(\overline{u},\overline{v})\big)(\cdot,\vartheta)\Big\|_F \\ &\leq |\lambda_4| \left(\Psi_0^{q_2}(T) + \frac{\Gamma(q_2+1)}{\Gamma(q_2+1-\delta_1)}\Psi_0^{q_2-\delta_1}(T)\right) l_\chi(\vartheta)\|v(\cdot,\vartheta) - \overline{v}(\cdot,\vartheta)\|_F,\end{aligned}$$

Thus

$$d\Big(A(u,v)(\cdot,\vartheta), A(\overline{u},\overline{v})(\cdot,\vartheta)\Big) \leq \overline{M}(\vartheta) d\Big(\big(u(\cdot,\vartheta),v(\cdot,\vartheta)\big),\big(\overline{u}(\cdot,\vartheta),\overline{v}(\cdot,\vartheta)\big)\Big),$$

where

$$\overline{M}(\vartheta) = \begin{pmatrix} N_{q_1} l_\varphi(\vartheta) & 0 \\ & \\ 0 & N_{q_2} l_\chi(\vartheta) \end{pmatrix}.$$

From (H6) and example 1.4.9, we conclude that the Matrix $\overline{M}(\vartheta)$ converges to zero, then the operator A is a $\overline{M}(\vartheta)$-contractive operators.
For the following steps, we show that B is completely continuous.
step 2. $B(\cdot,\cdot,\vartheta)$ is continuous operator.
Let (u_n,v_n) be a sequence such that

$$(u_n,v_n) \to (u,v) \in E \times F \qquad \text{as } n \to \infty.$$

Then, for each $\vartheta \in \Omega, t \in J$, we have

$$\begin{aligned}&\Big\|\big(B_1(u_n,v_n)\big)(t,\vartheta) - \big(B_1(u,v)\big)(t,\vartheta)\Big\| \\ &\leq \Big(K_1\|a_1(\cdot,\vartheta)\|_\infty + L_1\|k_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|a_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|k_2(\cdot,\vartheta)\|_\infty\Big)\Big\|u_n(\cdot,\vartheta) - u(\cdot,\vartheta)\Big\|_\infty\end{aligned}$$

$$
\begin{aligned}
&+\Big(K_1\|b_1(\cdot,\vartheta)\|_\infty + L_1\|l_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|b_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|n_2(\cdot,\vartheta)\|_\infty \\
&+ l_\chi(\vartheta)\|\lambda_1\Psi_0^{q_1}(T)+1\|\Big)\big\|v_n(\cdot,\vartheta)-v(\cdot,\vartheta)\big\|_\infty \\
&+ \widetilde{L}_1\|l_2(\cdot,\vartheta)\|_\infty\big\|{}^cD_{0^+}^{\delta_2}u_n(\cdot,\vartheta)-{}^cD_{0^+}^{\delta_2}u(\cdot,\vartheta)\big\|_\infty + L_1\|n_1(\cdot,\vartheta)\|_\infty\big\|{}^cD_{0^+}^{\delta_1}v_n(\cdot,\vartheta)-{}^cD_{0^+}^{\delta_1}v(\cdot,\vartheta)\big\|_\infty \\
&\le \Theta_1(\vartheta)\big\|u_n(\cdot,\vartheta)-u(\cdot,\vartheta)\big\|_E + \Theta_2(\vartheta)\big\|v_n(\cdot,\vartheta)-v(\cdot,\vartheta)\big\|_F.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\Big\|{}^cD_{0^+}^{\delta_2;\Psi}\big(B_1(u_n,v_n)\big)(t,\vartheta) - {}^cD_{0^+}^{\delta_2;\Psi}\big(B_1(u,v)\big)(t,\vartheta)\Big\| \\
&\le \frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Theta_1'(\vartheta)\big\|u_n(\cdot,\vartheta)-u(\cdot,\vartheta)\big\|_E + \frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Theta_2'(\vartheta)\big\|v_n(\cdot,\vartheta)-v(\cdot,\vartheta)\big\|_F.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&\Big\|\big(B_1(u_n,v_n)\big)(t,\vartheta) - \big(B_1(u,v)\big)(t,\vartheta)\Big\|_E \\
&\le \Bigg(\bigg(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Theta_1'(\vartheta)+\Theta_1(\vartheta)\bigg)\big\|u_n(\cdot,\vartheta)-u(\cdot\vartheta)\big\|_E \\
&+ \bigg(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Theta_2'(\vartheta)+\Theta_2(\vartheta)\bigg)\big\|v_n(\cdot,\vartheta)-v(\cdot,\vartheta)\big\|_F\Bigg) \to 0 \quad \textbf{as } n\to\infty.
\end{aligned}
$$

On the other hand, for any $\vartheta\in\Omega$ and each $t\in J$, we obtain

$$
\begin{aligned}
&\Big\|\big(B_2(u_n,v_n)\big)(t,\vartheta) - \big(B_2(u,v)\big)(t,\vartheta)\Big\|_F \\
&\le \Bigg(\bigg(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Theta_3'(\vartheta)+\Theta_3(\vartheta)\bigg)\big\|u_n(\cdot,\vartheta)-u(\cdot\vartheta)\big\|_E \\
&+ \bigg(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Theta_4'(\vartheta)+\Theta_4(\vartheta)\bigg)\big\|v_n(\cdot,\vartheta)-v(\cdot,\vartheta)\big\|_F\Bigg) \to 0 \quad \textbf{as } n\to\infty.
\end{aligned}
$$

Hence, $B(\cdot,\cdot)(t,\vartheta)$ is continuous
step 3. $B(\cdot,\cdot,\vartheta)$ maps bounded sets into bounded sets in $E\times F$.

Indeed, it is enough to show that for any $r>0$ there exists a positive constant R such that

$$\|(B(u,v))(\cdot,\vartheta)\|_{E\times F} \le R(\vartheta) = (R_1(\vartheta),R_2(\vartheta)).$$

For each $(u,v) \in B_r = \{(u,v) \in E \times F : \quad \|u\|_E \leq r, \|v\|_F \leq r\}$, and for each $t \in J$, we get

$$\begin{aligned}
&\left\|\big(B_1(u,v)\big)(t,\vartheta)\right\| \\
&\leq \|I_{0^+}^{q_1;\psi} h_{u,v}(t,\vartheta)\| + \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(t,\vartheta)\| + \frac{\Psi_0^{q_1}(t)}{|\Delta_1|\Gamma(q_1+1)}\left[\frac{\int_0^T \Psi_0^{q_2}(\tau)d\tau}{\Psi_0^{q_2}(\xi)}\right. \\
&\times\Bigg(\|I_{0^+}^{q_2;\psi} k_{u,v}(\xi,\vartheta)\| + \|I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\xi,\vartheta)\| + \frac{1}{|\kappa_2|}\bigg(\int_0^T I_{0^+}^{q_1;\psi} h_{u,v}(\tau,\vartheta)d\tau \\
&+ \int_0^T \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\tau,\vartheta)\|d\tau\bigg)\Bigg) + |\kappa_1|\bigg(\|I_{0^+}^{q_1;\psi} h_{u,v}(\rho,\vartheta)\| + \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\rho,\vartheta)\|\bigg) \\
&\left.+ \int_0^T \|I_{0^+}^{q_2;\psi} k_{u,v}(\tau,\vartheta)\|d\tau - \int_0^T \|I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\tau,\vartheta)\|d\tau\right] + (\Psi_0^{q_1}(t)|\lambda_1|+1)\|\chi(v(\vartheta))\| \\
&\leq \Phi_1(\vartheta) + r\big(\widetilde{\Theta}_1(\vartheta) + \widetilde{\Theta}_2(\vartheta)\big).
\end{aligned}$$

