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Some Remark of Harmonic Forms on Riemannian Manifold

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Dedication

I am dedicating this work :

To my dear father Mohamed and dear mother Fatna who never stopped to support and encourage me during my years of study. That they find here the testimony of my deep gratitude,

To my brothers Said, Abelkader and sisters Hadjira, Fatiha and family who give love and support.

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Daoudi Messaouda

Abstract

In this thesis, we will try to generalize the Laplacian on Euclidien space to operator of differential forms on a Riemannian manifold and prove the Hodge theory, with give a notion of the Riemannian manifold. Our goal is to understand how can a differential forms on manifold to be harmonic. The basic idea on harmonic forms on Riemannian manifold is that gives an information about Riemannian manifold and the Laplacian on compact Riemannian manifold and citation by the Hodge theory.

MSC2020 : 70G45, 35J91, 53C43, 58A10 .
Keywords— Differential geometry methods, Semilineair elliptic equations with Laplacian, Differential geometry aspects of harmonic maps, Differential forms

Résumé

Dans cette thèse, nous essaierons de généraliser le laplacien sur l'espace euclidien à un opérateur de formes différentielles sur une variété riemannienne et de prouver la théorie de Hodge, en donnant une notion de la variété riemannienne. Notre objectif est de comprendre comment une forme différentielle sur une variété peut être harmonique. L'idée de base sur les formes harmoniques sur la variété riemannienne est que cela donne une information sur la variété riemannienne et le laplacien sur la variété riemannienne compacte et la citation par la théorie de Hodge.

MSC2020 : 70G45, 35J91, 53C43, 58A10 .
Mots clés— Méthodes de géométrie différentielle, Equations elliptiques semi-lineaires avec Laplacien, Aspects géométriques differentiels.

ملخص

في هذه المذكرة سنحاول تعميم لابلاسيان على الفضاء الإقليدي لتشغيل الأشكال التفاضلية على مشعب ريماني و إثبات نظرية هودج , مع إعطاء فكرة عن المشعب الريماني . هدفنا هو فهم كيف يمكن أن تكون الأشكال التفاضلية في المشعب الريماني متناسقة. الفكرة الأساسية عن الأشكال التوافقية على مشعب ريماني و لابلاسيان على مشعب ريماني مضغوط واستشهاد بنظرية هودج.

MSC2020: 70G45, 35J91, 53C43, 58A10.

الكلمات المفتاحية— طرق الهندسة التفاضلية, المعادلات الإهليجية نصف الخطية مع لابلاسيان, الجوانب الهندسية التفاضلية للخرائط التوافقية, الأشكال التفاضلية

CONTENTS

INTRODUCTION

The differential geometry is a continuity of the infinitesimal calculus, it allows to study thanks to the techniques of the differential calculus a family of topological spaces called “differentiable manifold ”allowing the geometry renovation of curves and surfaces of real spaces, and placing it according to a contemporary context.
In this thesis we study the generalization of the Laplacian on Euclidien space to operator of differential forms on a Riemannian manifold, with the notion of Riemannian manifold, we gives some examples and some information on the connection , curvature and the tensor and Hodge theorey with citation of the Laplacian on Riemannian manifold.

The notion of a manifold needed an axiomatic definition by other mathematicians (of the Gottinger persuasion). D. Hilbert (1862–1943) sought an axiomatic characterization of the plane sufficient for the foundations of geometry in Appendix IV (1902) to the Grundlagen der Geometrie [Hil13].Hilbert define the plane by a system of neighborhoods that satisfy certain topological conditions. The locally Euclidean neighborhoods played a fondamental rule in many refinements that led to the definition of manifold used today. Among his topological axioms, Hilbert assumes the existence of large neighborhoods (for any pair of points in the plane, exist a neighborhood containing them). This assumption was dropped by H. Weyl (1885–1955) in his 1913 Der Idee der Riemannschen Flache. Such Weyl depened his definition of surface on a neighborhood system that gave a basis for the topology on the surface and satisfied an open map condition. Absent from the definition schemes of Hilbert and Weyl is the Hausdorff condition on the underlying topological space a fault pointed out by Hausdorff in his axiomatic treatment of topological spaces [HM27].

The important lemma of Riemannian geometry states that there exists a unique Riemannian connection ∇, i.e. a derivation on any Riemannian manifold (M^n, g) of vector fields with respect to vector fields following the rules of linearity and the product rule of Leibniz and which is compatible with the differential structure on M^n,(in that the commutator of this connection is identical with the Lie bracket of vector fields), and which, as well is compatible with the geometrical structure g on (M^n, g), (in that $\nabla g = 0$; -from deriving vector fields by ∇ one can normally get to deriving arbitrary tensor fields by ∇-). ∇ is given by the standard formula of Koszul and the corresponding expressions for the Riemann-Christoffel, for the Ricci and for the Weyl conformal curvature tensors R, S and C respectively, etc., were systematically developed by Nomizu in his thesis with Chern. The Riemannian or sectional curvatures $K(p, \pi)$ were known to be scalar valued isometric invariants of (M^n, g), determined at any point p and for any 2D tangent plane section π at p, right away since their introduction by Riemann, when their name curvature derived from the analogy of their calculation with the intrinsic formula for the Gauss curvature K of 2D surfaces M^2 in Euclidean 3D spaces E^3, now applied for the Gauss curvature at point p in M^2 of the 2D surface formed by the geodesics of (M^n, g) G^2 around p which are tangent to π at p . As regards further appreciations of curvatures K, one could also base e.g. on the formulas of Bertrand-Puisseux and of Diguet referring to the perimeters or the areas of geodesic circles or discs on (M^n, g) in comparison with the perimeters and areas of Euclidean circles and discs of the same radii. Butstriving for better truly geometrical insights in the curvature tensor R or equivalently in the sectional curvatures K, as already mentioned before, around the same time and independently, Levi-Civita and Schouten introduced the notion of parallel (or pseudo parallel) transport of vectors along curves in (M^n, g), -which is equivalent with the notion of Riemannian connection- to obtain their geometrical interpretations of R and K in terms of the lengths of the sides and the areas of parallelogramoids and of holonomy of vectors or of directions, respectively.[Ver14]

Riemann found the wright way to extend into n dimensions the differential geometry of surfaces. The fundamental object is called the Riemann curvature tensor. For the surface case, this can be reduced to a number (scalar), positive, negative or zero; the non-zero and constant cases being models of the known non-Euclidean geometries. contributions to analysis and differential geometry. He was first one to discover Riemannian geometry is the branch of differential geometry that studies Riemannian manifolds, smooth manifolds with a Riemannian metric, i.e. with an inner product on the tangent space at each point which varies smoothly from point to point. This gives in particular local notions of angle, length of curves, surface area, and volume.

In this thesis we will give the necessary background of Laplace equation.Pierre-Simon Laplace (1749-1827) was led to what is now known as Laplace's equation in three variables. The two-variable version of this equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The zeros operator on the equation is the Laplacian denoted by Δ.
The Laplacian differential operator, widely used in mathematics that is named after him for his memory and legacy. For example, the inverse mathematical problem of spectral theory identities features of the geometry from information about the eigenvalues of the Laplacian. Other examples that Laplacian is defined can be shown as follows; analysis on fractals, time scale calculus and discrete exterior calculus. In physics, the Laplacian occurs in a couple of partial differential equations that describe basic physical phenomena such as the propagation of waves or diffusion processes. In wave propagation, Laplace remodelled Newton's force law and stated that gravitational field has the same properties with radiation field or fluid. Theoretic representation of gravitational field is defined through radiation field. However, this approach is not accepted by classical physics especially with the contributions of Lorentz. In addition to that, The Laplacian plays key role in steady-state fluid flow, static electric field, heat diffusion, and quantum particles [Coo11].

The Hodge Decomposition Theorem or De Rham Decomposition Theorem is the main of this thesis. Also, Laplacian notion will be given in classical way. For generalization of Laplacian, differentiable manifold and Riemannian metric will be defined as well as introducing Laplacian on Riemannian manifold. In the final part demonstration of Hodge theory will take place with the Hodge Decomposition Theorem.

CHAPTER

1 PRELIMINARY

In this chapter we difined the notion of a manifold embedded in some ambient space $\mathbb{R}^n$.In order to give a maximization of the range of applications of the theory of manifolds it is important to generalize the consept of manifold to spaces that are not embedded in some $\mathbb{R}^n$. the basic idea is that any manifold is a topological spaces that can be covered by a collection of open subsets where is a isometric to some open set of $\mathbb{R}^n$. The manifold wold be duall without function defined on them and between them. Geometry arises from spaces and intersting classes of function between them. In this chapter , we use the following references ; [DCFF92], [Gud21], [KY85], [BGM71], [Can13],[Mas01].
Beginning with the concept of differential manifold

1.1 Differential manifold

Definition 1.2. ***(Topological Hausdorff space)***
Let X be a topological space, two points x and y in X are separable if can be separated by neighbourhoods i.e.: there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$). X is a topological Hausdorff space if all distinct points in X are pairwise neighbourhood-separable.

Definition 1.3. ***(Topological manifold)***
Let (M, τ) be a topological Hausdorff space. M is called a ***topological manifold*** *if there exists an $n \in \mathbb{N}$ such that for each point $p \in M$ we have an open neighbourhood U of p, an open subset V of $\mathbb{R}^n$and a homeomorphism $x : U \to V$.*
The pair (U,x) is called a ***local chart*** *(or local coordinates) on M.*
The integer n is called the ***dimension of M****. To denote that the dimension of M is n we write M^n.*
We can define the topological manifold in other way such there exist another homeomorphism $y : V \to U$ has the same as the other.

Definition 1.4. ***(C^r-atlas)*** *Let M be an n-dimensional topological manifold and a family of C^r-deffeomorphisms $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ of open sets u_α of $\mathbb{R}^n$ into M. A C^r***-atlas*** on M is a collection*

$$\mathcal{A} = \{(U_\alpha, x_\alpha) \mid \alpha \in I\}$$

of local charts on M such that $\mathcal{A}$ covers the whole of M i.e.

$$M = \bigcup_{\alpha} x_\alpha(U_\alpha)$$

for any pair α, β with

$$x_\alpha(U_\alpha) \bigcap x_\beta(U_\beta) = w \neq \emptyset$$

the sets $x_\alpha^{-1}(w)$ and $x_\beta^{-1}(w)$ are open sets in $\mathbb{R}^n$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are C^r differentiable .

Remark 1.5. *A local chart (U, x) on M is said to be* ***compatible*** *with a C^r-atlas $\mathcal{A}$ if the union $\mathcal{A} \cup \{(U, x)\}$ is a C^r-atlas.*
A C^r-atlas $\hat{\mathcal{A}}$ is said to be ***maximal*** *if it contains all the local charts that are compatible with it.*
*A maximal atlas $\hat{\mathcal{A}}$ on M is also called a C^r**-structure** on M.*
A differentiable manifold is said to be ***smooth*** *if it is of class C^∞.*

Definition 1.6. ***(Differential manifold)***
A ***differentiable manifold*** *of dimension n of class r is a pair $(M, \hat{\mathcal{A}})$ such that M is a topological manifold and $\hat{\mathcal{A}}$ is a C^r-structure on M. (fig 1.1).*

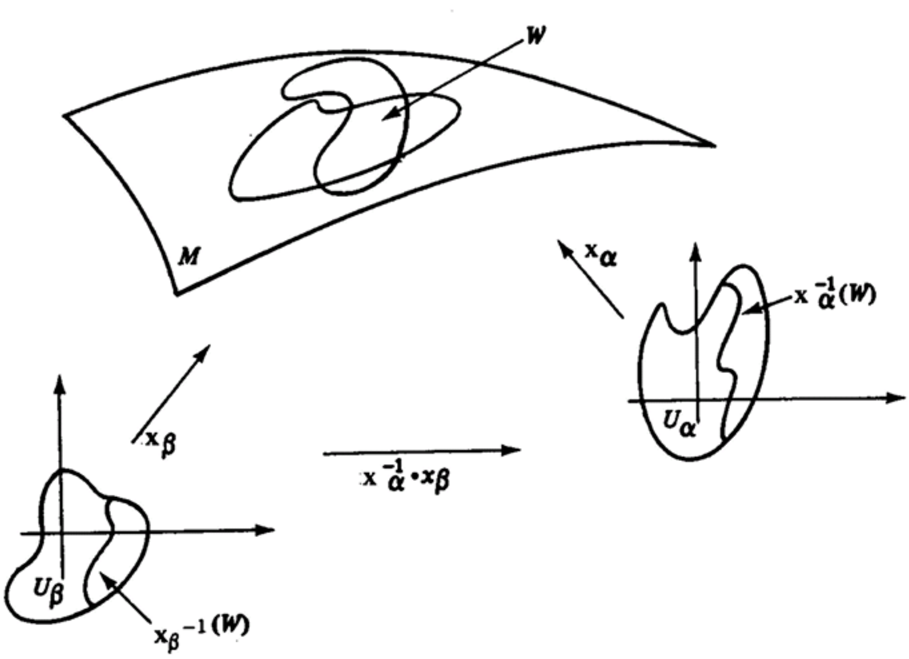

Figure 1.1:

Definition 1.7. ***(Other definition of differential manifold)***
A differentiable manifold of dimension n of class r is a topological Hausdorff space M and a family of deffeomorphisms $x_\alpha : u_\alpha \subset \mathbb{R}^n \to M$ of open sets u_α of $\mathbb{R}^n$ into M such that:

(1) $\bigcup_\alpha x_\alpha = M$.

(2) for any pair α, β with $x_\alpha(u_\alpha) \bigcap x_\beta(u_\beta) = w \neq \emptyset$ the sets $x_\alpha^{-1}(w)$ and $x_\beta^{-1}(w)$ are open sets in $\mathbb{R}^n$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are differentiable (fig 1.1).

(3) the familly $\{(u_\alpha, x_\alpha)\}$ is maximal relative to condition(1) and (2)

A family $\{(u_\alpha, x_\alpha)\}$ satisfying (1) and (2) is called a ***differentiable structure*** *on M.*

Definition 1.8. *The pair (u_α, x_α) (or mapping x_α) with $p \in x_\alpha(u_\alpha)$ is called a* ***parametrization*** *(or system of coordinates) of M at p.*
$x_\alpha(u_\alpha)$ is called a ***coordinate neighborhood*** *at p.*

Definition 1.9. ***(Mapping between manifolds)***
Let M_1^n and M_2^m be differentiable manifolds. A mapping $\varphi : M_1 \to M_2$ is differentiable at $p \in M_1$ if given a parametrization $y : v \subset \mathbb{R}^m \to M_2$ at $\varphi(p)$ there exists a parametrization $x : v \subset \mathbb{R}^n \to M_1$ at p such that $\varphi(x(u)) \subset y(v)$ and the mapping:

$$y^{-1} \circ \varphi \circ x : u \subset \mathbb{R}^n \to \mathbb{R}^m$$

is differentiable at $x^{-1}(p)$(fig1.2).
φ is differentiable on an open set of M_1 if differentiable at all of the points of this set.

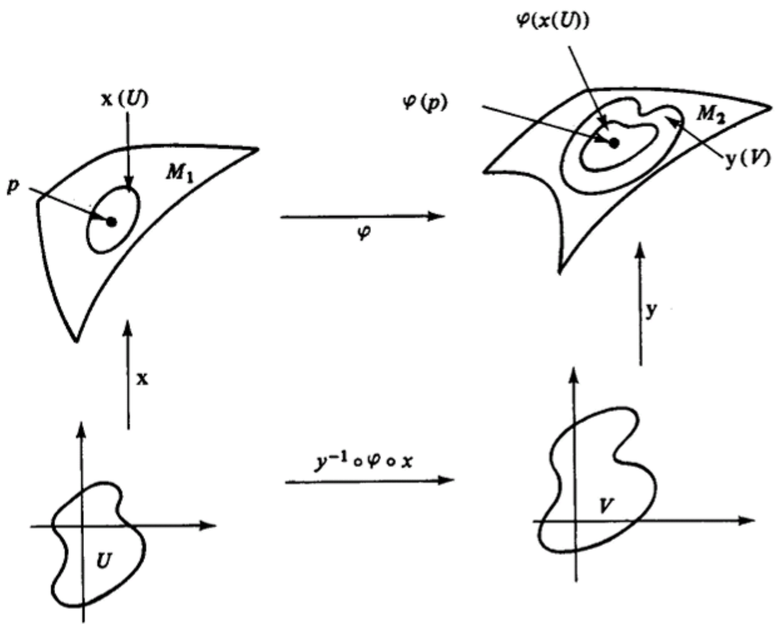

Figure 1.2:

Proposition 1.10. *It follous from (2) of definition 1.7 that the given definition is independent of the choise of the parametrizations.*

Proposition 1.11. *Let $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ be two differentiable manifolds of class C^r. Let $M = M_1 \times M_2$ be the product space with the product topology. Then there exists an atlas $\mathcal{A}$ on M turning $(M, \hat{\mathcal{A}})$ into a differentiable manifold of class C^r and the dimension of M satisfies:*

$$dim\ M = dim\ M_1 + dim\ M_2$$

1.12 Vector field, brackets

1.12.1 Tangent space

Definition 1.13. . *A differentiable mapping $c : I \to M$ of an open interval $I \subset \mathbf{R}$ into a differentiable manifold M is called a (parametrized)* ***curve****.(Fig.1.3).*

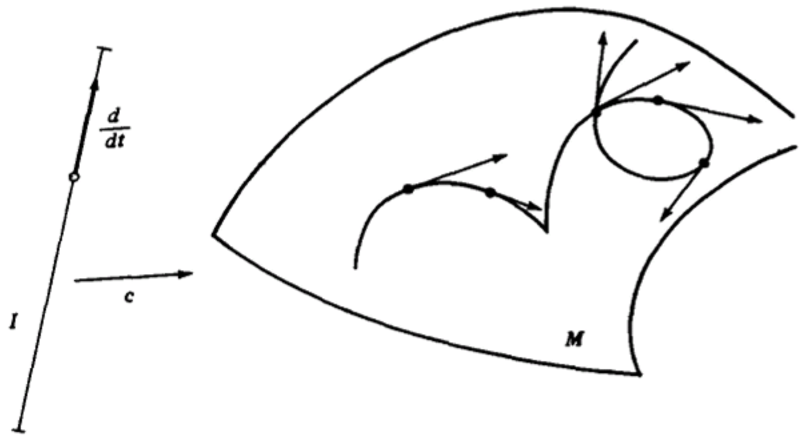

Figure 1.3:

Here, introducing the fundamental concept of a tangent vector on differentiable (smooth) manifolds.
The next considerations will motivate the definition that we are going to present below.
Let $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a differentiable curve in $\mathbb{R}^n$, with $\alpha(0) = p$ ($p \in \mathbb{R}^n$ such $p = (x_1(0), ..., x_n(0))$), write

$$\alpha(t) = (x_1(t), ..., x_n(t)), t \in (-\epsilon, \epsilon), \quad (x_1(t), ..., x_n(t))|_t \in \mathbb{R}^n$$

then,

$$\alpha'(t) = (x_1'(t), ..., x_n'(t)) = v \in \mathbb{R}^n.$$

Now let f ($f : U_p \subset \mathbb{R}^n \to C^\infty(M)$) be a differentiable function defined in a neighborhood of p. We can restrict f to the curve α and express the directional derivative with respect to the vector $v \in \mathbb{R}^n$ as

$$\left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0} = \sum_{i=1}^{n} \left.\frac{df}{dx_i}\right|_{t=0} \cdot \left.\frac{dx_i}{dt}\right|_{t=0} = (\sum_i x_i'(0) \frac{d}{dx_i}) f.$$

therefore, the directional derivative with respect to v is an operator on differentiable functions that depends uniquely on v. This is the characteristic property that we are going to use to define tangent vectors on manifold.