and

$$\begin{aligned}
&\left\|{}^cD_{0^+}^{\delta_2}\big(B(u,v)\big)(t,\vartheta)\right\| \\
&\leq \frac{q_1\Psi_0^{-\delta_2}(T)}{\Gamma(2-\delta_2)}\Phi_1(\vartheta) + \frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)} r\Big(\widetilde{\Theta}'_1(\vartheta) + \widetilde{\Theta}'_2(\vartheta)\Big),
\end{aligned}$$

therefore,

$$\begin{aligned}
&\left\|\big(B_1(u,v)\big)(\cdot,\vartheta)\right\|_E \\
&\leq \Phi_1(\vartheta)\left(\frac{q_1\Psi_0^{-\delta_2}(T)}{\Gamma(2-\delta_2)}+1\right) + r\left(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\big(\widetilde{\Theta}'_1(\vartheta)+\widetilde{\Theta}'_2(\vartheta)\big) + \widetilde{\Theta}_1(\vartheta)+\widetilde{\Theta}_2(\vartheta)\right) = R_1(\vartheta).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left\|\big(B_2(u,v)\big)(\cdot,\vartheta)\right\|_F \\
&\leq \Phi_2(\vartheta)\left(\frac{q_2\Psi_0^{-\delta_1}(T)}{\Gamma(1-\delta_1)}+1\right) + r\left(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\big(\widetilde{\Theta}'_3(\vartheta)+\widetilde{\Theta}'_4(\vartheta)\big) + \widetilde{\Theta}_3(\vartheta)+\widetilde{\Theta}_4(\vartheta)\right) = R_2(\vartheta).
\end{aligned}$$

Hence,

$$\|\big(B(u,v)\big)(\cdot,\vartheta)\|_{E\times F} = \|\big(\big(B_1(u,v)\big)(\cdot,\vartheta), \big(B_2(u,v)\big)(\cdot,\vartheta)\big)\|_{E\times F} \leq (R_1(\vartheta), R_2(\vartheta)) = R(\vartheta).$$

step 4. $B(\cdot,\cdot,\vartheta)$ maps bounded sets into equicontinuous sets of $E \times F$.
Let B_r be a bounded set of $E \times F$ as in **step.2**, let $t_1, t_2 \in J$ and $t_1 > t_2$ and any $(u,v) \in B_r$ and $\vartheta \in \Omega$, we have

$$\begin{aligned}
&\Big\|\big(B_1(u,v)\big)(t_1,\vartheta) - \big(B_1(u,v)\big)(t_2,\vartheta)\Big\| \\
&\leq \int_0^{t_2} \frac{\left|\Psi^{q_1}(t_1,s) - \Psi^{q_1}(t_2,s)\right|}{\Gamma(q_1)} \big\|h_{u,v}(s,\vartheta)\big\|ds + \int_{t_2}^{t_1} \frac{\Psi^{q_1}(t_1,s)}{\Gamma(q_1)} \big\|h_{u,v}(s,\vartheta)\big\|ds \\
&+ \int_0^{t_2} \frac{\left|\Psi^{p_1+q_1}(t_1,s) - \Psi^{p_1+q_1}(t_2,s)\right|}{\Gamma(p_1+q_1)} \big\|f_{u,v}(s,\vartheta)\big\|ds + \int_{t_2}^{t_1} \frac{\Psi^{p_1+q_1}(t_1,s)}{\Gamma(p_1+q_1)} \big\|f_{u,v}(s,\vartheta)\big\|ds \\
&+ \frac{|\Psi_0^{q_1}(t_1) - \Psi_0^{q_1}(t_2)|}{|\Delta_1|\Gamma(q_1+1)} \left[\frac{\int_0^T \Psi_0^{q_2}(\tau)d\tau}{\Psi_0^{q_2}(\xi)} \Bigg(\|I_{0^+}^{q_2;\psi} k_{u,v}(\xi,\vartheta)\| + \|I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\xi,\vartheta)\| \right. \\
&+ \frac{1}{|\kappa_2|}\left(\int_0^T \|I_{0^+}^{q_1;\psi} h_{u,v}(\tau,\vartheta)\|d\tau + \int_0^T \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\tau,\vartheta)\|d\tau\right)\Bigg) \\
&+ |\kappa_1|\left(\|I_{0^+}^{q_1;\psi} h_{u,v}(\rho,\vartheta)\| + \|I_{0^+}^{p_1+q_1;\psi} f_{u,v}(\rho,\vartheta)\|\right) + \int_0^T \|I_{0^+}^{q_2;\psi} k_{u,v}(\tau,\vartheta)\|d\tau \\
&\left. + \int_0^T \|I_{0^+}^{p_2+q_2;\psi} g_{u,v}(\tau,\vartheta)\|d\tau\right] + \left|\Psi_0^{q_1}(t_1) - \Psi_0^{q_1}(t_2)\right| \|\lambda_1 \chi(v(\vartheta))\| \to 0 \text{ as } t_2 \to t_1,
\end{aligned}$$

In same way, we find that

$$\Big\|{}^c D_{0^+}^{\delta_2}\big(B_1(u,v)\big)(t_1,\vartheta) - {}^c D^{\delta_2}\big(B_1(u,v)\big)(t_2,\vartheta)\Big\| \to 0 \text{ as } t_2 \to t_1,$$

Moreover;

$$\Big\|\big(B_1(u,v)\big)(t_1,\vartheta) - \big(B_1(u,v)\big)(t_2,\vartheta)\Big\|_E \to 0 \text{ as } t_1 \to t_2,$$

In similar manner, we have

$$\Big\|\big(B_2(u,v)\big)(t_1,\vartheta) - \big(B_2(u,v)\big)(t_2,\vartheta)\Big\|_F \to 0 \text{ as } t_1 \to t_2.$$

Thus the operators B_1 and B_2 are equicontinuous, and then B is also equicontinuous. Hence by the Ascoli-Arzila theorem, we deduce that B is compact. Therefore we conclude that B is

completely continuous. Also from the hypothesis (H6) and example 1.4.9, we can see that the matrix $I - \overline{M}$ has the absolute value property.
Now, it remains to show that the set

$$\mathcal{N} = \Big\{(u,v) : \Omega \to X \in Y \text{ is measurable } |\mu(\vartheta)A(u,v) + \mu(\vartheta)B\Big(\frac{u}{\mu(\vartheta)}, \frac{v}{\mu(\vartheta)}, \vartheta\Big) = (u,v)\Big\}$$

is bounded for some measurable mapping $\mu : \Omega \to \mathbb{R}$ with $0 < \mu(\vartheta) < 1$ on Ω, let $(u,v) \in \Lambda$. Then

$$\begin{aligned}
&\|u(t,\vartheta)\| \\
&\leq \Big(K_1\|\widetilde{a}_1(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{k}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{a}_2(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{k}_2(\cdot,\vartheta)\|_\infty + |\lambda_1|\Psi_0^{q_2}(T)l_\varphi(\vartheta)\Big) \\
&\times \|u(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{l}_2(\cdot,\vartheta)\|_\infty\|{}^cD^{\delta_2;\psi}u(\cdot,\vartheta)\|_\infty + \Big(K_1\|\widetilde{b}_1(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{l}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{b}_2(\cdot,\vartheta)\|_\infty \\
&+ \widetilde{L}_1\|\widetilde{n}_2(\cdot,\vartheta)\|_\infty + (|\lambda_2|\Psi_0^{q_2}(T)+1)l_\chi(\vartheta)\Big)\|v(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{n}_1(\cdot,\vartheta)\|_\infty\|{}^cD^{\delta_1;\psi}v(\cdot,\vartheta)\|_\infty \\
&+ K_1\|\widetilde{\theta}_1(\cdot,\vartheta)\|_\infty + \widetilde{K}_1\|\widetilde{\theta}_2(\cdot,\vartheta)\|_\infty + L_1\|\widetilde{\phi}_1(\cdot,\vartheta)\|_\infty + \widetilde{L}_1\|\widetilde{\phi}_2(\cdot,\vartheta)\|_\infty \\
&\leq \Phi_1(\vartheta) + \widetilde{\Upsilon}_1(\vartheta)\|u(\cdot,\vartheta)\|_E + \widetilde{\Upsilon}_2(\vartheta)\|v(\cdot,\vartheta)\|_F.
\end{aligned}$$

In addition, we obtain

$$\|{}^cD_{0^+}^{\delta_2;\psi}u(\cdot,\vartheta)\|_\infty \leq \Phi_1(\vartheta)\frac{q_1\Psi_0^{-\delta_2}(T)}{\Gamma(2-\delta_2)} + \frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Big(\Upsilon_1'(\vartheta)\|u(\cdot,\vartheta)\|_E + \Upsilon_2'(\vartheta)\|v(\cdot,\vartheta)\|_F\Big),$$

Furthermore, we get

$$\begin{aligned}
\| u(\cdot,\vartheta)\|_E \leq \Phi_1(\vartheta)\Big(\frac{q_1\Psi_0^{-\delta_2}(T)}{\Gamma(2-\delta_2)} + 1\Big) + \Big(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Upsilon_1'(\vartheta) + \Upsilon_1(\vartheta)\Big)\|u(\cdot,\vartheta)\|_E \\
+ \Big(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Upsilon_2'(\vartheta) + \Upsilon_2(\vartheta)\Big)\|v(\cdot,\vartheta)\|_F.
\end{aligned}$$

and

$$\begin{aligned}
\|v(\cdot,\vartheta)\|_F \leq \Phi_2(\vartheta)\Big(\frac{q_2\Psi_0^{-\delta_1}(T)}{\Gamma(2-\delta_1)} + 1\Big) + \Big(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Upsilon_3'(\vartheta) + \Upsilon_3(\vartheta)\Big)\|u(\cdot,\vartheta)\|_E \\
+ \Big(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Upsilon_4'(\vartheta) + \Upsilon_4(\vartheta)\Big)\|v(\cdot,\vartheta)\|_F.
\end{aligned}$$

This implies that

$$\begin{pmatrix} 1 - N_{\delta_2}\Upsilon_1'(\vartheta) - \Upsilon_1(\vartheta) & -N_{\delta_2}\Upsilon_2'(\vartheta) - \Upsilon_2(\vartheta) \\ \\ -N_{\delta_1}\Upsilon_3'(\vartheta) - \Upsilon_3(\vartheta) & 1 - N_{\delta_1}\Upsilon_4'(\vartheta) - \Upsilon_4(\vartheta) \end{pmatrix} \begin{pmatrix} \|u(\cdot,\vartheta)\|_E \\ \\ \|v(\cdot,\vartheta)\|_F \end{pmatrix} \leq \begin{pmatrix} \overline{N}_{\delta_2}\Phi_1(\vartheta) \\ \\ \overline{N}_{\delta_1}\Phi_2(\vartheta) \end{pmatrix},$$

where

$$N_{\delta_i} = \frac{\Psi_0^{1-\delta_i}(T)}{\Gamma(2-\delta_i)}; i = 1,2,$$

and

$$\overline{N}_{\delta_1} = \frac{q_2\Psi_0^{-\delta_1}(T)}{\Gamma(2-\delta_1)} + 1, \overline{N}_{\delta_2} = \frac{q_1\Psi_0^{-\delta_2}(T)}{\Gamma(2-\delta_2)} + 1.$$

Therefore

$$\widetilde{M}(\vartheta) \begin{pmatrix} \|u(\cdot,\vartheta)\|_E \\ \\ \|v(\cdot,\vartheta)\|_F \end{pmatrix} \leq \begin{pmatrix} \overline{N}_{\delta_2}\Phi_1(\vartheta) \\ \\ \overline{N}_{\delta_1}\Phi_2(\vartheta) \end{pmatrix}.$$

Since $\widetilde{M}(\vartheta)$ satisfies the hypotheses of Lemma 1.4.11, it follows that $\widetilde{M}^{-1}$ is order preserving. We apply $\widetilde{M}^{-1}$ to both sides of the above inequality, to obtain

$$\begin{pmatrix} \|u(\cdot,\vartheta)\|_E \\ \\ \|v(\cdot,\vartheta)\|_F \end{pmatrix} \leq \widetilde{M}^{-1}(\vartheta) \begin{pmatrix} \overline{N}_{\delta_2}\Phi_1(\vartheta) \\ \\ \overline{N}_{\delta_1}\Phi_2(\vartheta) \end{pmatrix}.$$

This shows that set $\mathcal{N}(\vartheta)$ is bounded, consequently of **steps 1-4** and theorem 1.5.7, we conclude the operator Q has at least one random fixed point, which is a solution of the system (5.1)-(5.2). □

5.4 Stability

In this section, we study Ulam-Hyres stability for the solutions of our proposed system.

Definition 5.4.1. [12] The system

$$\begin{cases} u(t,\vartheta) & = Q_1(u,v)(t,\vartheta), \\ \\ v(t,\vartheta) & = Q_2(u,v)(t,\vartheta), \end{cases} \tag{5.16}$$

is said to be Ulam-Hyres stable if there exist $C_1(\vartheta), C_2(\vartheta), C_3(\vartheta), C_4(\vartheta)$ such that, for each $\varepsilon_1, \varepsilon_2 \geq 0$ and solution (u^*, v^*) of the inequality system

$$\begin{cases} \|u^*(\cdot,\vartheta) - Q_1(u^*,v^*)(\cdot,\vartheta)\|_E & \leq \varepsilon_1, \\ \\ \|v^*(\cdot,\vartheta) - Q_2(u^*,v^*)(\cdot,\vartheta)\|_F & \leq \varepsilon_2, \end{cases}$$

there exist a solution $(\overline{u}, \overline{v})$ of system (5.16) such that

$$\|\overline{u}(\cdot,\vartheta) - u^*(\cdot,\vartheta)\|_E \leq C_1(\vartheta)\varepsilon_1 + C_2(\vartheta)\varepsilon_2,$$

$$\|\overline{v}(\cdot,\vartheta) - v^*(\cdot,\vartheta)\|_F \leq C_3(\vartheta)\varepsilon_1 + C_4(\vartheta)\varepsilon_2.$$

Theorem 5.4.2. *Suppose that the hypotheses (H1)-(H3) are achieved and M converge to zero. Then the system* (5.1)-(5.2) *is Ulam-Hyres stable.*

Proof. By the theorem (5.2.1) we deduce that there exits a unique element $(u^*, v^*) \in E \times F$ such that (u^*, v^*) is a solution for (5.1)-(5.2). let $\varepsilon_1, \varepsilon_2 \geq 0$ and $(\overline{u}, \overline{v}) \in E \times F$ such that

$$\begin{aligned} \left\|\overline{u}(\cdot,\vartheta) - Q_1(\overline{u},\overline{v})(\cdot,\vartheta)\right\|_E &\leq \varepsilon_1, \\ \left\|\overline{v}(\cdot,\vartheta) - Q_2(\overline{u},\overline{v})(\cdot,\vartheta)\right\|_F &\leq \varepsilon_2, \end{aligned}$$