Definition 1.14. ***(Tangent vector)***
let M be a differentiable manifold, and let $\alpha : (-\epsilon, \epsilon) \to M$ be a differentiable curve in M. Suppose that $\alpha(0) = p \in M$, and let $C^\infty(M)$ be the set of functions on M that are differentiable at p. The ***tangent vector*** *to the curve α at $t = 0$ is the function $\alpha'(0) : C^\infty(M) \to \mathbb{R}$ given by:*

$$\alpha'(0) f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}, \quad f \in C^\infty(M).$$

A tangent vector at p is the tangent vector at $t = 0$ of some curve $\alpha(-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p will be indicated by T_pM.

If we choose a parametrization $x : U \to M^n$ at $p = x(0)$, we can express the function f and the curve α in this parametrization by:

$$f \circ x(q) = f(x_1, ..., x_n), \quad q = (x_1, ..., x_n) \in U,$$

and,

$$x^{-1} \circ \alpha(t) = (x_1(t), ..., x_n(t)),$$

respectively. therfore, restricting f to α, we obtain

$$\alpha'(0)f = \frac{d}{dt}(f \circ \alpha)\bigg|_{t=0} = \frac{d}{dt} f\left(x_1(t), \ldots, x_n(t)\right)\bigg|_{t=0}$$
$$= \sum_{i=1}^{n} x_i'(0)\left(\frac{\partial f}{\partial x_i}\right) = \left(\sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right)\right) f$$

In other words, the vector $\alpha'(0)$can be expressed in the parametrization x by

$$\alpha'(0) = \sum_i x_i'(0)\left(\frac{\partial}{\partial x_i}\right) \tag{1.1}$$

Observe that $\left(\frac{\partial}{\partial x_i}\right)$ is tangent vector at p of the "coordinate curve" (fig1.4) :

$$x_i = x(0, ..., 0, x_i, 0, ..., 0)$$

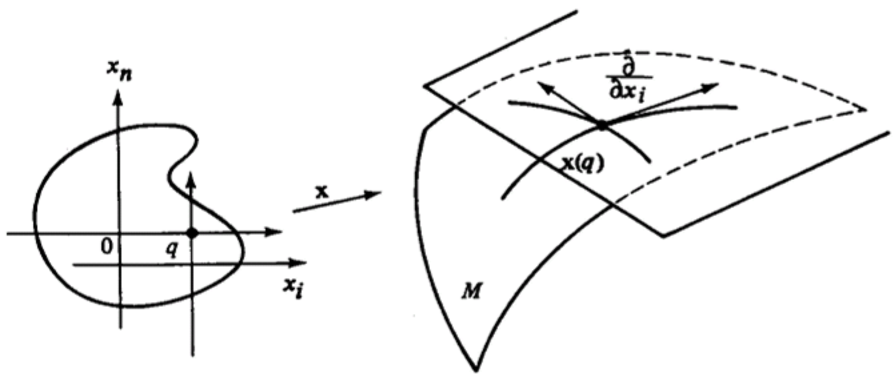

Figure 1.4:

The expression (1.1) shows that the tangent vector to the curve α at p depends only on the derivative of α in a coordinate system. It follows also from (1.1) that the T_pM, with the usual operations of functions forms a vector space of dimension n, and that the choice of a parametrization $x : U \to M$ determines an associated basis $\left\{\left(\frac{\partial}{\partial x_i}\right)_p, ..., \left(\frac{\partial}{\partial x_i}\right)_p\right\}$ in T_pM (fig1.4).
The linear structure in T_pM defined above does not depend on the parametrization x. The vector space T_pM is called the tangent space of M at p.

We can extend to differentiable manifolds the notion of the differential of differentiable mapping with the idea of tangent space.

Definition 1.15. ***(Tangent bundle)*** *The tangent bundle is the union of all tangent spaces at every point on the manifold M. It is denoted by TM*

$$TM = \{(p, v); p \in M, v \in T_pM\}$$

such that

$$TM = \bigcup_{p \in M} T_pM$$

Proposition 1.16. *Let M_1^n and M_2^m be differentiable manifolds and $\varphi : M_1 \to M_2$ be differentiable mapping. for every $p \in M_1$ and for each $v \in T_pM_1$, choose a differentiable curve $\alpha(-\epsilon, \epsilon) \to M_1$ with $\alpha(0) = p, \alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ given by $d\varphi_p(v) = \beta'(0)$ is linear mapping that does not depend on choice of α (fig 1.5).*

Proof. Let $x : U \to M_1$ and $y : V \to M_2$ be parametrizations at p and $\varphi(p)$, respectively. Expressing φ in these parametrizations, we can write:

$$\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x}(q) = (y_1 (x_1, \ldots, x_n), \ldots, y_m (x_1, \ldots, x_n))$$
$$q = (x_1, \ldots, x_n) \in U, \quad (y_1, \ldots, y_m) \in V$$

On the other hand, expressing α in the parametrization $\mathbf{x}$, we obtain:

$$\mathbf{x}^{-1} \circ \alpha(t) = (x_1(t), \ldots, x_n(t))$$

Therefore,

$$\mathbf{y}^{-1} \circ \beta(t) = (y_1 (x_1(t), \ldots, x_n(t)), \ldots, y_m (x_1(t), \ldots, x_n(t)))$$

It follows that the expression for $\beta'(0)$ with respect to the basis $\left\{(\frac{\partial}{\partial y_i})_0\right\}$ of $T_{\varphi(p)}M_2$, associated to the parametrization y, is given by:

$$\beta'(0) = \left(\sum_{i=1}^{n} \frac{\partial y_1}{\partial x_i} x_i'(0), \ldots, \sum_{i=1}^{n} \frac{\partial y_m}{\partial x_i} x_i'(0)\right) \tag{1.2}$$

The relation (1.2) shows that $\beta'(0)$ does not depend on the choice of α. In addition, (1.2) can be written as:

$$\beta'(0) = d\varphi_p(v) = \left(\frac{\partial y_i}{\partial x_j}\right) \left(x_j'(0)\right) \quad i = 1, \ldots, m; j = 1, \ldots, n$$

where $\left(\frac{\partial y_i}{\partial x_j}\right)$ denotes an $m \times n$ matrix and $\left(x_j'(0)\right)$ denotes column matrix with n elements. Therefore, $d\varphi_p$ is a linear mapping of T_pM_1 into $T_{\varphi(p)}M_2$ whose matrix in the associated bases obtained from the parametrizations x and y is precisely the matrix $(\frac{\partial y_i}{\partial x_j})$

■

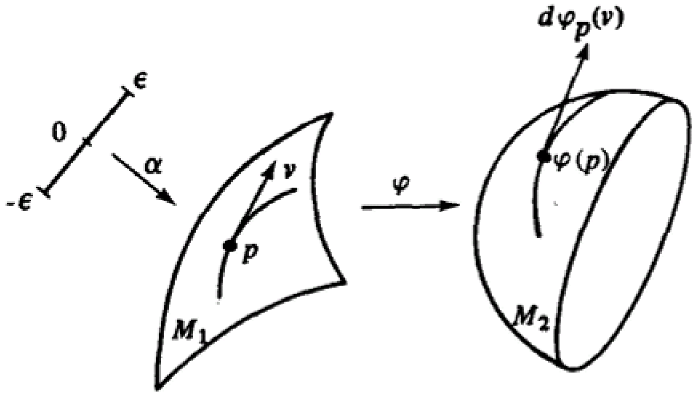

Figure 1.5:

Definition 1.17. *The linear mapping $d\varphi_p$ defined by proposition 1.16 is called the* ***differential*** *of φat p.*

Definition 1.18. *Let M_1, and M_2 be differentiable manifolds. A mapping $\varphi : M_1, \to M_2$ is a* ***diffeomorphism*** *if it is differentiable, bijective (bijective means every element of arrival set has a unique antecedent in departure set), and its inverse φ^{-1} is differentiable.*

Definition 1.19. *A mapping $\varphi : M_1, \to M_2$ is said to be a* ***local diffeomorphism*** *at $p \in M$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism.*

Remark 1.20. *The notion of diffeomorphism is the natural idea of equivalence between differentiable manifolds. It is consequence of the chain rule that if $\varphi : M_1 \to M_2$ is a diffeomorphism, then $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism for all $p \in M_1$, in particular, the dimensions of M_1 and M_2 are equal.*

A local converse to this fact is the following theorem.

Theorem 1.21. ***(A local converse theorem)*** *Let $\varphi : M_1^n \to M_2^n$ be a differentiable mapping and let $p \in M_1$ be such that $d\varphi(p) : T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism (an isomorphism is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping). Then φ is a local diffeomorphism at p.*

The proof follows from the application of the local inverse function theorem in $\mathbb{R}^n$.

1.21.1 Vector field

Definition 1.22. ***(Vector filed)*** *A vector field X on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into the tangent bundle TM ($TM = \{(s, v); s \in M, v \in T_sM\}$). The field is differentiable if the mapping $X : M \to TM$ is differentiable.*

Considering a parametrization $x : U \subset \mathbb{R}^n \to M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}, \;\; \forall p \in U \tag{1.3}$$

where each $a_i : U \subset \mathbb{R}^n \to \mathbb{R}$ is a function on U and $\left\{\frac{\partial}{\partial x_i}\right\}$ is the basis associated to x. X is differentiable if and only if the functions a_i are differentiable for for any parametrization. We denoted the set of all vertors fields by $\Gamma(TM)$.

Occasionally, it is convenient to use the idea suggested by (1.3) and think of a vector field as a mapping $X : C^\infty(M) \to F$ from the set $C^\infty(M)$ of differentiable functions on M to the set F of functions on M, defined in the following way

$$(Xf)(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial f}{\partial x_i}(p) \tag{1.4}$$

where f denotes, by abuse of notation, the expression of f in the parametrization x.
Indeed, this idea of a vector as a directional derivative was precisely what was used to define the notion of tangent vector.
The function Xf obtained in (1.4) does not depend on the choice of parametrization x. In this context, X is differentiable if and only if $X : C^\infty(M) \to C^\infty(M)$, that is , $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$.

If $\varphi : M \to M$ is a diffeomorphism , $v \in T_pM$ and f is a differentiable function in a neighborhood of $\varphi(p)$, we have

$$(d\varphi(v)f)\varphi(p) = v(f \circ \varphi)(p).$$

Indeed, let $\alpha : (-\epsilon, \epsilon) \to M$ be a differentiable curve with $\alpha'(0) = v, \alpha(0) = p$. Then

$$(d\varphi(v)f)\varphi(p) = \frac{d}{dt}(f \circ \varphi \circ \alpha)|_{t=0} = v(f \circ \varphi)(p).$$

Example 1.23. *(The tangent bundle) Let M^n be a differentiable manifold and let $TM = \{(p, v); p \in M, v \in T_pM\}$. We are going to provide the set TM with a differentiable structure (of dimension 2n). This is the natural space to work with when treating questions that involve positions and velocities, as in the case of mechanics.*

Let $\{(U_\alpha, x_\alpha)\}$ f be a maximal differentiable structure on M. Denote by $(x_1^\alpha, \ldots, x_n^\alpha)$ the coordinates of U_α and by $\left\{\frac{\partial}{\partial x_1^\alpha}, \ldots, \frac{\partial}{\partial x_n^\alpha}\right\}$ the associated bases to the tangent spaces of $\mathbf{x}_\alpha(U_\alpha)$. For every α, define:

$$\mathbf{y}_\alpha : U_\alpha \times \mathbf{R}^n \to TM$$

by:

$$\mathbf{y}_\alpha\left(x_1^\alpha, \ldots, x_n^\alpha, u_1, \ldots, u_n\right) = \left(\mathbf{x}_\alpha\left(x_1^\alpha, \ldots, x_n^\alpha\right) \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha}\right), \quad (u_1, \ldots, u_n) \in \mathbf{R}^n.$$

Geometrically, this means that we are taking as coordinates of a point $(p, v) \in TM$ the coordinates $x_1^\alpha, \ldots, x_n^\alpha$ of p together with the coordinates of v in the basis $\left\{\frac{\partial}{\partial x_1^\alpha}, \ldots, \frac{\partial}{\partial x_n^\alpha}\right\}$.
We are going to show that $\{(U_\alpha \times \mathbf{R}^n, \mathbf{y}_\alpha)\}$ is a differentiable structure on TM. Since:

$$\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$$

and,

$$(d\mathbf{x}_\alpha)_q(\mathbf{R}^n) = T_{\mathbf{x}_\alpha(q)}M, \quad q \in U_\alpha,$$

we have that:

$$\bigcup_\alpha \mathbf{y}_\alpha(U_\alpha \times \mathbf{R}^n) = TM$$

which verifies condition (1) of Definition 1.7. Now let:

$$(p, v) \in \mathbf{y}_\alpha(U_\alpha \times \mathbf{R}^n) \cap \mathbf{y}_\beta(U_\beta \times \mathbf{R}^n).$$

Then:

$$(p, v) = (\mathbf{x}_\alpha(q_\alpha), d\mathbf{x}_\alpha(v_\alpha)) = (\mathbf{x}_\beta(q_\beta), d\mathbf{x}_\beta(v_\beta))$$

where $q_\alpha \in U_\alpha, q_\beta \in U_\beta$, $v_\alpha, v_\beta \in \mathbf{R}^n$. Therefore,

$$\begin{aligned}\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha(q_\alpha, v_\alpha) &= \mathbf{y}_\beta^{-1}(\mathbf{x}_\alpha(q_\alpha), d\mathbf{x}_\alpha(v_\alpha)) \\ &= \left(\left(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha\right)(q_\alpha), d\left(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha\right)(v_\alpha)\right),\end{aligned}$$

Since $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ is differentiable, $d\left(\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha\right)$ is as well. It follows that $\mathbf{y}_\beta^{-1} \circ \mathbf{y}_\alpha$ is differentiable, which verifies condition (2) of the definition 1.7 and completes the example.

1.23.1 Lie Brackets

The interpretation of vector filed X as an operator on $C^\infty(M)$ permits us to consider the iterates of X. For example, if X and Y are vector fields on M and $f : M \to \mathbb{R}$ is a differentiable function, we can consider the functions $X(Yf)$ and $Y(Xf)$. In general, such operations do not lead to vector fields, because they involve derivatives of order highter than one. Nevertheless, we can affirm the following.

Lemma 1.24. *Let X and Y be differenltiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z such that, for all $f \in C^\infty(M)$,*

$$Zf = (XY - YX)f.$$

Proof. First, we prove that if Z exists, then it is unique. Assume, therefore, the existence of such a Z. Let $p \in M$ and let $x : U \to M$ be a parametrisation at p, and let:

$$X = \sum_i a_i \frac{\partial}{\partial x_i}, \quad Y = \sum_j b_j \frac{\partial}{\partial x_j}$$

be the expressions for N and Y in these parametrizations. Then for all $f \in C^\infty(M)$

$$XYf = X(\sum_j b_j \frac{\partial f}{\partial x_j}) = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$YXf = Y(\sum_i a_i \frac{\partial f}{\partial x_i}) = \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Therefore, Z in given, in the paramlrization x, by

$$Zf = XYf - YXf = \sum_{i,j} (a_i \frac{\partial b_j}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_i}) \frac{\partial f}{\partial x_j}$$

which proves the uniqueness of Z.

To show existence, define Z_α in each coordinate neighborhood $x_\alpha(U_\alpha)$ of a differentiable structure $\{(x_\alpha, U_\alpha)\}$ on M by the previous expression. By uniqueness,
$Z_\alpha = Z_\beta$ on $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$, which allows us to define Z over the entire manifold M.
■

Definition 1.25. *The vector field given by lemma* (1.24) *is called the* ***Lie Bracket*** *of X and Y, Z denoted $[X, Y] = XY - YX$*

The bracket operation has the following properties:

Proprety 1.26. *If X,Y and Z are differentiable vector fields on M, a, b are real nunbers, and f,g are differentiable functions, then:*

(a) $[X, Y] = -[Y, X]$ *(anticommutativity),*

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *(linearity),*

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity),*

(d) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$

Proof. To prove (a) we have

$$[X,Y] = XY - YX = -YX + XY = -(YX - XY) = -[Y,X]$$

To prove (b) we have

$$\begin{aligned}
[aX + bY, Z] &= (aX + bY)Z - Z(aX + bY) \\
&= aXZ + bYZ - Z(aX) - Z(bY) \\
&= aXZ + bYZ - aZX - bZY \\
&= aXZ - aZX + bYZ - bZY \\
&= a(XZ - ZX) + b(YZ - ZY) \\
&= a[X,Z] + b[Y,Z].
\end{aligned}$$

In order to prove (c), it suffices to observe that, on the one hand,

$$[[X,Y],Z] = [XY - YX, Z] = XYZ - YXZ - ZXY + ZYX$$

while, on the other hand,

$$[X,[Y,Z]] + [Y,[Z,X]] = XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY.$$

Because the second members of the expressions above are equal , (c) follows using (a). Finally, to prove (d), calculate

$$\begin{aligned}
[fX, gY] &= fX(gY) - gY(fX) \\
&= fgXY + fX(g)Y - gfYX - gY(f)X \\
&= fg[X,Y] + fX(g)Y - gY(f)X
\end{aligned}$$

■

Since a differentiable manifold is locally diffeomorphic to $\mathbb{R}^n$, the fundamental theorem on existence, uniqueness, and dependence on initial conditions of ordinary differential equations (which is a local theorem) extends naturally to differentiable manifolds.
For later use, it is convenient to state it explicitly here. The reader not familiar with differential equations can assume the statement below, which is all that we need.

Let X be a vector field on a differentiable manifold M, and let $p \in M$. Then there exist a neighborhood $U \subset M$ of p, an interval $(-\delta, \delta) \subset \mathbb{R}$, $\delta > 0$, and a differentiable mapping $\varphi : \mathbb{R} \times M \to M$ where $\varphi : (-\delta, \delta) \times U \to M$ such that the curve $t \to \varphi(t, q)$, $t \in (-\delta, \delta)$, $q \in U$ and $\varphi(0, q) = q$.

A curve $\alpha : (-\delta, \delta) \to M$ which satisfies the conditions $\alpha'(t) = X(\alpha(t))$ and $\alpha(0) = q$ is called a trajectory of the field X that passes through q for $t = 0$. The theorem above guarantees that for each point of a certain neighborhood there passes a unique trajectory of and on the "initial condition" q. It is common to use the notation $\varphi_t(q) = \varphi(t, q)$ and call $\varphi_t : U \to M$ the local flow of X.

The interpretation of the bracket $[X,Y]$, mentioned above, is contained in the following proposition.

Proposition 1.27. *Let X, Y be vector fields on a differentiable manifold M, let $p \in M$, and let φ_t be the local flow of X (flow of the vector field X is a differentiable function of the form $\varphi : U \subset \mathbb{R}^n \to M$ such $\forall_{t \in U}(d\varphi_t = X_{\varphi(t)})$) in a neighborhood U of p. Then*

$$[X,Y](p) = \lim_{t \to 0} \frac{1}{t} \left[Y - d\varphi_t Y\right](\varphi_t(p)).$$

For the proof, we need the following lemma from calculus.

Lemma 1.28. *Let $h : (-\delta, \delta) \times U \to \mathbf{R}$ be a differentiable mapping with $h(0, q) = 0$ for all $q \in U$. Then there exists a differentiable mapping $g : (-\delta, \delta) \times U \to \mathbf{R}$ with $h(t, q) = tg(t, q)$; in particular,*

$$g(0, q) = \left.\frac{\partial h(t, q)}{\partial t}\right|_{t=0}$$

Proof. of lemma 1.28. It suffices to define, for fixed t,

$$g(t, q) = \int_0^1 \frac{\partial h(ts, q)}{\partial (ts)} ds$$

and, after changing variables, observe that

$$tg(t, q) = \int_0^t \frac{\partial h(ts, q)}{\partial (ts)} d(ts) = h(t, q).$$

■

Proof. of the Proposition 1.27. Let f be a differentiable function in a neighborhood of p. Putting

$$h(t, q) = f(\varphi_t(q)) - f(q),$$

and applying the lemma we obtain a differentiable function $g(t, q)$ such that

$$f \circ \varphi_t(q) = f(q) + tg(t, q) \quad \text{and} \quad g(0, q) = Xf(q).$$

Accordingly

$$((d\varphi_t Y) f)(\varphi_t(p)) = (Y(f \circ \varphi_t))(p) = Yf(p) + t(Yg(t, p))$$

Therefore

$$\begin{aligned}
\lim_{t \to 0} \frac{1}{t} [Y - d\varphi_t Y] f(\varphi_t p) &= \lim_{t \to 0} \frac{(Yf)(\varphi_t p) - Yf(p)}{t} - (Yg(0, p)) \\
&= (X(Yf))(p) - (Y(Xf))(p) \\
&= ((XY - YX)f)(p) = ([X, Y]f)(p)
\end{aligned}$$

■

1.29 Submersion, Immersion, Embeddings

Definition 1.30. ***(Submersion)*** *Let M^m and N^n be differentiable manifolds ,a mapping φ from M to N is said to be a* ***submersion*** *if the differential $d\varphi = \varphi_* : T_pM \to T_{\varphi(p)}N (p \in M)$ of $\varphi : M \to N$ is sujective (surjective map means that for any element of the arrival set there exist at least one an element of the starting set that is the image of it.) for each $p \in M$*

Definition 1.31. ***(Immersion and embeddings)*** *Let M^m and N^n be differentiable manifolds, a differentiable mapping $\varphi : M \to N$ is said to be an* ***immersion*** *if $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective (injective means any element of the arrival set is the image of at most one point of the departure set, (perhaps none)), for all $p \in M$.*
If in addition φ is a homeomorphism onto $\varphi(M) \subset N$, where $\varphi(M)$ has the subspace topology induced from N, we say that y is an ***embedding****. If $M \subset N$ and the inclusion $\iota : M \subset N$ is*

an embedding, we say that M is a submanifold of N.
It can be seen that if $\varphi : M^m \to N^n$ is an immersion, then $m \leq n$; the difference $n - m$ is called the ***co-dimension*** *of the immersion φ.*

Example 1.32. *The curve $\alpha : \mathbf{R} \to \mathbf{R}^2$ given by $\alpha(t) = (t^3, t^2)$ is a differentiable mapping but is not an immersion. Indeed, the condition for the map to be an immersion in this case is equivalent to the fact that $\alpha'(t) \neq 0$, which does not occur for $t = 0$ (Fig. 1.6).*

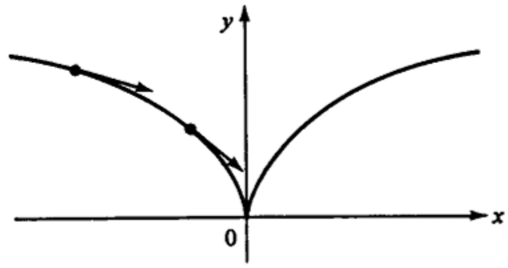

Figure 1.6:

Example 1.33. *The curve $\alpha(t) = (t^3 - 4t, t^2 - 4)$ (Fig. 1.7) is an immersion $\alpha : \mathbf{R} \to \mathbf{R}^2$ which has a self-intersection for $t = 2, t = -2$. Therefore, α is not an embedding.*

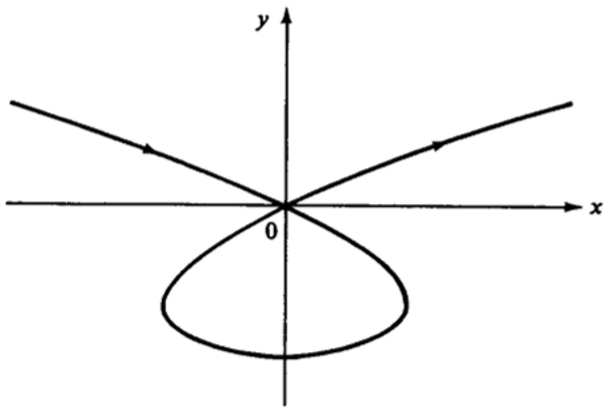

Figure 1.7:

1.34 Orientation

Definition 1.35. *(Orientation) Let M be a differentiable manifold. We say that M is orientable if M admits a differentiable structure $\{(U_\alpha, X_\alpha)\}$ such that:*

($\star$) *for every pair α, β, with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the differential of the change of coordinates $x_\beta^{-1} \circ x_\alpha$ has strict positive determinant .*

In the opposite case, we say that M is non-orientable.
If M is orientable, a choice of a differentiable structure satisfying ($\star$) is called an orientation of M.

Proposition 1.36. *Two differentiable structures that satisfy ($\star$) determine the same orientation if their union again satisfies ($\star$).*
If M is orientable and connected there exist exactly two distinct orientations on M.
Now let M_1 and M_2, be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a diffeomorphism. M_1 is orientable if and only if M_2 is orientable.

Corollary 1.37. *If M_1 and M_2 are connected and are oriented, φ induces an orientation on M_2 which may or may not coincide with the initial orientation of M_2. In the first case, we say that φ preserves the orientation and in the second case, that φ reverses the orientation.*

Example 1.38. *The simple criterion of the previous example can be used to show that the sphere*

$$S^n = \left\{ (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1}; \sum_{i=1}^{n+1} x_i^2 = 1 \right\} \subset \mathbf{R}^{n+1}$$

is orientable.

Using the stereographic projection, let $N = (0, \dots, 0, 1)$ be the north pole and $S = (0, \dots, 0, -1)$ the south pole of S^n.
Define a mapping $\pi_1 : S^n - \{N\} \to \mathbf{R}^n$ (stereographic projection from the north pole) that takes $p = (x_1, \dots x_{n+1})$ in $S^n - \{N\}$ into the intersection of the hyperplane $x_{n+1} = 0$ with the line that passes through p and N. It is easy to verify that (Fig. 1.8)

$$\pi_1 (x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right).$$

The mapping π_1 is differentiable, injective and maps $S^n - \{N\}$ onto the hyperplane $x_{n+1} = 0$. The stereographic projection $\pi_2 : S^n - \{S\} \to \mathbf{R}^n$ from the south pole onto the hyperplane $x_{n+1} = 0$ has the same properties.

Therefore, the parametrizations $\left(\mathbf{R}^n, \pi_1^{-1}\right), \left(\mathbf{R}^n, \pi_2^{-1}\right)$ cover S^n. In addition, the change of coordinates:

$$y_j = \tfrac{x_j}{1 - x_{n+1}} \leftrightarrow y'_j = \tfrac{x_j}{1 + x_{n+1}} \, (y_1, \dots, y_n) \in \mathbf{R}^n, \quad j = 1, \dots, n$$

is given by

$$y'_j = \frac{y_j}{\sum_{i=1}^n y_i^2}$$

(here we use the fact that $\sum_{k=1}^{n+1} x_k^2 = 1$). Therefore, the family $\left\{ \left(\mathbf{R}^n, \pi_1^{-1}\right), \left(\mathbf{R}^n, \pi_2^{-1}\right) \right\}$ is a differentiable structure on S^n. Observe that the intersection $\pi_1^{-1}(\mathbf{R}^n) \cap \pi_2^{-1}(\mathbf{R}^n) = S^n - \{N \cup S\}$ is connected, thus S^n is orientable and the family given determines an orientation of S^n.

Now let $A : S^n \to S^n$ be the antipodal map given by $A(p) = -p$, $p \in \mathbf{R}^{n+1}$. A is differentiable and $A^2 = Id$.
Therefore, A is a diffeomorphism of S^n. Observe that when n is even, A reverses the orientation of S^n and when n is odd, A preserves the orientation of S^n.

1.39 Affine Connections

Let us indicate by $C^\infty(M)$ the ring of real-valued functions of class C^∞ defined on M.

Definition 1.40. ***An affine connection*** ∇ *on a differentiable manifold M is a mapping*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

which is denoted by $(X, Y) \xrightarrow{\nabla} \nabla_X Y$ and which satisfies the following properties:

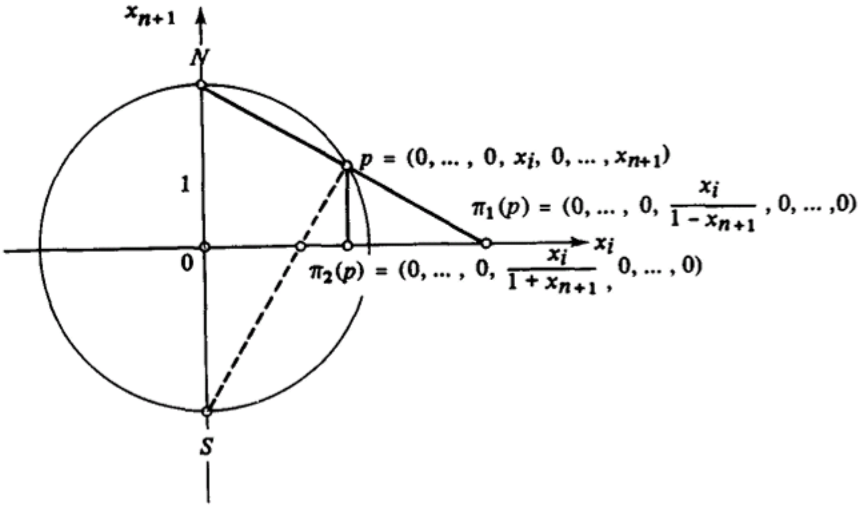

Figure 1.8:

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \Gamma(TM)$ *and* $f, g \in C^\infty(M)$.

Proposition 1.41. *Let M be a differentiable manifold with an affine connection ∇. There exists a unique correspondence which associates to a vector field V along the differentiable curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along c, called the* ***covariant derivative*** *of V along c, such that:*

a) $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$.

b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, *where W is a vector field along c and f is a differentiable function on I.*

c) If V is induced by a vector field $Y \in \Gamma(TM)$, i.e., $V(t) = Y(c(t))$, then

$$\frac{DV}{dt} = \nabla_{dc/dt} Y.$$

Remark 1.42. *The last line of (c) makes sense, since $\nabla_X Y(p)$ depends on the value of $X(p)$ and the value Y along a curve, tangent to X at p.*
In effect, part (iii) of Definition 1.40 allows us to show that the notion of affine connection is actually a local notion . Choosing a system of coordinates $(x_1, \ldots, x_n)$ about p and writing

$$X = \sum_i x_i X_i, \quad Y = \sum_j y_j Xj$$

where $X_i = \frac{\partial}{\partial x_i}$, *we have*

$$\nabla_X Y = \sum_i x_i \nabla_{X_i} \left(\sum_j y_j X_j \right) = \sum_{ij} x_i y_j \nabla_{X_i} X_j + \sum_{ij} x_i X_i \left(y_j\right) X_j$$

Setting $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$, *we conclude that the* Γ_{ij}^k *are differentiable functions and that*

$$\nabla_X Y = \sum_k \left(\sum_{ij} x_i y_j \Gamma_{ij}^k + X(y_k) \right) X_k$$

which proves that $\nabla_X Y(p)$ depends on $x_i(p), y_k(p)$ and the derivatives $X(y_k)(p)$ of y_k by X.

Proof. of Proposition 1.41. Let us suppose initially that there exists a correspondence satisfying (a), (b) and (c). Let $\mathbf{x} : U \subset \mathbf{R}^n \to M$ be a system of coordinates with $c(I) \cap \mathbf{x}(U) \neq \phi$ and let $(x_1(t), x_2(t), \ldots, x_n(t))$ be the local expression of $c(t), t \in I$. Let $X_i = \frac{\partial}{\partial x}$. Then we can express the field V locally as $V =$ Let $X_i = \frac{\partial}{\partial x_i}$. Then we can express the field V locally as $\sum_j v^j X_j, j = 1, \ldots, n$, where $v^j = v^j(t)$ and $X_j = X_j(c(t))$.

By a) and b), we have

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_j v^j \frac{DX_j}{dt}.$$

By c) and (i) of Definition 1.40,

$$\begin{aligned} \frac{DX_j}{dt} &= \nabla_{dc/dt} X_j = \nabla_{\left(\Sigma \frac{dx_i}{di} X_i\right)} X_j \\ &= \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j, \quad i, j = 1, \ldots, n. \end{aligned}$$

Therefore,

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{X_i} X_j. \tag{1.5}$$

The expression (1.5) shows us that if there is a correspondence satisfying the conditions of Proposition 1.41, then such a correspondence is unique.

To show existence, define $\frac{DV}{dt}$ in $\mathbf{x}(U)$ by (1.5). It is easy to verify that (1.5) possesses the desired properties. If $\mathbf{y}(W)$ is another coordinate neighborhood, with $\mathbf{y}(W) \cap \mathbf{x}(U) \neq \phi$ and we define $\frac{DV}{dt}$ in $\mathbf{y}(W)$ by (1.5), the definitions agree in $\mathbf{y}(W) \cap \mathbf{x}(U)$, by the uniqueness of $\frac{DV}{dt}$ in $x(U)$. It follows that the definition can be extended over all of M, and this concludes the proof. ■

The concept of parallelism now follows in a natural manner.

Definition 1.43. *Let M be a differentiable manifold with an affine connection ∇. A vector field V along a curve $c : I \to M$ is called* ***parallel*** *when $\frac{DV}{dt} = 0$, for all $t \in I$.*

Proposition 1.44. *Let M be a differentiable manifold with an affine connection ∇. Let $c : I \to M$ be a differentiable curve in M and let V_o be a vector tangent to M at $c(t_o), t_o \in I$ (i.e. $V_o \in T_{c(t_o)} M$). Then there exists a unique parallel vector field V along c, such that $V(t_o) = V_o$, (($V(t)$ is called the* ***parallel transport*** *of $V(t_o)$ along c).*

Proof. Suppose that the theorem was proved for the case in which $c(I)$ is contained in a local coordinate neighborhood.
By compactness, for any $t_1 \in I$, the segment $c([t_o, t_1]) \subset M$ can be covered by a finite number of coordinate neighborhoods, in each of which V can be defined, by hypothesis. From uniqueness, the definitions coincide when the intersections are not empty, thus allowing the definition of V along all of $[t_o, t_1]$.

we have only, therefore, to prove the theorem when $c(I)$ is contained in a coordinate neighborhood $\mathbf{x}(U)$ of a system of coordinates $\mathbf{x} : U \subset \mathbf{R}^n \to M$. Let $\mathbf{x}^{-1}(c(t)) = (x_1(t), \ldots, x_n(t))$

be the local expression for $c(t)$ and let $V_o = \sum_j v_o^j X_j$, where $X_j = \frac{\partial}{\partial x_j}(c(t_o))$.

Suppose that there exists a vector field V in $x(U)$ which is parallel along c with $V(t_o) = V_o$. Then $V = \sum v^j X_j$ satisfies

$$\frac{DV}{dt} = \sum_j \frac{dv^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{X_i} X_j = 0.$$

Putting $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$, and replacing j with k in the first sum, we obtain

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k \right\} X_k = 0$$

The system of n differential equations in $v^k(t)$,

$$\frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k v^j \frac{dx_i}{dt} = 0, \quad k = 1, \ldots, n,$$

possesses a unique solution satisfying the initial conditions $v^k(t_o) = v_o^k$. It then follows that, if V exists, it is unique.
Moreover, since the system is linear, any solution is defined for all $t \in I$, which then proves the existence (and uniqueness) of V with the desired properties.

■

1.45 Tensors on a differential manifold

Definition 1.46. ***((s,r) Tensor)***
For all $p \in M$, define the vectorial space

$$T_p^{(s,r)} M = \underbrace{T_p M \otimes \cdots \otimes T_p M}_{s \text{ times}} \otimes \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{r \text{ times}}$$

*An element $T \in T_p^{(s,r)} M$ is a **tensor of type** (s,r) above p. In a coordinate associated basis (x^i) on neighborhood of p, write*

$$T_{|p} = T_{j_1 \ldots j_r}^{i_1 \ldots i_o}(p) \frac{\partial}{\partial x^{i_1}}(p) \otimes \cdots \otimes \frac{\partial}{\partial x^{i_s}}(p) \otimes dx_{|p}^{j_1} \otimes \cdots \otimes dx_{|p}^{j_r}$$

Definition 1.47. ***(Tensor filed)***
We can consider the differentiable manifold

$$T^{(s,r)} M = \bigcup_{p \in M} T_p^{(s,r)} M$$

*which is a bundle over M, **the bundle of tensors** of type (s,r). C^∞ sections of this bundle will be called **tensor fields** of type (s,r). A tensor field T of type (s,r) above a local map of M, with coordinates (x^i), locally write as*

$$T = T_{j_1 \ldots j_r}^{i_1 \ldots i_\pi} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_r}$$

Remark 1.48. *Globally, a tensor of type (s,r) is an application $\mathscr{F}(M)$ multilinear on $A^1(M) \times \cdots \times A^1(M) \times \Gamma(TM) \times \cdots \times \Gamma(TM)$ with values in $\mathscr{F}(M)$.*
A tensor field of type $(0,0)$ is just a function on M.
A tensor of type $(1,0)$ is a vector field.
A tensor of type $(0,1)$ is a differential 1-form.