Thus

$$\begin{aligned} \|\overline{u}(\cdot,\vartheta) - u^*(\cdot,\vartheta)\|_E &\leq \left\|\overline{u}(\cdot,\vartheta) - Q_1(\overline{u},\overline{v})(\cdot,\vartheta)\right\|_E + \left\|u^*(\cdot,\vartheta) - Q_1(\overline{u},\overline{v})(\cdot,\vartheta)\right\|_E \\ &\leq \left\|\overline{u}(\cdot,\vartheta) - Q_1(\overline{u},\overline{v})(\cdot,\vartheta)\right\|_E + \left\|Q_1(u^*,v^*)(\cdot,\vartheta) - Q_1(\overline{u},\overline{v})(\cdot,\vartheta)\right\|_E \\ &\leq \varepsilon_1 + \left(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Upsilon_1'(\vartheta) + \Upsilon_1(\vartheta)\right)\left\|\overline{u}(.,\vartheta) - u^*(\cdot,\vartheta)\right\|_E \\ &+ \left(\frac{\Psi_0^{1-\delta_2}(T)}{\Gamma(2-\delta_2)}\Upsilon_2'(\vartheta) + \Upsilon_2(\vartheta)\right)\|\overline{v}(\cdot,\vartheta) - v^*(\cdot,\vartheta)\|_F. \end{aligned}$$

In the same way, we obtain

$$\|\overline{v}(\cdot,\vartheta)-v^*(\cdot,\vartheta)\|_F \leq \varepsilon_2+\left(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Upsilon_3'(\vartheta)+\Upsilon_3(\vartheta)\right)\left\|(\overline{u}(\cdot,\vartheta)-u^*(\cdot,\vartheta)\right\|_E$$
$$+\left(\frac{\Psi_0^{1-\delta_1}(T)}{\Gamma(2-\delta_1)}\Upsilon_4'(\vartheta)+\Upsilon_4(\vartheta)\right)\left\|(\overline{v}(\cdot,\vartheta)-v^*(\cdot,\vartheta)\right\|_F$$

$$\begin{pmatrix} d(u^*,\overline{u}) \\ d(v^*,\overline{v}) \end{pmatrix} \leq \varepsilon + M(\vartheta)\begin{pmatrix} d(u^*,\overline{u}) \\ d(v^*,\overline{v}) \end{pmatrix}.$$

Thus,

$$(I-M(\vartheta))\begin{pmatrix} d(u^*,\overline{u}) \\ d(v^*,\overline{v}) \end{pmatrix} \leq \varepsilon/$$

Since $(I-M(\vartheta))$ is invertible, we apply $(I-M(\vartheta))^{-1}$ to both sides of above inequality to obtain

$$\begin{pmatrix} d(u^*,\overline{u}) \\ d(v^*,\overline{v}) \end{pmatrix} \leq (I-M(\vartheta))^{-1}\varepsilon.$$

We denote $(I-M(\vartheta))^{-1}=\begin{pmatrix} C_1(\vartheta) & C_2(\vartheta) \\ & \\ C_3(\vartheta) & C_4(\vartheta) \end{pmatrix}$, we then obtain

$$\|\overline{u}(\cdot,\vartheta)-u^*(\cdot,\vartheta)\|_E \leq C_1(\vartheta)\varepsilon_1+C_2(\vartheta)\varepsilon_2,$$
$$\|\overline{v}(\cdot,\vartheta)-v^*(\cdot,\vartheta)\|_F \leq C_3(\vartheta)\varepsilon_1+C_4(\vartheta)\varepsilon_2.$$

Proving that the system is Ulam-Hyres stable. □

5.5 Example

We illustrate our results by an example. Let $\Omega=\mathbb{R}_-^*=(-\infty,0)$ be equipped by the usual $\sigma-$algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the following random coupled $\psi-$Caputo fractional differential system

$$\begin{cases} {}^cD_{0^+}^{\frac{3}{2};t}[{}^cD_{0^+}^{\frac{1}{4};t}u(t,\vartheta)-h(t,u_t(\vartheta),v_t(\vartheta),\vartheta)]=f(t,u_t(\vartheta),v_t(\vartheta),{}^cD_{0^+}^{\frac{1}{4};t}v(t,\vartheta),\vartheta); t\in[0,1],\vartheta\in\Omega,\\ {}^cD_{0^+}^{\frac{5}{4};t}[{}^cD_{0^+}^{\frac{1}{4};t}v(t,\vartheta)-k(t,u_t(\vartheta),v_t(\vartheta),\vartheta)]=g(t,u_t(\vartheta),{}^cD_{0^+}^{\frac{1}{7};t}u(t,\vartheta),v_t(\vartheta),\vartheta); t\in[0,1],\vartheta\in\Omega,\\ u(0,\vartheta)=\chi(v(\vartheta)),\quad D_\psi u(0,\vartheta)=0,\quad \int_0^1 v(\tau,\vartheta)d\tau=\frac{1}{2}u(\frac{1}{4},\vartheta),\\ v(0,\vartheta)=\varphi(u(\vartheta)),\quad D_\psi v(0,\vartheta)=0,\quad \int_0^1 u(\tau,\vartheta)d\tau=\frac{1}{2}u(\frac{3}{4},\vartheta), \end{cases}$$

where, $J=[0,1]$, $p_1=\frac{3}{2}$, $p_2=\frac{5}{4}$, $q_1=q_2=\delta_1=\frac{1}{4}$, $\delta_2=\frac{1}{7}$, $\kappa_1=\kappa_2=\frac{1}{2}$, $\xi=\frac{1}{4}$, $\rho=\frac{3}{4}$, $\psi(t)=t$ and

$$\begin{cases} f(t,u_t(\vartheta),v_t(\vartheta),w_t(\vartheta),\vartheta)=\dfrac{\vartheta^2e^{-t}}{4(\vartheta^2+2)\sqrt{400+t^2}}\left(\cos(u_t(\vartheta))+\dfrac{|v_t(\vartheta)|}{2+|v_t(\vartheta)|}+\tan^{-1}(w_t(\vartheta))\right),\\ g(t,u_t(\vartheta),v(t,\vartheta),w_t(\vartheta),\vartheta)=\dfrac{\vartheta^2\cos^2(t)}{100(\vartheta^2+2)}\left(\dfrac{|u_t(\vartheta)|+|v_t(\vartheta)|+|w_t(\vartheta)|}{1+|u_t(\vartheta)|+|v_t(\vartheta)|+|w_t(\vartheta)|}\right)-\sqrt{2},\\ h(t,u_t(\vartheta),v_t(\vartheta),\vartheta)=\dfrac{\vartheta^2e^{-3t}}{80(\vartheta^2+2)(1+|u_t(\vartheta)|+|v_t(\vartheta)|)},\\ k(t,u_t(\vartheta),v_t(\vartheta),\vartheta)=\dfrac{\vartheta^2\big(|u_t(\vartheta)|+|v_t(\vartheta)|\big)\sin(t)}{80(\vartheta^2+2)\big(1+|u_t(\vartheta)|+|v_t(\vartheta)|\big)},\\ \chi(v(\vartheta))=\dfrac{\vartheta^2}{81(\vartheta^2+2)}\sin\big(v(1,\vartheta)\big),\\ \varphi(u(\vartheta))=\dfrac{\vartheta^2|u(\frac{1}{2},\vartheta)|}{64\big(2+\vartheta^2+|u(\frac{1}{2},\vartheta)|\big)}. \end{cases}$$

Clearly, the functions f,g,h and k are Carathéodory. The hypothesis (H2) is satisfied with the following measurable functions:

$$k_1(t,\vartheta)=l_1(t,\vartheta)=n_1(t,\vartheta)=\frac{\vartheta^2e^{-t}}{80(\vartheta^2+2)},\quad k_2(t,\vartheta)=l_2(t,\vartheta)=n_2(t,\vartheta)=\frac{\vartheta^2\cos(t)}{100(\vartheta^2+2)},$$

$$a_1(t,\vartheta) = b_1(t,\vartheta) = \frac{\vartheta^2 e^{-3t}}{80(\vartheta^2+2)}, \quad a_2(t,\vartheta) = b_2(t,\vartheta) = \frac{\vartheta^2 \sin t}{80(\vartheta^2+2)},$$
$$l_\chi(\vartheta) = \frac{\vartheta^2}{81(\vartheta^2+2)}, \quad l_\varphi(\vartheta) = \frac{\vartheta^2}{64(\vartheta^2+2)}.$$