When changing coordinates $x^i \mapsto y^j\left(x^i\right)$, the components of the tensor change according to the relation

$$T'^{i_1 \dots i_s}_{j_1 \dots j_r} = \frac{\partial y^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial y^{i_s}}{\partial x^{k_s}} T^{k_1 \dots k_s}_{\ell_1 \dots \ell_r} \frac{\partial x^{\ell_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\ell_r}}{\partial y^{j_r}}$$

1.49 Differential forms

Let V be a real n-dimensional vectorial space and let V^* be its dual space.

Definition 1.50. *The space of alternating k-forms is defined as follows:*

$$\Lambda^k\left(V^*\right) = \{\omega : V \times \cdots \times V_{(k \text{ times })} \rightarrow \mathbb{R} : \omega \text{ is } k\text{-linears and alternating } \}. \tag{1.6}$$

were the form ω is linear and alternating if $\omega\left(v_1, \dots, v_n\right)$ is linear in each argument and

$$\omega\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right) = \varepsilon(\sigma)\omega\left(v_1, \dots, v_n\right). \tag{1.7}$$

such that σ is an permutation of symetric group, and $\varepsilon(\pi)$ is its signature.

Remark 1.51. $\dim \Lambda^k\left(V^*\right) = \frac{n!k!}{(n-k)!}$.

Definition 1.52. ***(k-differential form filed)*** *Choose a chart (U, φ) about x with local coordinates $(x_1, \dots, x_n)$.*
An element $\omega_x \in \Lambda^k\left(T^*_x M\right)$ is called differntial k-*form at x and can be written as*

$$\omega_x = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1 \dots i_k}\left(dx_{i_1}\right)_x \wedge \cdots \wedge \left(dx_{i_k}\right)_x.$$

We denoted the set of all differential $k-$form by $\Lambda^k\left(T^*M\right)$ where

$$\Lambda^k\left(T^*M\right) := \bigcup_{x \in M} \Lambda^k\left(T^*_x M\right).$$

and it is called k-differential form filed, Then all k-differential form is a C^∞section of this filed.

Proposition 1.53. *$\Lambda^k\left(T^*M\right)$ is a manifold of dimension $n + \frac{nlk!}{(n-k)!}$.*

Definition 1.54. ***(k-differential form)*** *A k-form on M is defined as a section of the bundle $\Lambda^k\left(T^*M\right)$. That is a C^∞ map $\omega : M \rightarrow \Lambda^k\left(T^*M\right)$.*
We denote the space of k-forms on M by $A^k(M)$.
We write $A(M) := \bigoplus_{k=0}^n A^k(M)$ and $A^0(M) = C^\infty(M, \mathbb{R})$.

Proposition 1.55. *$AA(M)$ is an algebra structure*

Proposition 1.56. *Let $\pi : \Lambda^k\left(T^*M\right) \rightarrow M$ and $\omega : M \rightarrow \Lambda^k\left(T^*M\right)$ so that $\pi \circ \omega = id_M$.*

Definition 1.57. ***(Local expressions)*** *If $\{dx^i\}$ is a local basis of differential 1-forms, over the open set U of a local map of M, with coordinates (x^i), for $i_1 < \cdots < i_r$. then the $dx^{in} \wedge \cdots \wedge dx^{ir}$ locally generate $A^r(M)$ on functions. That is to say that any r-form ω is written, above U,*

$$\omega = \omega_{i_1 \dots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

where the second sum is over $i_1 < \cdots < i_r$ and where the $\omega_{i_1 \dots i_r}$ are functions $U \rightarrow \mathbb{R}$. Sometimes, this second sum will relate to all the indices $i_1, \dots, i_r$, which supposes that we extend the definition of $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ to all $(i_1, \dots, i_r)$ and that $\omega_{i_1 \dots i_r} : U \rightarrow \mathbb{R}$ become completely antisymmetric functions on their indices; it will then also be necessary to place a factor $\frac{1}{r!}$ in front of the sum.

Definition 1.58. ***(Exterior product)***
For $\omega \in A^r(M)$ and $\eta \in A^s(M)$, we can define ***the exterior product*** *$\omega \wedge \eta \in A^{r+s}(M)$ by the formula:*

$$(\omega \wedge \eta)(X_1, \ldots, X_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{G}_{r+s}} (-1)^{\varepsilon(\sigma)} \omega\left(X_{\sigma(1)}, \ldots, X_{\sigma(r)}\right) \cdot \eta\left(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}\right)$$

such $\mathfrak{G}_{r+s}$ is the permutation of symetric group
This product gives the vector space $A(M)$. It has the commutative property

$$\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$$

Definition 1.59. *We defined the* ***differential*** *d on $A(M)$ by the linear maps*

$$d : A^k(M) \to A^{k+1}(M)$$

such $d : A^0(M) = C^\infty(M) \to A^1(M)$ the differential on the functions, and for all $\omega \in A^k(M)$

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \cdot \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k) + \sum_{i<j} (-1)^{i+j} \cdot \omega([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)$$

In the first term of the second member, X_i acts as a derivation on the function $\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k)$

Proprety 1.60. *We have the important relation (which makes d an antiderivation of the algebra $A(M)$) :*

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^r \omega \wedge d\eta$$

where $\omega \in A^r(M)$. Above an open U of a local M chart, if $\omega = \omega_{i_1 \ldots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, then

$$d\omega = \left(\frac{\partial}{\partial x^i} \omega_{i_1 \ldots i_r} \right) dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}$$

This summation relates to all the values of i and to $i_1 < \cdots < i_r$.

Definition 1.61. *Let $f : M \to N$ be a C^∞ map. We define the pull back of f as the map $f^* : A(N) \to A(M)$ so that:*

1 . *$f^*(g) = g \circ f$ for $g \in A^0(N) = C^\infty(N, \mathbb{R})$.*

2 . *$(f^*\omega)_x (X_1, \ldots, X_k) = \omega_{f(x)} (f_* X_1, \ldots, f_* X_k)$ for $\omega \in A^k(N)$ with $k \geq 1$. such that $f_* : \Gamma(TM) \to \Gamma(TN)$*

Proprety 1.62. *Properties of the pull-back map.*

1 . *$f^*(\omega \wedge \tau) = f^*\omega \wedge f^*\tau \quad \tau, \omega \in A(N)$*

2 . *$f^*(g\omega + h\tau) = f^*(g) f^*\omega \wedge f^*(h) f^*\tau \quad g, h \in A(N)$*

3 . *$(f \circ g)^* = g^* \circ f^*$*

Proposition 1.63. *A Pull-backs f^* and d commute:*

$$d(f^*\omega) = f^*(d\omega).$$

such, on one side, it is about the differential on M, and on the other side of the differential on N.

Proposition 1.64. *If $f : M \to N$ is a diffeomorphism, we can define the pull-back map on the tensor fields T of type (s, r) on N:*

$$(f^*T)\left(\alpha^1, \ldots, \alpha^s, X_1, \ldots, X_r\right) = T\left((f^*)^{-1}\alpha^1, \ldots, (f^*)^{-1}\alpha^s, f_*X_1, \ldots, f_*X_r\right)$$

where $(f^*)^{-1} : A(M) \to A(N)$, $\alpha^i \in A^1(M)$ and $X_i \in \Gamma(TM)$.

Proposition 1.65. ***Integral of n-forms**.*
Let M be an orientable manifold of dimension n.
1. If $\omega \in A^n(\mathbb{R}^n)$ has compact support, and $\omega = f dx^1 \wedge \cdots \wedge dx^n$ then

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} f dx^1 \ldots dx^n.$$

2. If $\omega \in A^n(M)$ we define

$$\int_{[M]} \omega = \sum_{i \in I} \int_{U_i} \rho_i \omega := \sum_{i \in I} \int_{\varphi(U_i)} (\varphi_i^*)^{-1}(\rho_i \omega)$$

where $\{(U_i, \varphi_i) : i \in I\}$ is a positively oriented atlas and $\{\rho_i : i \in I\}$ is a partition of unity subordinate to $\{(U_i, \varphi_i) : i \in I\}$.

Theorem 1.66. ***(Stokes theorem)**. Let M be a compact differentiable manifold of dimension n with boundary ∂M. Let $\omega \in A^{n-1}(M)$. Then,*

$$\int_M d\omega = \int_{\partial M} \omega.$$

where ∂M is provided with the canonical orientation induced by that of M.
In Particular, if M is a manifold without boundary, then

$$\int_M d\omega = 0.$$

CHAPTER

2

THE RIEMANNIAN MANIFOLDS

In this chapter we introduce the notion of Riemannian manifold and define the Riemannian metric with give the notion of levi-civita connection wich is play a important rule in the applications of Riemannian manifold.

the following references are used; [DCFF92] [BGM71] [Ura18][GY14]

2.1 Riemannian metric, Riemannian manifold

Definition 2.2. *A **Riemannian metric** g_p (or g) , on a differential manifold M, is a bilinear tensor field of type $(2,0)$ which is required to be symmetric and positive-definite .*

$$\begin{aligned} g : \Gamma(TM) \times \Gamma(TM) &\to \mathbb{R} \\ g_p : T_pM \times T_pM &\to \mathbb{R} \\ (X,Y) &\mapsto g_p(X,Y) \end{aligned}$$

It is bilinear in that the metric acts linearly on each of its two arguments,

$$g_p\left(aX_1+bX_2,Y\right)=ag_p\left(X_1,Y\right)+bg_p\left(X_2,Y\right) \ \forall X_1,X_2,Y\in T_pM$$
$$g_p\left(X,aY_1+bY_2\right)=ag_p\left(X,Y_1\right)+bg_p\left(X,Y_2\right) \ \forall X,Y_1,Y_2\in T_pM$$

It is symmetric in that the value given by the metric is independent on the order of operation,

$$g_p(X,Y)=g_p(Y,X) \quad \forall X,Y\in T_pM$$

It is definite that

$$g_p(X,Y)=0 \Rightarrow X=0 \ or \ Y=0$$

This implies that if $g_p(X,Y)=0$ and either X or Y are not equal to zero, then X,Y are orthogonal.
It is positive that

$$g_p(X,X)>0 \quad \forall X\in T_pM$$

More simply, it can be said that a Riemannian metric $g_p(\cdot,\cdot)$ is a inner product on the tangent space at each point , *The components g_{ij} of local representation*

$$g_p = \sum_{i,j} g_{ij}(p) dx^i|p \otimes dx^i|p \qquad g_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$$

where g_{ij} are the differentiable functions.

Definition 2.3. *A differentiable manifold with a given Riemannian metric g will be called a* ***Riemannian manifold****, denoted (M, g).*

Proposition 2.4. *The Riemannian metric varies differentiably in the following sense: If $\mathbf{x} : U \subset \mathbf{R}^n \to M$ is a system of coordinates around p, with $\mathbf{x}(x_1, x_2, \dots, x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \dots, 1, \dots, 0)$, then $g_{ij}(x_1, \dots, x_n) = g_q(\frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q))$ is a differentiable function on U.*

Definition 2.5. *The symetric matrix g_{ij} is called* ***the local representation of the Riemannian metric*** *(or "the g_{ij} of the metric") in the coordinate system* $\mathbf{x}$.

Proposition 2.6. *This definition does not depend on the choice of coordinate system.*

Definition 2.7. *Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $f : M \to N$ (that is, f is a differentiable bijection with a differentiable inverse) is called an* ***isometry*** *if:*

$$g_p(u, v) = h_{f(p)}(df_p(u), df_p(v)), \quad for \quad all \quad p \in M, \quad u, v \in T_pM. \tag{2.1}$$

Definition 2.8. *Let (M, g) and (N, h) be Riemannian manifolds. A differentiable mapping $f : M \to N$ is a* ***local isometry*** *at $p \in M$ if there is a neighborhood $U \subset M$ of p such that $f : U \to f(U)$ is a diffeomorphism satisfying (2.1).*

Proposition 2.9. *A Riemannian manifold (M, g) is locally isometric to a Riemannian manifold (N, h) if for every p in M there exists a neighborhood U of p in M and a local isometry $f : U \to f(U) \subset N$.*

What follows are some examples of the notion of Riemannian manifold.

Example 2.10. *The almost trivial example. $M = \mathbb{R}^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \dots, 1, \dots, 0)$. The metric is given by $g(e_i, e_j) = \delta_{ij}$ such that :*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is called the canonical metric of $\mathbb{R}^n$.

Definition 2.11. *(Immersed manifolds). Let $f : M^n \to N^{n+k}$ be an immersion, that is, f is differentiable and $df_p : T_pM \to T_{f(p)}N$ is injective for all p in M. If N has a Riemannian structure h, f induces a Riemannian structure g on M by defining*

$$g_p(u, v) = h_{f(p)}(df_p(u), df_p(v)) \quad u, v \in T_pM$$

Since df_p is injective, $g_p(. \ , \ .)$ is positive definite. The other conditions of Definition 2.2 are verified. This metric g on M is then called the metric induced by f, and f is an isometric immersion.

A particularly important case occurs when we have a differentiable function $\varphi : M^{n+k} \to N^k$ and $q \in N$ is a regular value of φ (that is, $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is surjective for all $p \in \varphi^{-1}(q)$). It is then that $\varphi^{-1}(q) \subset M$ is a submanifold of M of dimension n, hence, we can put a Riemannian metric on it induced by the inclusion.

Example 2.12. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be given by*

$$\varphi(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 - 1.$$

Then 0 *is a regular value of* φ *and*

$$\varphi^{-1}(0) = \left\{x \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1\right\} = S^{n-1}$$

is the unit sphere of $\mathbb{R}^n$*. The metric induced from* $\mathbb{R}^n$ *on* S^{n-1} *is called the canonical metric of* S^{n-1}*.*

Example 2.13. *The product metric. Let* (M_1, g) *and* (M_2, h) *be Riemannian manifolds and consider the cartesian product* $M_1 \times M_2$ *with the product structure* k*.*
Let $\pi_1 : M_1 \times M_2 \to M_1$ *and* $\pi_2 : M_1 \times M_2 \to M_2$ *be the natural projections. Introduce on* $M_1 \times M_2$ *a Riemannian metric* k *as follows:*

$$k_{(p,q)}(u, v) = g_p(d\pi_1(u), d\pi_1(v)) + h_q(d\pi_2(u), d\pi_2(v)) \quad \forall (p, q) \in M_1 \times M_2, \quad \forall (u, v) \in T_{(p,q)}(M_1 \times M_2)$$

This metric k *is really a Riemannian metric on the product.*
For example, the torus $S^1 \times \cdots \times S^1 = T^n$ *has a Riemannian structure obtained by choosing the induced Riemannian metric from* $\mathbf{R}^2$ *on the circle* $S^1 \subset \mathbf{R}^2$ *and then taking the product metric. The torus* T^n *with this metric is called the flat torus.*

Let us now prove a theorem on the existence of Riemannian metrics.

Theorem 2.14. *Every differentiable manifold has always a Riemannian metric.*

Proof. Let $\{f_\alpha\}$ be a differentiable partition of unity on (M, g) subordinate to a covering $\{V_\alpha\}$ of M by coordinate neighborhoods. This means that $\{V_\alpha\}$ is a locally finite covering (i.e., any point of M has a neighborhood U such that $U \cap V_\alpha \neq \phi$ at most for a finite number of indices) and $\{f_\alpha\}$ is a family of differentiable functions on M satisfying:

1) $f_\alpha \geq 0, f_\alpha = 0$ on the complement of the closed set $\bar{V}_\alpha$.

2) $\sum_\alpha f_\alpha(p) = 1$ for all p on M.

we can define a Riemannian metric $g^\alpha(. \, , \, .)$ on each V_α : the metric induced by the system of local coordinates. Let us then set

$$g_p(u, v) = \sum_\alpha f_\alpha(p) g_p^\alpha(u, v) \quad \text{for all} \quad p \in M, u, v \in T_pM.$$

This construction defines a Riemannian metric on M.

■

we are going to show how a Riemannian metric permits us to define a notion of volume on a given oriented manifold M^n.

Let $p \in M$ and let $\mathbf{x} : U \subset \mathbf{R}^n \to M$ be a parametrization about p which belongs to a family of parametrizations consistent with the orientation of M (we say that, any parametrizations are positive). Consider a orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM and write $X_i(p) = \frac{\partial}{\partial x_i}(p)$ in the basis $\{e_i\}$: $X_i(p) = \sum_j a_{ij} e_j$ (by changing the bases). Then

$$g_{ik}|_p = g_p(X_i, X_k) = \sum_{j\ell} a_{ij} \, a_{k\ell} \, g_p(e_j, e_\ell) = \sum_j a_{ij} \, a_{kj}. \tag{2.2}$$

Definition 2.15. *On any oriented Riemannian manifold (M, g), there is a unique n-form known as the* ***volume form*** ω $(\omega : (\Gamma(TM))^n \to \mathbb{R})$*, satisfying the property that* $\omega_p(e_1, \ldots, e_n) = 1$*, whenever* $(e_1, \ldots, e_n)$ *is an oriented orthonormal basis for a tangent space* T_pM*.*
Let (x^i) *a local map on* M *in point p, locally, the associated volume form is expressed by:*

$$\omega = \sqrt{\det(g_{ij}|_p)} \cdot dx^1 \wedge \ldots \wedge dx^n$$

Proposition 2.16. *Let* ω *be the volume form formed by the vectors* $X_1(p), \ldots, X_n(p)$ *in* T_pM*, we obtain*

$$\omega(X_1(p), \ldots, X_n(p)) = \det(a_{ij}) \cdot \omega(e_1, \ldots, e_n) = \det(a_{ij}) = \sqrt{\det(g_{ij}|_p)}$$

such that (a_{ij}) *is the matrix given by (2.2)*

Proposition 2.17. *If* $y : V \subset \mathbf{R}^n \to M$ *is another positive parametrization about p, with* $Y_i(p) = \frac{\partial}{\partial y_i}(p)$ *and* $h_{ij}|_p = h_p(Y_i, Y_j)$*, we obtain*

$$\begin{aligned} \sqrt{\det(g_{ij}|_p)} &= \omega(X_1(p), \ldots, X_n(p)) \\ &= J\omega(Y_1(p), \ldots, Y_n(p)) \\ &= J\sqrt{\det(h_{ij}|_p)} \end{aligned} \tag{2.3}$$

where $J = \det\left(\frac{\partial x_i}{\partial y_j}\right) = \det(d\mathbf{y}^{-1} \circ d\mathbf{x})|_p > 0$ *is the determinant of the derivative of the change of coordinates.*

Definition 2.18. *Now let* $R \subset M$ *be a region (an open connected subset), whose closure is compact. We suppose that* R *is contained in a coordinate neighborhood* $\mathbf{x}(U)$ *with a positive parametrization* $\mathrm{x} : U \to M$*, and that the boundary of* $\mathrm{x}^{-1}(R) \subset U$ *has measure zero in* $\mathbf{R}^n$ *(observe that the notion of measure zero in* $\mathbf{R}^n$ *is invariant by diffeomorphism). Let us define the* ***volume*** $\mathrm{vol}(R)$ *of* R *by the integral in* $\mathbf{R}^n$

$$\mathrm{vol}(R) = \int_{\mathbf{x}^{-1}(R)} \sqrt{\det(g_{ij})} dx_1 \ldots dx_n \tag{2.4}$$

The expression above is well-defined.

Proposition 2.19. *If* R *is contained in another coordinate neighborhood* $y(V)$ *with a positive parametrization* $\mathbf{y} : V \subset \mathbf{R}^n \to M$*, we obtain from the change of variable theorem for multiple integrals, (using the same notation as in (2.3),*

$$\mathrm{vol}(R) = \int_{\mathbf{x}^{-1}(R)} \sqrt{\det(g_{ij})} dx_1 \ldots dx_n = \int_{\mathbf{y}^{-1}(R)} \sqrt{\det h_{ij}} dy_1 \ldots dy_n$$

which proves that the definition given by (2.4) does not depend on the choice of the coordinate system (here the hypothesis of the orientability of M *enters by guaranteeing that* $\mathrm{vol}(R)$ *does not change sign).*

Definition 2.20. *In order to define the volume of a compact region* R*, which is not contained in a coordinate neighborhood it is necessary to consider a partition of unity* $\{\varphi_i\}$ *subordinate to a (finite) covering of* R *consisting of coordinate neighborhoods* $\mathbf{x}(U_i)$ *and to take*

$$\mathrm{vol}(R) = \sum_i \int_{x_i^{-1}(R)} \varphi_i \nu$$

such that ν *is a volume form where* $\nu = \sqrt{\det(g_{ij})} dx_1 \ldots dx_n$*.*
The expression above does not depend on the choice of the partition of unity.

Remark 2.21. *The existence of a globally defined positive differential form of degree n (volume element) leads to a notion of volume on a differentiable manifold. A Riemannian metric is only one of the ways through which a volume element can be obtained.*

2.22 Riemannian submersions

Definition 2.23. . *(Riemannian submersions) Let φ is a C^∞ map of a Riemannian manifold (M, g) into another Riemannian manifold (N, h) is called a* ***Riemannian submersion*** *if :*

(1) φ is surjective.

(2) the differential $\varphi_* = d\varphi : T_pM \to T_{\varphi(p)}N \, (p \in M)$ of $\varphi : M \to N$ is sujective for each $p \in M$.

(3) each tangent space T_pM at $p \in M$ has the direct decomposition:

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p$$

which is orthogonal decomposition with respect to g such that $\mathcal{V}_p = \operatorname{Ker}(\varphi_{*p}) \subset T_pM$ means that $\mathcal{V}_p = \varphi_*(0_p)^{-1}$ ($\mathcal{V}_p$ is the vertical subspace at p).
and is $\mathcal{H}_p := \mathcal{V}_p^\perp \subset T_pM$ ($\mathcal{H}_p$ is the horizontal subspace at p)

(4) the restriction of the differential φ_* to $\mathcal{H}_p$ is an isometry,
$\varphi_*|_{\mathcal{H}_p} : (\mathcal{H}_p, g_p) \to \left(T_{\varphi(p)}N, h_{\varphi(p)}\right)$ for each $p \in M$.

Definition 2.24. *Let $\varphi : M \to N$ a submersion of (M, g) in (N, h) is called Riemannian submersion if $\varphi_*|_{\mathcal{H}_p}$ induces an isometric of Euclidian spaces of $\mathcal{H}_p$ on $T_{\varphi(p)}N$.*

Proposition 2.25. *A manifold M is the total space of a Riemannian submersion over N with the projection $\pi : M \to N$ A Riemannian metric g on M, called adapted metric on M which satisfies*

$$g = \pi^*h + k$$

where k is the Riemannian metric on each fiber $\pi^{-1}(q), (q \in N)$. Then, T_pM has the orthogonal direct decomposition of the tangent space T_pM.

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p \quad p \in M \tag{2.5}$$

where $\mathcal{V}_p = \operatorname{Ker}(\pi_{*p}) \subset T_pM$, and $\mathcal{H}_p$ is $\mathcal{H} := \mathcal{V}^\perp \subset T_pM$.

Proposition 2.26. *fixing locally frame, called adapted local orthonormal frame field to the projection $\pi : M \to N$, Corresponding to the decomposition (2.5), the tangent vectors X_p, and Y_p in T_pM can be defined by*

$$X_p = X_p^{\mathrm{V}} + X_p^{\mathrm{H}}, \quad Y_p = Y_p^{\mathrm{V}} + Y_p^{\mathrm{H}} \qquad \begin{array}{l} X_p^{\mathrm{V}}, Y_p^{\mathrm{V}} \in \mathcal{V}_p, \\ X_p^{\mathrm{H}}, Y_p^{\mathrm{H}} \in \mathcal{H}_p \end{array}$$

for $p \in M$. Then, there exist a unique decomposition such that

$$g\left(X_p, Y_p\right) = h\left(\pi_* X_p, \pi_* Y_p\right) + k\left(X_p^{\mathrm{V}}, Y_p^{\mathrm{V}}\right), \quad X_p, Y_p \in T_pM, p \in M$$

2.27 Riemannian connections

Definition 2.28. *Let M be a differentiable manifold with an affine connection ∇ and a Riemannian metric g. A connection is said to be* ***compatible*** *with the metric g, when for any smooth curve c and any pair of parallel vector fields P and P' along c, we have $g(P, P') =$ constant.*

Definition 2.28 is justified by the following proposition which shows that if ∇ is compatible with g, then we are able to differentiate the inner product by the usual "product rule".

Proposition 2.29. *Let M be a Riemannian manifold. A connection ∇ on M is compatible with a metric if and only if for any vector fields V and W along the differentiable curve $c : I \to M$ we have*

$$\frac{d}{dt}g(V,W) = g(\frac{DV}{dt}, W) + g(V, \frac{DW}{dt}), \quad t \in I. \tag{2.6}$$

Proof. It is obvious that equation (2.6) implies that ∇ is compatible with g . Therefore, let us prove the converse. Choose an orthonormal basis $\{P_1(t_o), \ldots, P_n(t_o)\}$ of $T_{x(t_o)}(M), t_o \in I$. Using Proposition 1.44, we can extend the vectors $P_i(t_o), i = 1, \ldots, n$, along c by parallel translation. Because ∇ is compatible with the metric, $\{P_1(t), \ldots, P_n(t)\}$ is an orthonormal basis of $T_{c(t)}(M)$, for any $t \in I$. Therefore, we can write

$$V = \sum_i v^i P_i, \quad W = \sum_i w^i P_i, \quad i = 1, \ldots, n$$

where v^i and w^i are differentiable functions on I. It follows that

$$\frac{DV}{dt} = \sum_i \frac{dv^i}{dt}P_i, \quad \frac{DW}{dt} = \sum_i \frac{dw^i}{dt}P_i$$

Therefore,

$$\begin{aligned} g(\frac{DV}{dt}, W) + g(V, \frac{DW}{dt}) &= \sum_i \left\{ \frac{dv^i}{dt}w^i + \frac{dw^i}{dt}v^i \right\} \\ &= \frac{d}{dt}\left\{ \sum_i v^i w^i \right\} = \frac{d}{dt}g(V,W). \end{aligned}$$

■

Corollary 2.30. *A connection ∇ on a Riemannian manifold M is compatible with the metric if and only if*

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \Gamma(TM) \tag{2.7}$$

Proof.. Suppose that ∇ is compatible with the metric. Let $p \in M$ and let $c : I \to M$ be a differentiable curve with $c(t_o) = p, t_o \in I$, and with $\frac{dc}{dt}\big|_{t=t_0} = X(p)$. Then

$$X(p)g(Y,Z) = \frac{d}{dt}g(Y,Z)\bigg|_{t=t_0} = g_p\left(\nabla_{X(p)}Y, Z\right) + g_p\left(Y, \nabla_{X(p)}Z\right)$$

Since p is arbitrary, (2.7) follows. The converse is obvious.

■

Definition 2.31. *A* ***torsion*** *of an affine connection ∇ on a smooth manifold M is defined as*

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X,Y) &\to T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \end{aligned}$$

T is called ***torsion-free*** *if $T(X,Y) = 0 \quad \forall X, Y \in \Gamma(TM)$*

Definition 2.32. . *An affine connection ∇ on a smooth manifold M is said to be* ***torsion-free*** *when*

$$\nabla_X Y - \nabla_Y X = [X, Y] \qquad for \quad all \quad X, Y \in \Gamma(TM).$$

Remark 2.33. *In a coordinate system $(U, \mathbf{x})$, the fact that ∇ is free torsion implies that for all $i, j = 1, \ldots, n$,*

$$\nabla_{X_i} X_j - \nabla_{X_j} X_i = [X_i, X_j] = 0, \quad X_i = \frac{\partial}{\partial x_i}, \tag{2.8}$$

which justifies the terminology (observe that (2.8) is equivalent to the fact that $\Gamma^k_{ij} = \Gamma^k_{ji}$).

Theorem 2.34. ***(Levi-Civita)*** *Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions*

a) ∇ is torsion-free,.

b) ∇ is compatible with the Riemannian metric.

The connection called ***Levi-Civita (or Riemannian) connection*** *on M.*

Proof. Suppose initially the existence of such a ∇. Then

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \tag{2.9}$$

$$g(YZ, X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X), \tag{2.10}$$

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y). \tag{2.11}$$

Adding (2.9) and (2.10) and subtracting (2.11), we have, using the symmetry of ∇, that

$$Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) = g([X, Z], Y) + g([Y, Z], X) + g([X, Y], Z) + 2g(Z, \nabla_Y X)$$

Therefore

$$g(Z, \nabla_Y X) = \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z)) \tag{2.12}$$

The expression (2.12) shows that ∇ is uniquely determined from the metric g. Hence, if it exists, it will be unique.

To prove existence, define ∇ by (2.12). It is easy to verify that ∇ is well-defined and that it satisfies the desired conditions.

■

In a coordinate system $(U, \mathbf{x})$, we can say that the functions Γ^k_{ij} defined on U by $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$, the coefficients of the connection ∇ on U or the Christoffel symbols of the connection. From (2.12) it follows that

$$\sum_\ell \Gamma^\ell_{ij} g_{\ell k} = \frac{1}{2}\left\{\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right\}$$

where $g_{ij} = g(X_i, X_j)$.
Since the matrix (g_{km}) admits an inverse (g^{km}), we obtain that

$$\Gamma^m_{ij} = \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right\} g^{km}. \tag{2.13}$$

The equation (2.13) is a classical expression for the Christoffel symbols of the Riemannian connection in terms of the g_{ij} (given by the metric).

Observe that for the Euclidean space $\mathbf{R}^n$, we have $\Gamma_{ij}^k = 0$.
In terms of the Christoffel symbols, the covariant derivative has the classical expression

$$\frac{DV}{dt} = \sum_k \left\{ \frac{dv^k}{dt} + \sum_{i,j} \Gamma_{ij}^k v^j \frac{dx_i}{dt} \right\} X_k$$

which follows from (1.5). Observe that $\frac{DV}{dt}$ differs from the usual derivative in Euclidean space by terms which involve the Christoffel symbols. Therefore, in Euclidean spaces the covariant derivative coincides with the usual derivative.

2.35 curvature

Definition 2.36. *The curvature R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \Gamma(TM)$ a mapping $R(X,Y) : \Gamma(TM) \to \Gamma(TM)$ given by*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \Gamma(TM)$$

where ∇ is the Riemannian (Levi-Civita) connection of M.

Example 2.37. *Observe that if $M = R^n$, then $R(X,Y)Z = 0$ for all $X, Y, Z \in \Gamma(TR^n)$. In fact, if the vector field Z is given by $Z = (z_1, \ldots, z_n)$, with the components of Z coming from the natural coordinates of R^n, we obtain*

$$\nabla_X Z = (Xz_1, \ldots, Xz_n) \tag{2.14}$$

hence

$$\nabla_Y \nabla_X Z = (YXz_1, \ldots, YXz_n) \tag{2.15}$$

which implies that

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z = 0$$

as was stated. We are able, therefore, to think of R as a way of measuring how much M deviates from being Euclidean.

Remark 2.38. *Another way of viewing definition 2.36 is to consider a system of coordinates $\{x_i\}$ around $p \in M$. Since $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, we obtain*

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \left(\nabla_{\partial/\partial x_j} \nabla_{\partial/\partial x_i} - \nabla_{\partial/\partial x_i} \nabla_{\partial/\partial x_j}\right)\frac{\partial}{\partial x_k}$$

that is the curvature measures the non-commutativity of the covariant derivative.

Proposition 2.39. *The curvature R of a Riemannian manifold has the following properties:*

(i) R is bilinear in $\Gamma(TM) \times \Gamma(TM)$, that is,

$$\begin{aligned} R(fX_1 + gX_2, Y_1) &= fR(X_1, Y_1) + gR(X_2, Y_1) \\ R(X_1, fY_1 + gY_2) &= fR(X_1, Y_1) + gR(X_1, Y_2) \end{aligned}$$

$f, g \in C^\infty(M), \quad X_1, X_2, Y_1, Y_2 \in \Gamma(TM).$

(ii) For any $X, Y \in \Gamma(TM)$, the curvature operator $R(X,Y) : \Gamma(TM) \to \Gamma(TM)$ is linear, that is,

$$\begin{aligned} R(X,Y)(Z + W) &= R(X,Y)Z + R(X,Y)W, \\ R(X,Y)fZ &= fR(X,Y)Z, \end{aligned}$$

$f \in C^\infty(M), \quad Z, W \in \Gamma(TM).$

Proof. Let us verify (i), by the properties of the connection and Lie bracket we have the first part of (i),

$$\begin{aligned}
R\left(fX_1+gX_2,Y_1\right) &= \nabla_{Y_1}\nabla_{fX_1+gX_2} - \nabla_{fX_1+gX_2}\nabla_{Y_1} + \nabla_{[fX_1+gX_2,Y_1]}\\
&= \nabla_{Y_1}(f\nabla_{X_1}+g\nabla_{X_2}) - (f\nabla_{X_1}+g\nabla_{X_2})\nabla_{Y_1} + \nabla_{[fX_1+gX_2,Y_1]}\\
&= \nabla_{Y_1}f\nabla_{X_1}+\nabla_{Y_1}g\nabla_{X_2} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_2}\nabla_{Y_1} + \nabla_{[fX_1+gX_2,Y_1]}\\
&= \nabla_{Y_1}f\nabla_{X_1}+\nabla_{Y_1}g\nabla_{X_2} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_2}\nabla_{Y_1} + \nabla_{f[X_1,Y_1]} + \nabla_{g[X_2,Y_1]}\\
&= \nabla_{Y_1}f\nabla_{X_1}+\nabla_{Y_1}g\nabla_{X_2} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_2}\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_2,Y_1]}\\
&= f\nabla_{Y_1}\nabla_{X_1} + -f\nabla_{X_1}\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{Y_1}\nabla_{X_2} - g\nabla_{X_2}\nabla_{Y_1} + g\nabla_{[X_2,Y_1]}\\
&= f(\nabla_{Y_1}\nabla_{X_1} + -\nabla_{X_1}\nabla_{Y_1} + \nabla_{[X_1,Y_1]}) + g(\nabla_{Y_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{Y_1} + \nabla_{[X_2,Y_1]})\\
&= fR\left(X_1,Y_1\right) + gR\left(X_2,Y_1\right).
\end{aligned}$$

The second part of (i)

$$\begin{aligned}
R\left(X_1,fY_1+gY_2\right) &= \nabla_{fY_1+gY_2}\nabla_{X_1} - \nabla_{X_1}\nabla_{fY_1+gY_2} + \nabla_{[X_1,fY_1+gY_2]}\\
&= (f\nabla_{Y_1}+g\nabla_{Y_2})\nabla_{X_1} - \nabla_{X_1}(f\nabla_{Y_1}+g\nabla_{Y_2}) + \nabla_{[X_1,fY_1+gY_2]}\\
&= f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_2}\nabla_{X_1} - \nabla_{X_1}f\nabla_{Y_1} - \nabla_{X_1}g\nabla_{Y_2} + \nabla_{[X_1,fY_1+gY_2]}\\
&= f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_2}\nabla_{X_1} - \nabla_{X_1}f\nabla_{Y_1} - \nabla_{X_1}g\nabla_{Y_2} + \nabla_{[X_1,fY_1]} + \nabla_{[X_1,gY_2]}\\
&= f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_2}\nabla_{X_1} - \nabla_{X_1}f\nabla_{Y_1} - \nabla_{X_1}g\nabla_{Y_2} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_1,Y_2]}\\
&= f\nabla_{Y_1}\nabla_{X_1} + g\nabla_{Y_2}\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} - g\nabla_{X_1}\nabla_{Y_2} + f\nabla_{[X_1,Y_1]} + g\nabla_{[X_1,Y_2]}\\
&= f\nabla_{Y_1}\nabla_{X_1} - f\nabla_{X_1}\nabla_{Y_1} + f\nabla_{[X_1,Y_1]} + g\nabla_{Y_2}\nabla_{X_1} - g\nabla_{X_1}\nabla_{Y_2} + g\nabla_{[X_1,Y_2]}\\
&= f(\nabla_{Y_1}\nabla_{X_1} - \nabla_{X_1}\nabla_{Y_1} + \nabla_{[X_1,Y_1]}) + g(\nabla_{Y_2}\nabla_{X_1} - \nabla_{X_1}\nabla_{Y_2} + \nabla_{[X_1,Y_2]})\\
&= fR\left(X_1,Y_1\right) + gR\left(X_1,Y_2\right)
\end{aligned}$$

and verify (ii) The first part of (ii) is obvious. As for the second, we have

$$\begin{aligned}
\nabla_Y\nabla_X(fZ) = \nabla_Y\left(f\nabla_X Z + (Xf)Z\right) = f\nabla_Y\nabla_X Z + (Yf)\left(\nabla_X Z\right)\\
+ (Xf)\left(\nabla_Y Z\right) + (Y(Xf))Z.
\end{aligned}$$

Therefore,

$$\nabla_Y\nabla_X(fZ) - \nabla_X\nabla_Y(fZ) = f\left(\nabla_Y\nabla_X - \nabla_X\nabla_Y\right)Z + ((YX-XY)f)Z$$

hence

$$\begin{aligned}
R(X,Y)fZ &= f\nabla_Y\nabla_X Z - f\nabla_X\nabla_Y Z + ([Y,X]f)Z + f\nabla_{[X,Y]}Z + ([X,Y]f)Z\\
&= fR(X,Y)Z.
\end{aligned}$$

■

Remark 2.40. *An analysis of the proof above shows that the necessity of the appearance of the term $\nabla_{[X,Y]}Z$ in the definition of the curvature is connected to the fact that we want the mapping $R(X,Y):\Gamma(TM)\to\Gamma(TM)$ to be linear.*

Proposition 2.41. *(Bianchi identity)*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Proof. From the symmetry of the Riemannian comection, we have,

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y\\
&\quad= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z + \nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X + \nabla_X\nabla_Z Y\\
&\qquad\quad - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}Y\\
&\quad= \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] - \nabla_{[X,Z]}Y - \nabla_{[Y,X]}Z - \nabla_{[Z,Y]}X\\
&\quad= [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]]\\
&\quad= 0,
\end{aligned}$$

where the last equality follows from the Jacobi identity for vector fields.