With the given data, we find that

$$\lambda_1 \simeq -1.31069491, \quad \lambda_2 \simeq 0.09768142, \quad \lambda_3 \simeq -1.19318575, \quad \lambda_4 \simeq -0.13733691,$$
$$\Delta_1 \simeq 1.48376793, \quad \Delta_2 \simeq 1.27419825, \quad K_1 \simeq 3.80522808, \quad K_2 \simeq 1.91051664,$$
$$\widetilde{K}_1 \simeq 1.64067230, \quad \widetilde{K}_2 \simeq 3.49398120, \quad L_1 \simeq 2.13327452, \quad L_2 \simeq 0.50098430,$$
$$\widetilde{L}_1 \simeq 0.39551389, \quad \widetilde{L}_2 \simeq 2.27045515, \quad \Upsilon_1 \simeq \frac{\vartheta^2}{2+\vartheta^2}0.11917443, \quad \Upsilon_2 \simeq \frac{\vartheta^2}{2+\vartheta^2}0.11224645,$$
$$\Upsilon_3 \simeq \frac{\vartheta^2}{2+\vartheta^2}0.12802312, \quad \Upsilon_4 \simeq \frac{\vartheta^2}{2+\vartheta^2}0.10309264, \quad \Upsilon_1' \simeq \frac{\vartheta^2}{2+\vartheta^2}0.02979361,$$
$$\Upsilon_2' \simeq \frac{\vartheta^2}{2+\vartheta^2}0.02497896, \quad \Upsilon_3' \simeq \frac{\vartheta^2}{2+\vartheta^2}0.02891936,$$
$$\Upsilon_4' \simeq \frac{\vartheta^2}{2+\vartheta^2}0.02577316,$$

and

$$M(\vartheta) \simeq \frac{\vartheta^2}{2+\vartheta^2}\begin{pmatrix} 0.15060891 & 0.13860111 \\ 0.15948927 & 0.13113553 \end{pmatrix}.$$

Where the eigenvalues of matrix $M(\vartheta)$ are:

$$\eta_1 \simeq \frac{\vartheta^2}{2+\vartheta^2}0.28986951 < 1, \quad \eta_2 \simeq \frac{-\vartheta^2}{2+\vartheta^2}0.00812508 < |\eta_2| < 1.$$

Thus, $M(\vartheta)$ is converges to zero, then by Theorem 5.3.1 the coupled system (5.5) has unique random solution on $[0,1]$ and is Ulam–Hyers stable.

CHAPTER 6

RANDOM FRACTIONAL DIFFERENTIAL COUPLED SYSTEM WITH RETARDED AND ADVANCED ARGUMENTS

6.1 Introduction

In this chapter, we investigate the existence and uniqueness of the following nonlinear random coupled system of ψ-Caputo fractional integro-differential equations[1]

$$\begin{cases} {}^{c}D_{a^+}^{p_1;\psi}u(t,\vartheta) + \sum_{i=1}^{m} I_{a^+}^{\gamma_{1,i};\psi} g_{1,i}(t,u^t(\vartheta),v^t(\vartheta),\vartheta) = f_1(t,u^t(\vartheta),v^t(\vartheta),\vartheta) \\ {}^{c}D_{a^+}^{p_2;\psi}v(t,\vartheta) + \sum_{i=1}^{m} I_{a^+}^{\gamma_{2,i};\psi} g_{2,i}(t,u^t(\vartheta),v^t(\vartheta),\vartheta) = f_2(t,u^t(\vartheta),v^t(\vartheta),\vartheta) \end{cases} ;t \in J, \vartheta \in \Omega, \tag{6.1}$$

with

$$\begin{cases} (u(t,\vartheta),v(t,\vartheta)) = (\eta_1(t,\vartheta),\eta_2(t,\vartheta)); \quad t \in [a-r,a], r > 0, \\ (u(t,\vartheta),v(t,\vartheta)) = (\xi_1(t,\vartheta),\xi_2(t,\vartheta)); \quad t \in [T,T+l], l > 0, \end{cases} ;\vartheta \in \Omega, \tag{6.2}$$

[1]**F. Fredj**, H. Hammouche. Existence of random coupled system of fractional differential equations in generalized Banach space with retarded and advanced arguments. International Conference on Mathematics and Application (ICMA' 2021) organized on the 7 – 8 December 2021 at the University of Blida 1.

where $J=[a,T]$, $D_{a^+}^{p_j;\psi}$ denotes the ψ-Caputo fractional derivative of order $1<p_j\leq 2$, $I_{a^+}^{\gamma_{i,j};\psi}$ is the ψ-Riemann-Liouville fractional integral of orders $\gamma_{i,j}>0$. $(\Omega,\mathcal{A})$ is measurable space. $g_i, f_{i,j}: J\times C([-r,l],\mathbb{R}^n)\times C([-r,l],\mathbb{R}^n)\times\Omega\to\mathbb{R}^n$ are given functions. $\eta_j\in C([a-r,a],\mathbb{R}^n)$ with $\eta_j(a,\vartheta)=0$ and $\xi\in C([T,T+l],\mathbb{R}^n)$ with $\xi(T,\vartheta)=0$; $j=1,2,\ i=1\cdots m$.
We denote by $x^t(s)$ the element of $C([-r,l])$ defined by

$$u^t(s)=u(t+s),\quad s\in[-r,l.]$$

Main results

Lemma 6.1.1. *Let $1<p\leq 2$. For any functions $G,F_i\in C(J,\mathbb{R})$, $\eta\in C([a-r,a],\mathbb{R})$ with $\eta(a)=0$ and $\xi\in C([T,T+l],\mathbb{R})$ with $\xi(T)=0$. Then the following linear problem*

$$\begin{cases} {}^cD_{a^+}^{p;\psi}u(t)+\sum_{i=1}^{m}I_{a^+}^{\gamma_i;\psi}F_i(t)=G(t), & t\in J,\\ u(t)=\eta(t); & t\in[a-r,a],r>0,\\ u(t)=\xi(t); & t\in[T,T+l],l>0.\end{cases} \tag{6.3}$$

has a unique solution, which is given by

$$u(t)=\begin{cases}\eta(t); \quad \text{if } t\in[a-r,a],\\ \\ I_{a^+}^{p;\psi}Q(t)-\sum_{i=1}^{m}I_{a^+}^{\gamma_i+p;\psi}F_i(t)\\ +\frac{\Psi_a(t)}{\Psi_a(T)}\left(\sum_{i=1}^{m}I_{a^+}^{\gamma_i+p;\psi}F_i(T)-I_{a^+}^{p;\psi}G(T)\right); \quad \text{if } t\in J.\\ \\ \xi(t); \quad \text{if } t\in[T,T+l].\end{cases} \tag{6.4}$$

Proof. Applying the ψ-Riemann-Liouville fractional integral of order p to both side of the equation in (6.3), and using Lemma 1.2.10, we get

$$u(t)=I_{a^+}^{p;\psi}G(t)-\sum_{i=1}^{m}I_{a^+}^{\gamma_i+p;\psi}F_i(t)+c_0+c_1\Psi_a(t);\qquad c_0,c_1\in\mathbb{R}. \tag{6.5}$$

Using the fact that $u(a) = \eta(a) = 0$, $u(T) = \xi(T) = 0$, and from (6.5), we find

$$u(a) = c_0 = 0,$$

and

$$u(T) = I_{a^+}^{p;\psi} G(T) - \sum_{i=1}^{m} I_{a^+}^{\gamma_i+p;\psi} F_i(T) + c_1 \Psi_a(t) = 0.$$