■

From now on, we shall write $g(R(X,Y)Z,T) = R(X,Y,Z,T)$.

Proposition 2.42.

$$R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T) = 0 \tag{2.16}$$

$$R(X,Y,Z,T) = -R(Y,X,Z,T) \tag{2.17}$$

$$R(X,Y,Z,T) = -R(X,Y,T,Z) \tag{2.18}$$

$$R(X,Y,Z,T) = R(Z,T,X,Y). \tag{2.19}$$

Proof.

(2.16) is just the Bianchi identity again, such

$$\begin{aligned}
R(X,Y,Z,T) &+ R(Y,Z,X,T) + R(Z,X,Y,T) \\
&= g(R(X,Y)Z,T) + g(R(Y,Z)X,T) + g(R(Z,X)Y,T) \\
&= g(R(X,Y)Z + R(Y,Z)X + R(Z,X)Y,T) \\
&= g(0,T) \\
&= 0
\end{aligned}$$

(2.17) follows directly from Definition 2.36;

$$\begin{aligned}
R(X,Y,Z,T) &= g(R(X,Y)Z,T) \\
&= g(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z,T) \\
&= g(-(-\nabla_Y\nabla_X Z + \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z),T) \\
&= -g((-\nabla_Y\nabla_X Z + \nabla_X\nabla_Y Z - \nabla_{[X,Y]}Z),T) \\
&= -g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{-[Y,X]}Z),T) \\
&= -g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[Y,X]}Z),T) \\
&= -g(R(Y,X)Z,T) \\
&= -R(Y,X,Z,T)
\end{aligned}$$

(2.18) is equivalent to $R(X,Y,Z,Z) = 0$, whose proof follows:

$$R(X,Y,Z,Z) = g\left(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z,Z\right)$$

But

$$g\left(\nabla_Y\nabla_X Z,Z\right) = Y\left(\nabla_X Z,Z\right) - g\left(\nabla_X Z,\nabla_Y Z\right)$$

and

$$g\left(\nabla_{[X,Y]}Z,Z\right) = \frac{1}{2}[X,Y]g(Z,Z)$$

Hence

$$\begin{aligned}
R(X,Y,Z,Z) &= Yg\left(\nabla_X Z,Z\right) - Xg\left(\nabla_Y Z,Z\right) + \frac{1}{2}[X,Y]g(Z,Z) \\
&= \frac{1}{2}Y(Xg(Z,Z)) - \frac{1}{2}X(Yg(Z,Z)) + \frac{1}{2}[X,Y]g(Z,Z) \\
&= -\frac{1}{2}[X,Y]g(Z,Z) + \frac{1}{2}[X,Y]g(Z,Z) \\
&= 0,
\end{aligned}$$

which proves (2.18).

In order to prove (2.19), we use (2.16), and write:

$$
\begin{aligned}
R(X,Y,Z,T)+R(Y,Z,X,T)+R(Z,X,Y,T)&=0\\
R(Y,Z,T,X)+R(Z,T,Y,X)+R(T,Y,Z,X)&=0\\
R(Z,T,X,Y)+R(T,X,Z,Y)+R(X,Z,T,Y)&=0\\
R(T,X,Y,Z)+R(X,Y,T,Z)+R(Y,T,X,Z)&=0
\end{aligned}
$$

Summing the equations above, we obtain

$$2R(Z,X,Y,T)+2R(Y,T,Z,X)=0$$

and, therefore,

$$R(Z,X,Y,T)=R(Y,T,Z,X)$$

■

It convenient to express what was seen above in coordonate system $(U,\mathbf{x})$ based at the point $p\in M$. Let us indicate, as usual, $\frac{\partial}{\partial x_i}=X_i$. We put

$$R\left(X_i,X_j\right)X_k=\sum_{\ell} R_{ijk}^{\ell}X_{\ell}.$$

Thus R_{ijk}^{ℓ} are the components of the curvature R in $(U,\mathbf{x})$. If

$$X=\sum_i u^iX_i,\quad Y=\sum_j v^jX_j,\quad Z=\sum_k w^kX_k$$

we obtain, from the linearity of R,

$$R(X,Y)Z=\sum_{i,j,k,\ell} R_{ijk}^{\ell}u^iv^jw^kX_{\ell}. \tag{2.20}$$

To express R_{ijk}^{ℓ} in terms of the coefficients Γ_{ij}^{k} of the Riemannian connection, we write,

$$
\begin{aligned}
R\left(X_i,X_j\right)X_k&=\nabla_{X_j}\nabla_{X_i}X_k-\nabla_{X_i}\nabla_{X_j}X_k\\
&=\nabla_{X_j}\left(\sum_{\ell}\Gamma_{ik}^{\ell}X_{\ell}\right)-\nabla_{X_i}\left(\sum_{\ell}\Gamma_{jk}^{\ell}X_{\ell}\right)
\end{aligned}
$$

which by a direct calculation yields

$$R_{ijk}^{s}=\sum_{\ell}\Gamma_{ik}^{\ell}\Gamma_{j\ell}^{s}-\sum_{\ell}\Gamma_{jk}^{\ell}\Gamma_{i\ell}^{s}+\frac{\partial}{\partial x_j}\Gamma_{ik}^{s}-\frac{\partial}{\partial x_i}\Gamma_{jk}^{s}.$$

Putting

$$R_{ijks}=g\left(R\left(X_i,X_j\right)X_k,X_s\right)=\sum_{\ell} R_{ijk}^{\ell}g_{\ell s}$$

we can write the ideritities of Proposition 2.42 as:

$$
\begin{aligned}
R_{ijks}+R_{jkis}+R_{kijs}&=0\\
R_{ijks}&=-R_{jiks}\\
R_{ijks}&=-R_{ijsk}\\
R_{ijks}&=R_{ksij}.
\end{aligned}
$$

Remark 2.43. *The equation (2.20), which depends on the linearity of the operator R, shows that the value of $R(X,Y)Z$ at the point p depends uniquely on the values of X,Y,Z at p and the values of the functions R_{ijk}^{ℓ} at p. Observe that this contrasts with the behavior of the covariant derivative , the reason being that the covariant derivative is not linear in all of its arguments. In general, entities, such as the curvature, that are linear, are called tensors on M*

2.44 Sectional curvature

Closely related to the curvature operator is the sectional curvature that we are now going to define.

In what follows it is convenient to use the following notation.

Definition 2.45. *Given a vector space V, we denote by the expression*

$$|x \wedge y| = \sqrt{|x|^2|y|^2 - \langle x, y\rangle^2},$$

which represents the area of a two-dimensional parallelogram determined by the pair of vectors $x, y \in V$.

Proposition 2.46. *Let $\sigma \subset T_pM$ be a two-dimensional subspace of the tangent space T_pM and let $x, y \in \sigma$ be two linearly independent vectors. Then*

$$K(x, y) = \frac{R(x, y, x, y)}{|x \wedge y|^2}$$

does not depend on the choice of the vectors $x, y \in \sigma$.

Proof. To avoid calculating, we observe that we can pass from the basis $\{x, y\}$ of σ to any other basis $\{x', y'\}$ by iterating the following . elementary transformations:

(a) $\{x, y\} \to \{y, x\}$, such from 2.17 and 2.18

$$\begin{aligned} K(y, x) &= \frac{R(y, x, y, x)}{|y \wedge x|^2} \\ &= \frac{-R(x, y, y, x)}{|y \wedge x|^2} \\ &= \frac{-(-R(x, y, x, y))}{|y \wedge x|^2} \\ &= \frac{R(x, y, x, y)}{|x \wedge y|^2} \end{aligned}$$

(b) $\{x, y\} \to \{\lambda x, y\}$, by the linearity of R

$$\begin{aligned} K(\lambda x, y) &= \frac{R(\lambda x, y, \lambda x, y)}{|\lambda x \wedge y|^2} \\ &= \frac{g_p(R(\lambda x, y)\lambda x, y)}{|\lambda x|^2|y|^2 - g_p(\lambda x, y)^2} \\ &= \frac{g_p(\lambda R(x, y)\lambda x, y)}{\lambda^2|x|^2|y|^2 - g_p(\lambda x, y)^2} \\ &= \frac{\lambda^2 g_p(R(x, y)x, y)}{\lambda^2|x|^2|y|^2 - g_p(\lambda x, y)^2} \\ &= \frac{\lambda^2 g_p(R(x, y)x, y)}{\lambda^2|x|^2|y|^2 - \lambda^2 g_p(x, y)^2} \\ &= \frac{\lambda^2 g_p(R(x, y)x, y)}{\lambda^2(|x|^2|y|^2 - g_p(x, y)^2)} \\ &= \frac{g_p(R(x, y)x, y)}{(|x|^2|y|^2 - g_p(x, y)^2)} \\ &= \frac{R(x, y, x, y)}{|x \wedge y|^2} \end{aligned}$$

(c) $\{x, y\} \to \{x + \lambda y, y\}$.

$$\begin{aligned}
K(x+\lambda y, y) &= \frac{R(x+\lambda y, y, x+\lambda y, y)}{|x+\lambda y \wedge y|^2} \\
&= \frac{g_p(R(x+\lambda y, y)(x+\lambda y), y)}{|x+\lambda y|^2|y|^2 - g_p(x+\lambda y, y)^2} \\
&= \frac{g_p(R(x+\lambda y, y)x + R(x, +\lambda y, y)\lambda y), y)}{|x+\lambda y|^2|y|^2 - g_p(x+\lambda y, y)^2} \\
&= \frac{g_p(R(x, y)x + \lambda R(y, y)x + \lambda R(x, y)y + \lambda^2 R(y, y)y, y)}{g_p(x+\lambda y, x+\lambda y)|y|^2 - g_p(x+\lambda y, y)^2}
\end{aligned}$$

we have $R(y, y)x$ and $R(y, y)y$ are equal to 0 then

$$\begin{aligned}
K(x+\lambda y, y) &= \frac{g_p(R(x, y)x + \lambda R(x, y)y, y)}{(g_p(x+\lambda y, x) + \lambda g_p(x+\lambda y, y))|y|^2 - g_p(x+\lambda y, y)^2} \\
&= \frac{R(x, y, x, y) + \lambda R(x, y, y, y)}{(g_p(x, x) + \lambda g_p(y, x) + \lambda g_p(x, y) + \lambda^2 g_p(x, y))|y|^2 - (g_p(x, y) + \lambda g_p(y, y))^2} \\
&= \frac{R(x, y, x, y)}{(g_p(x, x) + \lambda g_p(y, x) + \lambda g_p(x, y) + \lambda^2 g_p(x, y))|y|^2 - (g_p(x, y) + \lambda g_p(y, y))^2}
\end{aligned}$$

$$= \frac{R(x, y, x, y)}{((g_p(x, x) + \lambda g_p(y, x) + \lambda g_p(x, y) + \lambda^2 g_p(x, y))|y|^2 - (g_p(x, y)^2 + \lambda|y|)^4 + 2g_p(x, y)\cdot\lambda|y|^2)}$$

$$\begin{aligned}
&= \frac{R(x, y, x, y)}{|x|^2|y|^2 + 2\lambda g_p(x, y)|y|^2 + \lambda^2 g_p(x, y))|y|^2 - g_p(x, y)^2 - \lambda^2|y|^4 - 2g_p(x, y)\cdot\lambda|y|^2)} \\
&= \frac{R(x, y, x, y)}{|x|^2|y|^2 - g_p(x, y)^2} \\
&= \frac{R(x, y, x, y)}{|x \wedge y|^2}
\end{aligned}$$

We can see that $K(x, y)$ is invariant by such transformations and that completes the proof.

■

Definition 2.47. *Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$, the real number $K(x, y) = K(\sigma)$, where $\{x, y\}$ is any basis of σ, is called the **sectional curvature** of σ at p.*

Lemma 2.48. *Let V be a vector space of dimension ≥ 2, provided with an inner product $\langle\ ,\ \rangle$. Let $R : V \times V \times V \to V$ and $R' : V \times V \times V \to V$ be tri-linear mappings such that conditions of Proposition 2.42 are satisfied by*

$$R(x, y, z, t) = \langle R(x, y)z, t\rangle, \quad R'(x, y, z, t) = \langle R'(x, y)z, t\rangle.$$

If x, y are two linearly independent vectors, we may write,

$$K(\sigma) = \frac{R(x, y, x, y)}{|x \wedge y|^2}, \quad K'(\sigma) = \frac{R'(x, y, x, y)}{|x \wedge y|^2}$$

where σ is the bi-dimensional subspace generated by x and y. If for all $\sigma \subset V, K(\sigma) = K'(\sigma)$, then $R = R'$.

Proof. It suffices to prove that $R(x, y, z, t) = R'(x, y, z, t)$ for any $x, y, z, t \in V$. Observe first that, by hypothesis, we have $R(x, y, x, y) = R'(x, y, x, y)$, for all $x, y \in V$. Then

$$\begin{aligned}
R(x+z, y, x+z, y) &= R'(x+z, y, x+z, y) \\
g(R(x+z, y)x+z, y) &= g(R'(x+z, y)x+z, y)
\end{aligned}$$

$$
\begin{aligned}
&g(R(x,y)x,y)+g(R(x,y)z,y)+g(R(z,y)x,y)+g(R(z,y)z,y)\\
&\qquad =g(R'(x,y)x,y)+g(R'(x,y)z,y)+g(R'(z,y)x,y)+g(R'(z,y)z,y)
\end{aligned}
$$

because of 2.19 we have

$$g(R(x,y)x,y)+2g(R(x,y)z,y)+g(R(z,y)z,y)=g(R'(x,y)x,y)+2g(R'(x,y)z,y)+g(R'(z,y)z,y)$$

hence

$$R(x,y,x,y)+2R(x,y,z,y)+R(z,y,z,y)=R'(x,y,x,y)+2R'(x,y,z,y)+R'(z,y,z,y)$$

and, therefore

$$R(x,y,z,y)=R'(x,y,z,y)$$

for all x,y,z $\in$ V Using what we have just proved, we obtain

$$
\begin{aligned}
R(x,y+t,z,y+t)&=R'(x,y+t,z,y+t)\\
g(R(x,y+t)z,y+t))&=g(R'(x,y+t)z,y+t)
\end{aligned}
$$

$$
\begin{aligned}
R(x,y,z,t)+R(x,t,z,y)+R(x,y,z,y)+R(x,t,z,t)=&\ R'(x,y,z,t)+R'(x,t,z,y)\\
&+R'(x,y,z,y)+R'(x,t,z,t)
\end{aligned}
$$

hence

$$R(x,y,z,t)+R(x,t,z,y)=R'(x,y,z,t)+R'(x,t,z,y),$$

which can be written further as

$$R(x,y,z,t)-R'(x,y,z,t)=R(y,z,x,t)-R'(y,z,x,t).$$

It follows that, the expression $R(x,y,z,t)-R'(x,y,z,t)$ is invariant by cyclic permutations of the first three elements. Therefore, by (a) of Proposition 2.42, we have

$$3\left[R(x,y,z,t)-R'(x,y,z,t)\right]=0,$$

hence

$$R(x,y,z,t)=R'(x,y,z,t)$$

for all $x,y,z,t\in V$.

■

Definition 2.49. *M has* ***constant sectional curvature*** *if there exist $K_o\in\mathbb{R}$ such that*

$$K(p,\sigma)=K_o\ \forall p\in M,\sigma\subset T_pM$$

Lemma 2.50. *Let M be a Riemannian manifold and p a point of M. Define a tri-linear mapping $R':T_pM\times T_pM\times T_pM\times T_pM\to T_pM$ by*

$$R'(X,Y,W,Z)=g(X,W)g(Y,Z)-g(Y,W)g(X,Z)$$

for all $X,Y,W,Z\in T_pM$. Then M has constant sectional curvature K_o if and only if $R=K_oR'$, where R is the curvature of M.

Proof. Assume that $K(p,\sigma)=K_o$ for all $\sigma\subset T_pM$, and set $R(X,Y,W,Z)$. Observe that R' satisfies the properties of Proposition 2.42. Since

$$R'(X,Y,X,Y)=g(X,X)g(Y,Y)-g(X,Y)^2,$$

we have that, for all pairs of vectors $X,Y\in T_pM$,

$$R(X,Y,X,Y) = K_o\left(|X|^2|Y|^2 - \langle X,Y\rangle^2\right) = K_oR'(X,Y,X,Y)$$

Lemma 2.48 implies that, for all X, Y, W, Z,

$$R(X,Y,W,Z) = K_oR'(X,Y,W,Z),$$

hence $R = K_oR'$.

■

Corollary 2.51. *Let M be a Riemannian manifold of dimension n, p a point of M and $\{e_1,\dots,e_n\}$, an orthonormal basis of T_pM. Define $R_{ijk\ell} = g_p\left(R\left(e_i,e_j\right)e_k, e_\ell\right)$, $i,j,k,\ell = 1,\dots,n$. Then $K(p,\sigma) = K_o$ for all $\sigma \subset T_pM$, if and only if $R_{ijk\ell} = K_o\left(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}\right)$ In other words, $K(p,\sigma) = K_o$ for all $\sigma \subset T_pM$ if and only if $R_{ijij} = -R_{ijji} = K_o$ for all $i \neq j$, and $R_{ijk\ell} = 0$ in the other cases.*

2.52 Ricci curvature and scalar curvature

Definition 2.53. *Let (M,g) be a Riemannian manifold, then we define*
*(i) the **Ricci operator** $\mathrm{Ric} : \Gamma(TM) \to C_1^\infty(M)$ by*

$$\mathrm{Ric}(X) = \sum_{i=1}^n R\left(X, e_i\right)e_i,$$

*(ii) the **Ricci curvature** $\mathrm{Ric} : \Gamma(TM) \times \Gamma(TM) \to \mathbb{R}$ by*

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^n g\left(R\left(X,e_i\right)e_i, Y\right),$$

*(iii) the **scalar curvature** $Scal \in C^\infty(M)$ by*

$$\mathrm{Scal} = \sum_{j=1}^n \mathrm{Ric}\left(e_j,e_j\right) = \sum_{j=1}^n\sum_{i=1}^n g\left(R\left(e_i,e_j\right)e_j, e_i\right).$$

Here $\{e_1,\dots,e_n\}$ is any local orthonormal frame for the tangent bundle.