Some simple computations give us

$$c_1 = \frac{1}{\Psi_a(T)} \left(\sum_{i=1}^{m} I_{a^+}^{\gamma_i+p;\psi} F_i(T) - I_{a^+}^{p;\psi} G(T) \right).$$

Inserting c_0 and c_1 in eqution (6.5), which leads to solution (6.4). □

By $C([-r,l], \mathbb{R}^n)$ we denote the Banach space of all continuous functions from $[-r,l]$ into $\mathbb{R}^n$ provided with the norm

$$\|u\|_{[-r,l]} = \sup\{\|u(t)\| : -r \leq t \leq l\}.$$

$C([a,T], \mathbb{R}^n)$ is Banach space equipped with norm

$$\|x\|_{[a,T]} = \sup\{\|u(t)\| : a \leq t \leq T\}.$$

$AC(J, \mathbb{R}^n)$ is the space of absolutely continuous functions on J, and we denote by $AC^1(J, \mathbb{R}^n)$ the space of functions $u(t)$ which have continuous derivatives on J:

$$AC^1(J, \mathbb{R}^n) = \{u : J \to \mathbb{R}^n : u' \in AC(J, \mathbb{R}^n)\},$$

where

$$u'(t) = t\frac{d}{dt} g(t), \quad t \in J.$$

We denote the space X by

$$X = \{u : [a-r, T+l] \to \mathbb{R}^n : u_{|[a-r,a]} \in C([a-r,a]), u_{|[a,T]} \in AC([a,T]) \text{ and } u_{|[T,T+l]} \in C([T,T+l])\};$$

where the aforementioned space are supplemented with the following norms

$$\|u\|_{[a-r,a]} = \sup\{\|u(t)\| : a-r \leq t \leq a\},$$

$$\|u\|_{[T,T+l]} = \sup\{\|u(t)\| : T \le t \le T+l\},$$
$$\|u\|_X = \sup\{\|u(t)\| : a-r \le t \le T+l\}.$$

The product space $X \times X$ is provided with the norm

$$\|(u,v)\|_{X\times X} := \|u\|_X + \|v\|_X.$$

Lemma 6.1.2. *For given functions $g_{j,i}, f_j \in C(J,\mathbb{R}^n)$, $j=1,2$ and $i=1,\cdots,m$. A functions $u,v \in C^2$ is a random solution of systems (6.1)-(6.2) if and only if u,v satisfies the following random coupled system integral equations*

$$u(t,\vartheta) = \begin{cases} \eta_1(t,\vartheta); \quad \text{if } t \in [a-r,a], \\ I_{a^+}^{p_1;\psi} f_1(t,u^t(\vartheta),v^t(\vartheta),\vartheta) - \sum_{i=1}^{m} I_{a^+}^{\gamma_{1,i}+p_1;\psi} g_{1,i}(t,u^t(\vartheta),v^t(\vartheta),\vartheta) + \frac{\Psi_a(t)}{\Psi_a(T)} \\ \times\left(\sum_{i=1}^{m} I_{a^+}^{\gamma_{1,i}+p;\psi} g_{1,i}(T,u^T(\vartheta),v^T(\vartheta),\vartheta) - I_{a^+}^{p_1;\psi} f_1(T,u^T(\vartheta),v^T(\vartheta),\vartheta)\right); \text{ if } t \in J, \\ \xi_1(t,\vartheta); \quad \text{if } t \in [T,T+l], \end{cases} \tag{6.6}$$

and

$$v(t,\vartheta) = \begin{cases} \eta_2(t,\vartheta); \quad \text{if } t \in [a-r,a], \\ I_{a^+}^{p_2;\psi} f_2(t,u^t(\vartheta),v^t(\vartheta),\vartheta) - \sum_{i=1}^{m} I_{a^+}^{\gamma_{2,i}+p_2;\psi} g_{2,i}(t,u^t(\vartheta),v^t(\vartheta),\vartheta) + \frac{\Psi_a(t)}{\Psi_a(T)} \\ \times\left(\sum_{i=1}^{m} I_{a^+}^{\gamma_{2,i}+p_2;\psi} g_{2,i}(T,u^T(\vartheta),v^T(\vartheta),\vartheta) - I_{a^+}^{p_2;\psi} f_2(T,u^T(\vartheta),v^T(\vartheta),\vartheta)\right); \text{ if } t \in J, \\ \xi_2(t,\vartheta); \quad \text{if } t \in [T,T+l]. \end{cases} \tag{6.7}$$

To define a fixed point problem equivalent to the system (6.1)-(6.1), we introduce the operator

$$Q : J \times X \times X \times \Omega \to X \times X,$$

defined by

$$\big(Q(u,v)\big)(t,\vartheta) = \begin{pmatrix} \big(Q_1(u,v)\big)(t,\vartheta) \\ \\ \big(Q_2(u,v)\big)(t,\vartheta) \end{pmatrix},$$

where

$$\big(Q_j(u,v)\big)(t,\vartheta) = \begin{cases} \eta_j(t,\vartheta); \quad \text{if } t \in [a-r,a], \\ \\ I_{a^+}^{p_j;\psi} f_j(t,u^t(\vartheta),v^t(\vartheta),\vartheta) - \sum_{i=1}^{m} I_{a^+}^{\gamma_{j,i}+p_j;\psi} g_{j,i}(t,u^t(\vartheta),v^t(\vartheta),\vartheta) + \frac{\Psi_a(t)}{\Psi_a(T)} \\ \times \left(\sum_{i=1}^{m} I_{a^+}^{\gamma_{j,i}+p_j;\psi} g_{j,i}(T,u^T(\vartheta),v^T(\vartheta),\vartheta) - I_{a^+}^{p_j;\psi} f_j(T,u^T(\vartheta),v^T(\vartheta),\vartheta) \right); \text{ if } t \in J, \\ \\ \xi_j(t,\vartheta); \quad \text{if } t \in [T,T+l]. \end{cases} \tag{6.8}$$

6.2 Existence and Uniqueness

The first result is concerned with the uniqueness of random solution for the system (6.1)-(6.2) and is based on random versions of the Pervo fixed point theorem.

Theorem 6.2.1. *We assume that the following hypotheses holds.*

(H1) The functions $g_{j,i}$ and f_j are Carathéodory; $j = 1,2$ and $i = 1,\cdots,m$.

(H2) There exist measurable bounded functions $\mathcal{K}_j, \mathcal{L}_j, \mathcal{M}_{j,i}, \mathcal{N}_{j,i} : \Omega \to (0,\infty); j = 1,2$ and $i = 1,\cdots,m$ such that:

$$\|f_j(t,u,v,\vartheta) - f_j(t,\overline{u},\overline{v},\vartheta)\| \le \mathcal{K}_j(t,\vartheta)\|u-\overline{u}\|_{[-r,l]} + \mathcal{L}_j(t,\vartheta)\|v-\overline{v}\|_{[-r,l]},$$

and

$$\|g_{j,i}(t,u,v,\vartheta)-g_{j,i}(t,\overline{u},\overline{v},\vartheta)\| \leq \mathcal{M}_{j,i}(t,\vartheta)\|u-\overline{u}\|_{[-r,l]}+\mathcal{N}_{j,i}(t,\vartheta)\|v-\overline{v}\|_{[-r,l]},$$

for a.e.t $\in J$, and each $u,v,\overline{u},\overline{v} \in C([-r,l])$.