In the case of constant sectional curvature we have the following result.

Corollary 2.54. *Let (M^n,g) be a Riemannian manifold of constant sectional curvature κ. Then its scalar curvature satisfies the following*

$$\mathrm{Scal} = n\cdot(n-1)\cdot\kappa.$$

Proof. Let $\{e_1,\dots,e_n\}$ be any local orthonormal frame. Then

$$\begin{aligned}
\mathrm{Ric}\left(e_j,e_j\right) &= \sum_{i=1}^n g\left(R\left(e_j,e_i\right)e_i, e_j\right)\\
&= \sum_{i=1}^n g\left(\kappa\left(g\left(e_i,e_i\right)e_j - g\left(e_j,e_i\right)e_i\right), e_j\right)\\
&= \kappa\left(\sum_{i=1}^n g\left(e_i,e_i\right)g\left(e_j,e_j\right) - \sum_{i=1}^n g\left(e_i,e_j\right)g\left(e_i,e_j\right)\right)\\
&= \kappa\left(\sum_{i=1}^n 1 - \sum_{i=1}^n \delta_{ij}\right)\\
&= (n-1)\cdot\kappa.
\end{aligned}$$

then

$$\begin{aligned}
\text{Scal} &= \sum_{j=1}^{n} \text{Ric}\left(e_j, e_j\right) \\
&= \sum_{j=1}^{n} (n-1) \cdot \kappa \\
&= n \cdot (n-1) \cdot \kappa.
\end{aligned}$$

■

Let $x = z_n$ be a unit vector in T_pM; we take an orthonormal basis $\{z_1, z_2, \ldots, z_{n-1}\}$ of the hyperplane in T_pM orthogonal to x . We are going to prove that the expressions above do not depend on the choice of the corresponding orthonormal basis.
To prove these facts, we give an intrinsic characterization of the expressions above. First, define a bilinear form on T_pM as follows: let $x, y \in T_pM$ and put

$$Q(x,y) = \text{ trace of the mapping } z \mapsto R(x,z)y.$$

Q is obviously bilinear. Choosing x a unit vector and then completing it to an orthonormal basis $\{z_1, \ldots, z_{n-1}, z_n = x\}$ of T_pM we have

$$\begin{aligned}
Q(x,y) &= \sum_i g\left(R\left(x, z_i\right) y, z_i\right) \\
&= \sum_i g\left(R\left(y, z_i\right) x, z_i\right) = Q(y,x)
\end{aligned}$$

that is, Q is symmetric and $Q(x,x) = (n-1)\,\text{Ric}_p(x)$; this proves that $\text{Ric}_p(x)$ is intrinsically defined.

On the other hand, the bilinear form Q on T_pM corresponds to a linear self-adjoint mapping K, given by

$$g(\text{Scal}(x), y) = Q(x,y).$$

Taking an orthonormal basis $\{z_1, \ldots, z_n\}$, we have

$$\begin{aligned}
Trace \quad of \quad \text{Scal} &= \sum_j g\left(\text{Scal}\left(z_j\right), z_j\right) \\
&= \sum_j Q\left(z_j, z_j\right) \\
&= (n-1) \sum_j \text{Ric}_p\left(z_j\right) \\
&= n(n-1)\text{Scal},
\end{aligned}$$

which proves the statement.

The bilinear form $\frac{1}{n-1}Q$ is, at times, called the **Ricci tensor**.

As usual we should express what was done above in a coordinate system (x_i). Let $X_i = \frac{\partial}{\partial x_i}$, $g_{ij} = g\left(X_i, X_j\right)$, and g^{ij} the inverse matrix of g_{ij} (i.e., $\sum_k g_{ik} g^{k\ell} = \delta_i^{\ell}$). Then the coefficients of the bilinear form $\frac{1}{n-1}Q$ in the basis $\{X_i\}$ are given by

$$\frac{1}{n-1} R_{ik} = \frac{1}{n-1} \sum_j R_{ijk}^{j} = \frac{1}{n-1} \sum_{sj} R_{ijks} g^{sj}.$$

We observe now that if $A : T_pM \to T_pM$ is a linear self-adjoint mapping and
$B : T_pM \times T_pM \to \mathbf{R}$ is the associated bilinear form,

i.e., $B(X,Y)=g(A(X),Y)$, then the trace of A is equal to $\sum_{ik} B\left(X_i, X_k\right) g^{ik}$. Thus, the scalar curvature in the coordinate system (x_i) is given by

$$K=\frac{1}{n(n-1)} \sum_{ik} R_{ik} g^{ik}$$

Let $f: A \subset R^2 \rightarrow M$ be a parametrized surface and let (s,t) be the usual coordinates of R^2. Let $V=V(s,t)$ be a vector field along f. For each (s,t), it is possible to define $R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) V$ in an obvious manner.

Lemma 2.55.

$$\frac{D}{\partial t} \frac{D}{\partial s} V-\frac{D}{\partial s} \frac{D}{\partial t} V=R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) V$$

.

Proof. The proof is a long calculation. Choose a system of coordinates $(U, \mathbf{x})$ based at $p \in M$. Let $V=\sum_i v^i X_i$, where $v^i=v^i(s,t)$e $X_i=\frac{\partial}{\partial x_i}$. Then

$$\frac{D}{\partial s} V=\frac{D}{\partial s}\left(\sum_i v^i X_i\right)=\sum_i v^i \frac{D}{\partial s} X_i+\sum_i \frac{\partial v^i}{\partial s} X_i$$

and

$$\frac{D}{\partial t}\left(\frac{D}{\partial s} V\right)=\sum_i v^i \frac{D}{\partial t} \frac{D}{\partial s} X_i+\sum_i \frac{\partial v^i}{\partial t} \frac{D}{\partial s} X_i+\sum_i \frac{\partial v^i}{\partial s} \frac{D}{\partial t} X_i+\sum_i \frac{\partial^2 v^i}{\partial t \partial s} X_i$$

Therefore, interchanging the roles of s and t in the expression above, and subtracting, we obtain

$$\frac{D}{\partial t} \frac{D}{\partial s} V-\frac{D}{\partial s} \frac{D}{\partial t} V=\sum v^i\left(\frac{D}{\partial t} \frac{D}{\partial s} X_i-\frac{D}{\partial s} \frac{D}{\partial t} X_i\right).$$

Let us now calculate $\frac{D}{\partial t} \frac{D}{\partial s} X_i$. Put

$$f(s,t)=\left(x_1(s,t), \ldots, x_n(s,t)\right).$$

Then $\quad \frac{\partial f}{\partial s}=\sum_j \frac{\partial x_j}{\partial s} X_j$ and $\quad \frac{\partial f}{\partial t}=\sum_k \frac{\partial x_k}{\partial t} X_k$. Thus, we have

$$\frac{D}{\partial s} X_i=\nabla_{\Sigma_j\left(\partial x_j / \partial s\right) X_j}\left(X_i\right)=\sum_j \frac{\partial x_j}{\partial s} \nabla_{X_j} X_i$$

and

$$\begin{aligned}
\frac{D}{\partial t} \frac{D}{\partial s} X_i & =\frac{D}{\partial t}\left(\sum_j \frac{\partial x_j}{\partial s} \nabla_{X_j} X_i\right) \\
& =\sum_j \frac{\partial^2 x_j}{\partial t \partial s} \nabla_{X_j} X_i+\sum_j \frac{\partial x_j}{\partial s} \nabla_{\Sigma_k\left(\partial x_k / \partial t\right) X_k}\left(\nabla_{X_j} X_i\right) \\
& =\sum_j \frac{\partial^2 x_j}{\partial t \partial s} \nabla_{X_j} X_i+\sum_{jk} \frac{\partial x_j}{\partial s} \frac{\partial x_k}{\partial t} \nabla_{X_k} \nabla_{X_j} X_i
\end{aligned}$$

or

$$\left(\frac{D}{\partial t} \frac{D}{\partial s}-\frac{D}{\partial s} \frac{D}{\partial t}\right) X_i=\sum_{jk} \frac{\partial x_j}{\partial s} \frac{\partial x_k}{\partial t}\left(\nabla_{X_k} \nabla_{X_j} X_i-\nabla_{X_j} \nabla_{X_k} X_i\right)$$

Joining everything together, we finally get

$$\begin{aligned}\left(\frac{D}{\partial t}\frac{D}{\partial s}-\frac{D}{\partial s}\frac{D}{\partial t}\right)V &= \sum_{ijk} v^i \frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial t} R\left(X_j, X_k\right) X_i \\ &= R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) V.\end{aligned}$$

■

2.56 Tensors on Riemannian manifolds

For what follows it is useful to observe that $\Gamma(TM)$ has a linear structure when we take as "scalars" the elements of $C^\infty(M)$.

Definition 2.57. *A* ***covariant tensor*** T *of order* r *on a Riemannian manifold is a multilinear mapping*

$$T : \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r \text{ factors}} \to C^\infty(M)$$

This means that given $Y_1, \ldots, Y_r \in \Gamma(TM), T\left(Y_1, \ldots, Y_r\right)$, is a differentiable function on M, and that T is linear in each argument, that is,

$$T\left(Y_1, \ldots, fX + gY, \ldots, Y_r\right) = fT\left(Y_1, \ldots, X, \ldots, Y_r\right) + gT\left(Y_1, \ldots, Y, \ldots, Y_r\right)$$

for all $X, Y \in \Gamma(TM), f, g \in C^\infty(M)$.

A tensor T is a pointwise object in a sense that we now explain. Fix a point $p \in M$ and let U be a neighborhood of p in M on which it is possible to define vector fields $E_1 \ldots, E_n \in \Gamma(TM^n)$, in such a fashion that at each $q \in U$, the vectors $\{E_i(q)\}, i = 1, \ldots, n$, form a basis of T_qM, we say, in this case, that $\{E_i\}$ is a moving frame on U. Let

$$Y_1 = \sum_{i_1} y_{i_1} E_{i_1}, \ldots, Y_r = \sum_{i_r} y_{i_r} E_{i_r}, \quad i_1, \ldots, i_r = 1, \ldots, n$$

be the restrictions to U of the vector fields $Y_1, \ldots, Y_r$, expressed in the moving frame $\{E_i\}$. By linearity,

$$T\left(Y_1, \ldots, Y_r\right) = \sum_{i_1, \ldots, i_r} y_{i_1} \ldots y_{i_r} T\left(E_{i_1}, \ldots, E_{i_r}\right)$$

The functions $T\left(E_{i_1}, \ldots E_{i_r}\right) = T_{i_1, \ldots, i_r}$ is on U are called the components of T in the frame $\{E_i\}$

The expression above implies that the value of $T\left(Y_1, \ldots, Y_r\right)$ at a point $p \in M$ depends only on the values at p of the components say that T is a pointwise object.

Example 2.58. *The curvature tensor*

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

is defined by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W), \quad X, Y, Z, W \in \Gamma(TM)$$

We can verify that R *is a covariant tensor of order* 4*, whose components in the frame* $\left\{X_i = \frac{\partial}{\partial x_i}\right\}$ *associated with the system of coordinates* (x_i) *is*

$$R_{ijk\ell} = R\left(X_i, X_j, X_k, X_\ell\right).$$

Example 2.59. *The "metric tensor" $G : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$ is defined by $G(X,Y) = \langle X, Y \rangle, X, Y \in \Gamma(TM)$. G is a covariant tensor of order 2 and its components in the frame $\{X_i\}$ are the coefficients g_{ij} of the Riemannian metric in the given system of coordinates.*

Example 2.60. *The Riemannian connection ∇ defined by:*

$$\begin{gathered}\nabla : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M) \\ \nabla(X,Y,Z) = g\left(\nabla_X Y, Z\right), \quad X, Y, Z \in \Gamma(TM)\end{gathered}$$

is not a tensor, because ∇ is not linear with respect to the argument Y.

Definition 2.61. *Let T be a tensor of order r. The* ***covariant differential*** *∇T of T is a tensor of order $(r+1)$ given by*

$$\nabla T\left(Y_1, \ldots, Y_r, Z\right) = Z\left(T\left(Y_1, \ldots, Y_r\right)\right) - T\left(\nabla_Z Y_1, \ldots, Y_r\right) - \cdots - T\left(Y_1, \ldots, Y_{r-1}, \nabla_Z Y_r\right)$$

For each $Z \in \Gamma(TM)$, the covariant derivative $\nabla_Z T$ of T relative to Z is a tensor of order r given by

$$\nabla_Z T\left(Y_1, \ldots, Y_r\right) = \nabla T\left(Y_1, \ldots, Y_r, Z\right)$$

We are going to show that, in a convenient frame, the definition of the covariant derivative of a tensor T relative to $Z \in \Gamma(TM)$ becomes quite natural. For this, let $p \in M$ and let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve with $\alpha(0) = p, \alpha'(t) = Z(\alpha(t))$. Let $\{e_1, \ldots, e_n\}$ be a basis of T_pM and let $e_i(t)$ be the parallel transport of e_i along $\alpha = \alpha(t)$, for $i = 1, \ldots, n$. Let $T_{i_1 \ldots i_r}(t)$ be the components, in the basis $\{e_i(t)\}$, of the restriction $T(\alpha(t))$ of T to the curve α. Then, by the definition of $\nabla_z T$,

$$\left(\nabla_Z T\right)\left(e_{i_1}(t), \ldots, e_{i_r}(t)\right) = \frac{d}{dt} T_{i_1 \ldots i_r}(t) - T\left(\nabla_Z e_{i_1}(t), \ldots, e_{i_r}(t)\right) - \cdots - T\left(e_{i_1}(t), \ldots, \nabla_Z e_{i_r}(t)\right)$$

Since $\nabla_Z e_i(t) = 0$, we have, by linearity,

$$\left(\nabla_z T\right)_{i_1 \ldots i_r} = \left(\nabla_z T\right)\left(e_{i_1}(t), \ldots, e_{i_r}(t)\right) = \frac{d}{dt} T_{i_1 \ldots i_r}$$

In other words, in this frame, the components of the corariant derivative of T are the usual derivatives of the components of T.

Example 2.62. *The covariant differential of the metric tensor is the zero tensor. Indeed, for all $X, Y, Z \in \Gamma(TM)$,*

$$\nabla G(X,Y,Z) = Zg(X,Y) - g\left(\nabla_Z X, Y\right) - g\left(X, \nabla_Z Y\right) = 0$$

because ∇ is the Riemannian connection.

CHAPTER

3

THE LAPLACIAN IN RIEMANNIAN MANIFOLD

The Laplacian is the most important operator such it used in many of the equations in physics to descibe natural phenomena .It generalize to functions defined on a manifold. Were George de Rham realized that it was fruitful to define a version of Laplacian operating on differential forms, because of a fondamental relationship between harmonic forms and the de Rham colomology groups on a smooth manifold.
In this chapter , the following references are used; [BGM71] [Can13] [RS97] [JJ08] [CS08] [GQ12]

3.1 Divergence of vector field, δ operator

On the Riemannian manifold (M, g), we define a linear mapping of $\Gamma(TM)$ in $C^\infty(M)$, called divergence, and defined as follos:

Definition 3.2. *the divergent of vector field ξ on M is the function $\operatorname{div}\xi$ locally define by :*

$$\operatorname{div}_\xi : \Gamma(TM) \to C^\infty(M)$$
$$\operatorname{div}_\xi \omega = d(\xi \lrcorner\, \omega) \tag{3.1}$$

such ω design the volume form wich correspondent in a local orientation, and $\xi \lrcorner\, \omega$ design the contract product of ξ and of the n-form ω (n is the dimension of Riemannian manifold M), means that the (n-1)-form defined by:

$$\xi \lrcorner\, \omega(X_1, \ldots, X_{n-1}) = \omega(\xi, X_1, \ldots, X_{n-1}), \forall X_1, \ldots, X_{n-1} \in \Gamma(TM)\,.$$

Remark 3.3. *The $\operatorname{div}\xi$ does not depend on the choose of the volume form ω, so the divergence is defined globally in (M,g).*

Proposition 3.4.

$$\operatorname{div}(f.\xi) = f.\operatorname{div}\xi + df(\xi) \qquad \forall f \in C^\infty(M)$$
$$\forall \xi \in \Gamma(TM)\,.$$

by duality, we obtain from the divergence an operator on the space $A^1(M)$ of 1-forms, called δ and defined as follows:

Definition 3.5. *δ is operator of $A^1(M)$ in $C^\infty(M)$ defined by :*

$$\delta\alpha = -\operatorname{div}(\alpha^*) \qquad \alpha \in A^1(M) \tag{3.2}$$

such $\alpha^* = \sum_{i,j} (g^{ij}\alpha_j) \cdot X_i$.

Remark 3.6. *On 0-forms, δ is simply the zero linear functional.*

If M is compact, the spaces $A^P(M)$ are endowed with a structure pre-Hilbertian, defined from the inner product $(.|.)$ on the euclidean space $\Gamma(\wedge^p T^* M)$, and from the canonical measure v_g, on (M,g) , the global inner product is denoted $< .,. >$. We therefore have, if α and β are two-forms on M :

$$< \alpha, \beta > = \int_M (\alpha|\beta) \cdot v_g.$$

Proposition 3.7.

$$< df, a > = < f, \delta a > \qquad \begin{array}{l} \forall f \in A^0(M) \\ \forall a \in A^1(M) \end{array}$$

this equality also written :

$$\int_M (df \mid \alpha) \cdot v_g = \int_M f \cdot \delta\alpha \cdot v_g.$$

Proof. To prove it, we must prove that :

$$\int_{\mathrm{M}} (df \mid \alpha) \cdot \omega = \int_{\mathrm{M}} f\delta\alpha \cdot \omega.$$

where ω is form volume deffned locally near any point of the manifold.
Posed:

$$\begin{aligned} I &= \int_M (df \mid \alpha) \cdot \omega - \int_M (f\delta\alpha) \cdot \omega \\ &= \int_M ((df \mid \alpha) - f\delta\alpha) \cdot \omega. \end{aligned}$$

Because of the definitions of the operators δ and div, we have:

$$I = \int_M ((df \mid \alpha) + f \operatorname{div} a^*) \cdot \omega$$

However, by proposition3.4, we have:

$$f \operatorname{div} \alpha^* = \operatorname{div}(f\alpha^*) - df\,(\alpha^*)$$

and , by definition,

$$(\mathrm{df} \mid \alpha) = df\,(\alpha^*)\,,$$

so that,

$$\begin{aligned} I &= \int_M ((df \mid \alpha) + \operatorname{div}(f\alpha^*) - df\,(\alpha^*)) \cdot \omega \\ &= \int_M (df\,(\alpha^*) + \operatorname{div}(f\alpha^*) - df\,(\alpha^*)) \cdot \omega \\ &= \int_M \operatorname{div}(f\alpha^*) \cdot \omega \\ &= \int_M d(f\alpha^* \lrcorner\, \omega), \end{aligned}$$

that is, because of stokes

$$\begin{aligned} I &= \int_M d(f\alpha^* \lrcorner\, \omega) \\ &= \int_M d\omega(f\alpha^*, X_1, \ldots, X_{m-1}) \quad \forall X_1, \ldots, X_{m-1} \in \Gamma(TM) \\ &= 0 \end{aligned}$$

which demonstrates the proposition previous.