(H3) $M(\vartheta) \in M_{n\times n}(\mathbb{R}_+)$ *is random variable matrix, such that for every* $\vartheta \in \Omega$, *the matrix*

$$M(\vartheta)=2\begin{pmatrix} C_{p_1}\mathcal{K}_1^*(\vartheta)+C_{\gamma_1+p}\mathcal{M}_1^*(\vartheta) & C_{p_1}\mathcal{L}_1^*(\vartheta)+C_{\gamma_1+p_1}\mathcal{N}_1^*(\vartheta) \\ \\ C_{p_2}\mathcal{K}_2^*(\vartheta)+C_{\gamma_2+p_2}\mathcal{M}_2^*(\vartheta) & C_{p_2}\mathcal{L}_2^*(\vartheta)+C_{\gamma_2+p_2}\mathcal{N}_2^*(\vartheta) \end{pmatrix},$$

converges to zero.
Then the coupled system (6.1)-(6.2) *has a unique random solution.*
With $C_{p_j}=\dfrac{\Psi_a^p(T)}{\Gamma(p_j+1)}, C_{p_j+\gamma_j}=\dfrac{\Psi_a^p(T)}{\Gamma(\gamma_j+p_j+1)}; j=1,2,$ *and*

$$\mathcal{K}_j^*(\vartheta)=\|\mathcal{K}_j(\cdot,\vartheta)\|_{[a,T]}, \mathcal{L}_j^*(\vartheta)=\|\mathcal{L}_j(\cdot,\vartheta)\|_{[a,T]}, \mathcal{M}_j^*(\vartheta)=\sum_{i=1}^{m}\|\mathcal{M}_{j,i}(\cdot,\vartheta)\|_{[a,T]},$$

$$\mathcal{N}_j^*(\vartheta)=\sum_{i=1}^{m}\|\mathcal{N}_{j,i}(\cdot,\vartheta)\|_{[a,T]}, \gamma_j=\sup_i\{\gamma_{j,i}: i=1,\cdots m,\}.$$

Proof. The operator Q is a random operator on $X\times X$. Now, we prove that the operator T is contractive.
For all $\vartheta\in\Omega$, $(u,v),(\overline{u},\overline{v})\in X\times X$,. If $t\in[a-r,a]$ or $[T,T+l]$, then

$$\|(Q_1(u,v))(t,\vartheta)-(Q_1(\overline{u},\overline{v}))(t,\vartheta)\|=0.$$

For $t\in J$, we have

$$\begin{aligned}
&\big\|\big(Q_1(u,v)\big)(\cdot,\vartheta)-\big(Q_1(\overline{u},\overline{v})\big)(\cdot,\vartheta)\big\|_X \\
&\leq 2\bigg(\frac{\Psi_a^{p_1}(T)}{\Gamma(p_1+1)}\mathcal{K}_1^*(\vartheta)+\frac{\Psi_a^{\gamma_{1,i}+p_1}(T)}{\Gamma(\gamma_{1,i}+p_1+1)}\mathcal{M}_1^*(\vartheta)\bigg)\|u(\cdot,\vartheta)-\overline{u}(\cdot,\vartheta)\|_X \\
&+2\bigg(\frac{\Psi_a^{p_1}(T)}{\Gamma(p_1+1)}\mathcal{L}_1^*(\vartheta)+\frac{\Psi_a^{\gamma_{1,i}+p_1}(T)}{\Gamma(\gamma_{1,i}+p_1+1)}\mathcal{N}_1^*(\vartheta)\bigg)\|v(\cdot,\vartheta)-\overline{v}(\cdot,\vartheta)\|_X,
\end{aligned}$$

and

$$\begin{aligned}&\left\|\left(Q_2(u,v)\right)(\cdot,\vartheta)-\left(Q_2(\overline{u},\overline{v})\right)(\cdot,\vartheta)\right\|_X\\&\leq 2\left(\frac{\Psi_a^{p_2}(T)}{\Gamma(p_2+1)}\mathcal{K}_2^*(\vartheta)+\frac{\Psi_a^{\gamma_{2,i}+p_2}(T)}{\Gamma(\gamma_{2,i}+p_2+1)}\mathcal{M}_2^*(\vartheta)\right)\|u(\cdot,\vartheta)-\overline{u}(\cdot,\vartheta)\|_X\\&+2\left(\frac{\Psi_a^{p_2}(T)}{\Gamma(p_2+1)}\mathcal{L}_2^*(\vartheta)+\frac{\Psi_a^{\gamma_{2,i}+p_2}(T)}{\Gamma(\gamma_{2,i}+p_2+1)}\mathcal{N}_2^*(\vartheta)\right)\|v(\cdot,\vartheta)-\overline{v}(\cdot,\vartheta)\|_X.\end{aligned}$$

Thus,

$$d\Big(\big(Q(u,v)\big)(\cdot,\vartheta),\big(Q(\overline{u},\overline{v})\big)(\cdot,\vartheta)\Big)\leq M(\vartheta)d\Big(\big(u(\cdot,\vartheta),v(\cdot,\vartheta)\big),\big(\overline{u}(\cdot,\vartheta),\overline{v}(\cdot,\vartheta)\big)\Big),$$

where

$$d\Big(\big(u(\cdot,\vartheta),u(\cdot,\vartheta)\big),\big(\overline{v}(\cdot,\vartheta),\overline{v}(\cdot,\vartheta)\big)\Big)=\begin{pmatrix}\|u(\cdot,\vartheta)-\overline{u}(\cdot,\vartheta)\|_X\\ \\ \|v(\cdot,\vartheta)-\overline{v}(\cdot,\vartheta)\|_X\end{pmatrix}.$$

As for every $\vartheta\in\Omega$, the matrix $M(\vartheta)$ converges to zero, this implies that the operator Q is a $M(\vartheta)-$contractive operator. Consequently, by theorem 1.5.4, we conclude that Q has a unique fixed point, which is a random solution of systems (6.1)-(6.2). This completes the proof. □

6.3 Existence result

In the next result, we prove the existence of solution for the system (6.1)-(6.2) by applying a random version of a fixed point theorem.

Theorem 6.3.1. *Assume that (H1)-(H2) and the following hypotheses holds.*

(H4) there exist measurable functions $\varphi_j,\chi_j,\omega_j,\lambda_{j,i},\rho_{j,i},\mu_{j,i}:J\to(0,X);i=1,2$ *and* $i=1,\cdots,m$ *such that:*

$$\|f_j(t,u,v,\vartheta)\|\leq\varphi_j(t,\vartheta)+\chi_j(t,\vartheta)\|u\|_{[-r,l]}+\omega_j(t,\vartheta)\|v\|_{[-r,l]},$$

$$\|g_{j,i}(t,u,v,\vartheta)\|\leq\lambda_{j,i}(t,\vartheta)+\rho_{j,i}(t,\vartheta)\|u\|_{[-r,l]}+\mu_{j,i}(t,\vartheta)\|v\|_{[-r,l]},$$

for a.e.t $\in J$*, and each* $u,v\in\mathbb{R}^n$.

Then the coupled system (6.1)-(6.2) *has at least a random solution.*

Proof. We need to prove that the operators Q satisfies all conditions of the Theorem 1.5.5. The proof is divided into several steps.

step 1. $Q(\cdot,\cdot,\vartheta)$ is continuous operator: Let (u_n,v_n) be a sequence such that

$$(u_n,v_n) \to (u,v) \in X \times X \qquad \text{as } n \to \infty.$$

Then, for each $\vartheta \in \Omega$ and for , If $t \in [a-r,a]$ or $[T,T+l]$, we have

$$\Big\|\big(Q(u_n,v_n)\big)(t,\vartheta) - \big(Q(u,v)\big)(t,\vartheta)\Big\| = 0.$$

For $t \in J$, we get

$$\begin{aligned}
&\Big\|\big(Q(u_n,v_n)\big)(t,\vartheta) - \big(Q(u,v)\big)(t,\vartheta)\Big\| \\
&\leq \Bigg(2\bigg(\frac{\Psi_a^{p_j}(T)}{\Gamma(p_j+1)}\mathcal{K}_j^*(\vartheta) + \frac{\Psi_a^{\gamma_{j,i}+p_j}(T)}{\Gamma(\gamma_{j,i}+p_j+1)}\mathcal{M}_j^*(\vartheta)\bigg)\|u(\cdot,\vartheta)-\overline{u}(\cdot,\vartheta)\|_X \\
&+2\bigg(\frac{\Psi_a^{p_j}(T)}{\Gamma(p_j+1)}\mathcal{L}_j^*(\vartheta) + \frac{\Psi_a^{\gamma_{j,i}+p_j}(T)}{\Gamma(\gamma_{j,i}+p_j+1)}\mathcal{N}_j^*(\vartheta)\bigg)\|v(\cdot,\vartheta)-\overline{v}(\cdot,\vartheta)\|_X\Bigg) \to 0 \quad \textbf{as } n \to \infty.
\end{aligned}$$

Hence, $Q(\cdot,\cdot)(t,\vartheta)$ is continuous.

step 2. $Q(\cdot,\cdot,\vartheta)$ maps bounded sets into bounded sets in $X \times X$.

First, we set

$$\varphi_j^*(\vartheta) = \|\varphi_j(\cdot,\vartheta)\|_{[a,T]}, \chi_j^*(\vartheta) = \|\chi_j(\cdot,\vartheta)\|_{[a,T]}, \omega_j^*(\vartheta) = \|\omega_j(\cdot,\vartheta)\|_{[a,T]},$$

$$\lambda_j^*(\vartheta) = \sum_{i=1}^m \|\lambda_{j,i}(\cdot,\vartheta)\|_{[a,T]}, \rho_j^*(\vartheta) = \sum_{i=1}^m \|\rho_{j,i}(\cdot,\vartheta)\|_{[a,T]}, \mu_j^*(\vartheta) = \sum_{i=1}^m \|\mu_{j,i}(\cdot,\vartheta)\|_{[a,T]}.$$

Indeed, it is enough to show that for any $r > 0$ there exists a positive constant R such that

$$\|(Q(u,v)))(\cdot,\vartheta)\|_{X\times X} \leq R(\vartheta) = (R_1(\vartheta), R_2(\vartheta))$$

for each $(u,v) \in B_r = \{(u,v) \in X \times X : \quad \|u\|_X \leq r, \|v\|_X \leq r\}$, for all $t \in J$ and for $j = 1,2$, we get

$$\|\big(Q_j(u,v)\big)(t,\vartheta)\|$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} I_{a^+}^{\gamma_{j,i}+p_j;\psi}\|g_{j,i}(s,u^s(\vartheta),v^s(\vartheta),\vartheta)\|(t) + \sum_{i=1}^{m} I_{a^+}^{\gamma_{j,i}+p_j;\psi}\|g_{j,i}(s,u^s(\vartheta),v^s(\vartheta),\vartheta)\|(t)\\
&\leq 2\frac{\Psi_a^{\gamma_{j,i}+p_j}(T)}{\Gamma(\gamma_{j,i}+p_j+1)}\Big(\lambda_j^*(\vartheta)+\rho_j^*(\vartheta)\|u(\cdot,\vartheta)\|_X+\mu_j^*(\vartheta)\|v(\cdot,\vartheta)\|_X\Big)\\
&\leq 2\frac{\Psi_a^{\gamma_{j,i}+p_j}(T)}{\Gamma(\gamma_{j,i}+p_j+1)}\Big(\lambda_j^*(\vartheta)+r\Big(\rho_j^*(\vartheta)+\mu_j^*(\vartheta)\Big)\Big) = R_j(\vartheta).
\end{aligned}
$$

Hence

$$
\Big\|\big(Q(u,v)\big)(\cdot,\vartheta)\Big\|_{X\times X} = \Big\|\Big(\big(Q_1(u,v)\big)(\cdot,\vartheta),\big(Q_2(u,v)\big)(\cdot,\vartheta)\Big)\Big\|_{X\times X} \leq (R_1(\vartheta),R_2(\vartheta)) = R(\vartheta).
$$

step 3. $Q(\cdot,\cdot,\vartheta)$ maps bounded sets into equicontinuous sets of $X\times X$.
Let B_r be a bounded set of $X\times X$ as in **step.2**, let $t_1,t_2\in J$, where $t_1>t_2$, and any $(u,v)\in B_r$, $\vartheta\in\Omega$ and for $j=1,2$, we have

$$
\begin{aligned}
&\Big\|\big(Q_j(u,v)\big)(t_1,\vartheta)-\big(Q_j(u,v)\big)(t_2,\vartheta)\Big\|\\
&\leq \left|\frac{\Psi_a(t_1)-\Psi_a(t_2)}{\Psi_a(T)}\right|\sum_{i=1}^{m} I_{a^+}^{\gamma_{j,i}+p_j;\psi}\|g_{j,i}(s,u^s(\vartheta),v^s(\vartheta),\vartheta)\|(T)\\
&+\sum_{i=1}^{m}\int_{t_2}^{t_1}\frac{\psi'(s)\big(\psi(t_1)-\psi(s)\big)^{\gamma_{j,i}+p_j-1}}{\Gamma(\gamma_{j,i}+p_j)}\Big\|g_{j,i}(s,u^s(\vartheta),v^s(\vartheta),\vartheta)\Big\|ds\\
&+\sum_{i=1}^{m}\int_{a}^{t_2}\frac{\psi'(s)\Big(\big(\psi(t_1)-\psi(s)\big)^{\gamma_{j,i}+p_j-1}-\big(\psi(t_2)-\psi(s)\big)^{\gamma_{j,i}+p_j-1}\Big)}{\Gamma(\gamma_{j,i}+p_j)}\Big\|g_{j,i}(s,u^s(\vartheta),v^s(\vartheta),\vartheta)\Big\|ds\\
&\leq \Bigg(\left|\frac{\Psi_a(t_1)-\Psi_a(t_2)}{\Psi_a(T)}\right|\frac{\Psi_a^{\gamma_{j,i}+p_j}(T)}{\Gamma(\gamma_{j,i}+p_j+1)}+\frac{\big(\psi(t_1)-\psi(t_2)\big)^{\gamma_{j,i}+p_j}}{\Gamma(\gamma_{j,i}+p_j+1)}\\
&+\left|\frac{\big(\psi(t_1)-\psi(a)\big)^{\gamma_{j,i}+p_j}}{\Gamma(\gamma_{j,i}+p_j+1)}-\frac{\big(\psi(t_2)-\psi(a)\big)^{\gamma_{j,i}+p_j}}{\Gamma(\gamma_{j,i}+p_j+1)}\right|\Bigg)\Big(\lambda_j^*(\vartheta)+r\Big(\rho_j^*(\vartheta)+\mu_j^*(\vartheta)\Big)\Big)\to 0 \text{ as } t_2\to t_1.
\end{aligned}
$$

Thus the operators Q_1 and Q_2 are equicontinuous. Moreover Q is also equicontinuous. Hence by the Ascoli-Arzila theorem, we deduce that Q is compact. We conclude the operator Q has at least one random fixed point, which is a solution of the system (6.1)-(6.2). □

Conclusion and perspectives.

In this thesis, we have successfully investigated the existence, uniqueness and stability in the sense of Ulam of the solutions for a new classes ψ-Caputo type hybrid fractional differential equation with hybrid conditions. The existence of solutions is provided by using Dhage fixed point theorem [16], whereas the uniqueness result is achieved by Banach's fixed point theorem. After that, we have studied the concept of Ulam-Hyres and generalized Ulam-Hyres stabilities in third chapter. While in the rest of chapters, we have studied the existence, uniqueness and stability of random fractional differential systems by the use of generalized random fixed point theorems in generalized Banach spaces. Also, we have presented an illustrative examples to support our main results.

In future works, many results can be established when one takes a more generalized operator. Precisely, it will be of interest to study the current problem in this work for the fractional operator with variable order [54], and ψ-Hilfer fractional operator [49].

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