■

3.7.1 The Divergence and δ operator of differential forms

Definition 3.8. *The divergent is tace operator wich define by:*

$$\begin{aligned} \mathrm{div} : A^p(M) &\to A^{p-1}(M) \\ \omega &\to \mathrm{div}\, \omega \end{aligned}$$

such that for all p-form :

$$\mathrm{div}\, \omega(X, X_1, \ldots, X_{p-1}) = \nabla_X \omega\,(X, X_1, X_2, \ldots, X_{p-1})$$

Proposition 3.9. *We can express the differential d in function of the connection ∇ by:* $\forall \omega \in A^p(M), \forall,\ i = 1, 2, \ldots, p$

$$d\omega\,(X, X_1, X_2, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X^i} \omega \left(X_1, X_2, \ldots, \hat{X}_i, \ldots, X_{p+1}\right).$$

Definition 3.10. *δ operator of p-form define as:*

$$\begin{aligned} \delta : \mathrm{A}^p(M) &\longrightarrow \mathrm{A}^{p-1}(M) \\ \alpha &\longmapsto \delta\alpha \end{aligned}$$

such that for all $\alpha \in \mathrm{A}^{p-1}(M)$ and $\beta \in \mathrm{A}^p(M)$ we have:

$$(d\alpha, \beta) = (\alpha, \delta\beta)\,.$$

wich have the propriety :

$$\delta\alpha\,(X_2, X_3, \ldots, X_p) = -\sum_{i=1}^{n} \nabla_{X_i} \alpha\,(X, X_2, \ldots, , X_p)\,.$$

The definition of the divergent we have :

$$\forall \alpha \in \mathrm{A}^p(M),\ \delta\alpha = -\,\mathrm{div}\, \alpha.$$

3.10.1 Calculation of divergence and δ in local coordinates:

Let (x^i) a local map on M in point m , to which is attached a real function $\theta = \sqrt{\det{(g_{ij})}}$, such that, locally, the associated volume form is expressed by:

$$\omega = \theta \cdot dx^1 \wedge \ldots \wedge dx^n$$

.

If (x_i) is the local filed of the frame associated with the map (x^i), (means that $x_i = \frac{\partial}{\partial x^i}$), we have:

$$
\begin{aligned}
\xi \lrcorner \omega\left(x_1, \ldots, \hat{x}_i, \ldots, x_n\right) &= \omega\left(\xi, x_1, \ldots, \hat{x}_i, \ldots, x_n\right) \\
&= (-1)^{i-1} \omega\left(x_1, \ldots, \xi, \ldots, x_n\right). \\
&= (-1)^{i-1} \theta \cdot d x^1 \wedge \ldots \wedge d x^n\left(x_1, \ldots, \xi, \ldots, x_n\right) \\
&= (-1)^{i-1} \theta . \xi^i
\end{aligned}
$$

such , ξ^i design the i^{th} component of ξ. then we have:

$$
\xi \lrcorner \omega=\sum_i^n(-1)^{i-1}\left(\theta . \xi^i\right) d x^i \wedge \ldots \wedge \widehat{d x^i} \wedge \ldots d x^n
$$

and therefore :

$$
\begin{aligned}
d(\xi \lrcorner \omega) &= d\left(\sum_i^n(-1)^{i-1}\left(\theta . \xi^i\right) d x^i \wedge \ldots \wedge \widehat{d x^i} \wedge \ldots d x^n\right) \\
&= \sum_i^n(-1)^{i-1} \frac{\partial\left(\theta \cdot \xi^i\right)}{\partial x^i} d x^i \wedge d x^1 \wedge \cdots \wedge \widehat{d x^i} \wedge \ldots d x^n \\
&= \sum_i^n \frac{\partial\left(\theta \cdot \xi^i\right)}{\partial x^i} d x^1 \wedge \cdots \wedge d x^n \\
&= \left(\sum_i^n \frac{\partial\left(\theta \cdot \xi^i\right)}{\partial x^i} \cdot \theta^{-1} \cdot \omega\right) \\
&= \theta^{-1}\left(\sum_i^n \frac{\partial\left(\theta \cdot \xi^i\right)}{\partial x^i} \cdot \omega\right) \\
&= \theta^{-1}\left(\sum_i^n \frac{\partial\left(\theta \cdot \xi^i\right)}{\partial x^i}\right) \cdot \omega .
\end{aligned}
$$

by (3.1), so that:

$$
\operatorname{div} \xi=\theta^{-1} \sum_i \frac{\partial\left(\theta \cdot \xi^i\right)}{\partial x^i} . \tag{3.3}
$$

In the same map, a 1 -form α is expressed by:

$$
\alpha=\sum_i \alpha_i d x^i
$$

and therefore :

$$
\alpha^*=\sum_{i, j}\left(g^{i j} \alpha_j\right) \cdot X_i \tag{3.4}
$$

where g^{ij} is the generic element of the inverse matrix of (g_{ij}). It then follows from (3.4)that:

$$
\begin{aligned}
\delta \alpha &= -\operatorname{div}\left(\alpha^*\right) \\
&= -\theta^{-1}\left(\sum_{i, j} \frac{\partial\left(\theta g^{i j} \alpha_j\right)}{\partial x^i}\right) .
\end{aligned} \tag{3.5}
$$

Proposition 3.11. *From (3.3) , for any $X, Y \in \Gamma(TM)$ and $\omega \in A^p(M)$ we have:*

$$
\operatorname{div}(X+Y)=\operatorname{div} X+\operatorname{div} Y .
$$

and, from (3.3) and (3.1) we have:

$$
(X+Y) \lrcorner \omega=X \lrcorner \omega+Y \lrcorner \omega
$$

3.11.1 Geometric formulation of δ

The 2-form $D\alpha$ is a covariant derivative of 1-form α such that :

$$\text{trace } D\alpha = \sum_i D\alpha(X_i, X_i)$$

such (X_i) is the orthonormal fram .
as we have :

$$D\alpha(X_i, X_i) = (D_{X_i}\alpha^* | X_i)$$

Proposition 3.12. *For all 1-form α define on the Riemannian manifold (M, g) :*

$$\delta\alpha = -\,\text{trace } \alpha$$

3.13 The Laplacien operator of a function

Definition 3.14. *The Laplacien, noted Δ, is an operator of $A^0(M)$ in $A^0(M)$ define by :*

$$\Delta f = \delta df \qquad f \in A^0(M)\,.$$

3.14.1 Expression in local coordonnates:

It given by the equation (3.5) such we replace α by df , means that α_j by $\frac{\partial f}{\partial x^j}$. It comes :

$$\Delta f = -\theta^{-1} \sum_{i,j} \frac{\partial \left(\theta\, g^{ij} \frac{df}{dx^j}\right)}{dx^i}$$

So the Laplacian is a second-order differential operator , its homogeneous part of the second order is written :

$$\sigma = -\sum_{i,j} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}\,.$$

Proposition 3.15. *In n-dimensional Euclidean space, the Laplace operator or Laplacian Δ is differentiable operator is the divergence of the gradient such that :*

$$\text{div} = \left(\frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_n}\right) \, and \, \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

$$\begin{aligned} \Delta &= -\,\text{div}\,.\nabla \\ &= -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \end{aligned}$$

Example 3.16. *Let $(\mathbb{R}^n, g_0)$be connected (g_0 is the metric on R^n) the g_{ij} are constants, and consider the operator acting on $A^0(R^n)$ such that simply differentiates a function $f \in A^0(\mathbb{R}^n)$ two two times with respect to each position variable, the expression of the Laplacian is :*

$$\Delta f = -\sum_i \frac{\partial^2 f}{\partial x^{i^2}}, \qquad f \in A^0(\mathbb{R}^n)$$

means that the opposite of the usual Laplacian.
Extending the known notion on $\mathbb{R}^n$, we will say that a real function defined on a Riemannian manifold is harmonic if it verifies equality:
$\Delta f = 0.$

3.16.1 Geometric formulation of the Laplacien

Let $D\,df$ is the second covariant derivative of f , For all f of $A^0(M)$:

$$\Delta f = -\operatorname{trace}(D\,df)$$

means that

$$\Delta f = -\operatorname{trace}(\text{Hess } f)$$

such Hess f design the Hessien of f (the Hessien of f is the second covariant derivative of f such Hess $f = d^2 f = D\,df$).
If X_i is the orthonorms frame then :

$$\Delta f = \sum_i \text{Hess } f(X_i, X_i)$$

Example 3.17. *(Laplacien on the sphere) The sphere (S^n, g) being considered as immersed in $(\mathbb{R}^{n+1}, g_0)$, compare the two following applications of S^n in $\mathbb{R}$:*

$$\Delta^{S^n}\left(f|_{s^n}\right) \text{ and } \left(\Delta^{R^{n+1}} f\right)|_{s^n}$$

such f is an application C^∞ of $\mathbb{R}^n$ in $\mathbb{R}$.
we have the quality:

$$\left(\Delta^{R^{n+1}} f\right)|_{s^n} = \Delta^{S^n}\left(f|_{s^n}\right) - \frac{\partial^2 f}{\partial r^2}|_{S^n} - n \cdot \frac{\partial f}{\partial r^1}|_{S^n} \tag{3.6}$$

for all $f : \mathbb{R}^{n+1} \to \mathbb{R}$

A point p of s^n determines a unit vector x on $\mathbb{R}^{n+1}$. We complete it with the x_i, $i = 2, \ldots, n+1$, so as to obtain an orthonormal basis $\{x, x_i\}_{i=2,\ldots,n+1}$ of $\mathbb{R}^{n+1}$ and therefore an orthonormal basis $\{x_i\}_{i=2,\ldots,n+1}$ of $T_n s^n$.
The geodesic γ_i (is a curvature $\gamma_i : I \to M$; $I \subset \mathbb{R}$ such that $\frac{D}{dt}\gamma_i(t) = 0$; $t \in I$ where D is curvature derivation.) , determined on (S^n, g_0) par x_i, written:

$$\gamma_i : a \to \cos a.x + \sin a \cdot x_i \quad i = 2, \ldots, n+1$$

(ou x and x_i are considered as points of R^{n+1} and therefore $\gamma_i(a)$ as a point of s^n).

The map f has partial derivatives $\left(\frac{\partial f}{\partial x^i}\right)_{i=1,\ldots,n+1}$ corresponding to the basis of $\mathbb{R}^{n+1}$ $\{x = x_1, x_i\}_{i=2,\ldots,n+1}$. With cos notations the first derivative, with respect to a, of $f \circ \gamma_i$ written, at the point $\gamma_i(a)$:

$$\frac{d\,(f \circ \gamma_i)}{da}(a) = -\sin a \cdot \frac{\partial f}{\partial x^1} + \cos a \cdot \frac{\partial f}{\partial x^i}$$

and the second derivative, at the point $n = \gamma_i(0)$:

$$\frac{d^2\,(f \circ \gamma_i)}{da^2}(0) = \frac{\partial f}{\partial x^1}(p) + \frac{\partial^2 f}{\partial x^{i^2}}(p).$$

It follows, the following value of $\Delta^{s^n}(f|s^n)$:

$$\begin{aligned}\Delta^{s^n}\left(f|_{s^n}\right)(p) &= \sum_{i=2}^{n+1} \frac{d^2}{da^2}\left(f \circ \gamma_1\right)(0) \\ &= \sum_{i=2}^{n+1} \frac{\partial^2 f}{\partial x^{i^2}}(p) + p \cdot \frac{\partial f}{\partial x^1}(p).\end{aligned}$$

While we have

$$\left(\Delta^{\mathbb{R}^{n+1}} f\right)(p) = -\sum_{i=2}^{n+1} \frac{\partial^2 f}{\partial x^{i^2}}(p)$$
$$= -\sum_{i=2}^{p+1} \frac{\partial^2 f}{\partial x^{i^2}}(p) - \frac{\partial^2 f}{\partial x^1}(p).$$

So that:

$$\left(\Delta^{R^{n+1}} f\right)\big|_{s^n}(p) = \Delta^{s^n}(f\big|_{s^n})(p) - \frac{\partial^2 f}{\partial x^2}(p) - n \cdot \frac{\partial f}{\partial x^1}(p),$$

that is to say precisely (3.6)

3.18 The Laplacien of compact manifold

As the operators d and δ are adjuncts on the compact Riemannian manifold (M, g), for all f, for all g in $A^0(M)$ we have :

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$$

$$\langle \Delta f, f\rangle = \|df\|^2 \tag{3.7}$$

Definition 3.19. ***(The Hodge-de Rham Laplacian)*** *On a n-dimensional compact Riemannian manifold (M, g) the Laplacian is defined on the $A^p(M)$, for all p, by the formula:*

$$\Delta : \mathrm{A^p(M)} \to \mathrm{A^p(M)}$$
$$\alpha \to d\delta(\alpha) + \delta \mathrm{d}(\alpha)$$

Proposition 3.20. *The Laplacien of a n-dimensional compact Riemannian manifold (M, g) is an operator positive-definite and self-adjoint that is:*

$$\langle \Delta\alpha, \beta\rangle = \langle \alpha, \Delta\beta\rangle \qquad \forall \alpha, \beta \in A^p(M); \quad 0 < p < n$$

Remark 3.21. *From (3.7) we deduce that a harmonic function is locally constant, means that constant on each connected component of M.*

Proposition 3.22. ***(Bochner-Liohnerowicz formula)***
For all $f \in A^0(M)$, we have :

$$-\frac{1}{2}\Delta\left(|\mathrm{df}|^2\right) = |\text{ Hess } f|^2 - |\Delta f|^2 + \rho\left(df^*, df^*\right)$$

when , ρ design the Ricci courvature of the Riemannian manifold (M, g).

Lemma 3.23. *For all form $\alpha \in A^1(M)$ and all $X, Y \in \Gamma(TM)$, we have :*

$$D_X D_Y \alpha^* - D_Y D_X \alpha^* - D_{[X,Y]}\alpha^* = (R(X, Y)\alpha^*)$$

The lemma follows from the definition of curvature.

Proposition 3.24. *For the Laplacian, thus defined on the p-forms, we have the following generalized Bochner-Lichnerowicz formula:*

$$-\frac{1}{2}\Delta\left(|\alpha|^2\right) = |D\alpha|^2 - (\alpha \mid \Delta\alpha) + F(\alpha) \quad \forall \alpha \in A^p(M)$$

where $F(\alpha)$ is quadratic in α and linear in the curvature tensor R*, such that, for all $X_a, X_b, X_{a_2}, X_{a_3}, \cdots, X_{a_p} \in \{X_i\}_{i \le n}$ an orthonormal and parallal frame :*

$$F(\alpha) = \frac{1}{(p-1)!} \sum_{a,b,a_2,\ldots,a_p} \left(\nabla_{X_a}\nabla_{X_b} - \nabla_{X_b}\nabla_{X_a}\right)\alpha\left(X_a, X_{a_2}, X_{a_3}, \cdots, X_{a_p}\right)\cdot\alpha\left(X_b, X_{a_2}, X_{a_3}, \cdots, X_{a_p}\right).$$

3.25 Hodge Theory

Definition 3.26. ***(Hodge star operator*** *)is an isomorphism (the unique isomorphism) between smooth p forms to smooth $n-p$ forms on compact n-dimensional Riemannian manifold, defined by:*

$$* : A^p(M) \to A^{n-p}(M) \quad 0 < p < n$$
$$\alpha \longmapsto *\alpha$$

such that for all pour $\alpha, \beta \in \mathrm{A}^p(M)$:

$$\alpha \wedge *\beta =< \alpha, \beta > \omega.$$

such ω is a volume form.

Definition 3.27. *We can define an inner product on the vector space $A^p(M)$ of p-forms on M by setting*

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta \quad \text{for } \alpha, \beta \in A^p(M) \tag{3.8}$$

and we denote the corresponding norm by $\|\alpha\|$.

Proprety 3.28. *Let $\{X_i\}_{1\leq i\leq n}$ the orthonormal fram of M and $\{X_i^*\}_{1\leq i\leq n}$ the associated orthonormal basis, then for all $\sigma \in S(k,n)$:*

$$*\left(X^*_{\sigma(1)} \wedge X^*_{\sigma(2)} \wedge \cdots \wedge X^*_{\sigma(k)}\right) = \mathrm{Sign}(\sigma)\left(X^*_{\sigma(k+1)} \wedge X^*_{\sigma(k+2)} \wedge \cdots \wedge X^*_{\sigma(n)}\right)$$

Proprety 3.29. *For all forms $\alpha, \beta \in \mathrm{A}^p(M)$*

$$\begin{array}{lll}
- & (\alpha, \beta) & = \int_M \alpha \wedge *\beta \\
- & **\alpha & = (-1)^{p(n-p)}\alpha \\
- & \delta\alpha & = (-1)^{np-n+1} * d * \alpha \\
- & *\Delta\alpha & = \Delta * \alpha \\
- & \alpha\Lambda * \beta & = \beta\Lambda * \alpha \\
- & *1 & = \omega \quad , \quad *\omega = 1 \\
- & \langle *\alpha, *\beta \rangle & = \langle \alpha, \beta \rangle.
\end{array}$$

Theorem 3.30. *We can also define the operator δ from p-forms to $(p-1)$ forms by setting*

$$\delta = (-1)^{n(p+1)+1} * d*$$

where d denotes exterior derivative

Proof. For $\alpha \in A^{p-1}(M), \beta \in A^p(M)$

$$\begin{aligned}
d(\alpha \wedge *\beta) =& d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta \\
=& d\alpha \wedge *\beta + (-1)^{p-1}(-1)^{(p-1)(n-p+1)}\alpha \wedge **(d * \beta)
\end{aligned}$$

by propriety 3.29 ($d * \beta$ is a $(n-p+1)$-form) then

$$\begin{aligned}
d(\alpha \wedge *\beta) =& d\alpha \wedge *\beta - (-1)^{n(p+1)+1}\alpha \wedge ** d * \beta \\
=& \pm * \left(\langle d\alpha, \beta \rangle - (-1)^{n(p+1)+1}\langle \alpha, *d * \beta \rangle\right).
\end{aligned}$$

We integrate this formula. By Stokes' theorem, the integral of the left hand side vanishes,we have

$$\int_M \pm * \left(\langle d\alpha, \beta \rangle - (-1)^{n(p+1)+1}\langle \alpha, *d * \beta \rangle\right) = 0$$

then

$$\int_M \pm * \langle d\alpha, \beta \rangle = \int_M (-1)^{n(p+1)+1} \langle \alpha, *d * \beta \rangle$$

so

$$\pm * \langle d\alpha, \beta \rangle = (-1)^{n(p+1)+1} \langle \alpha, *d * \beta \rangle$$

then

$$\pm * \langle \alpha, \delta\beta \rangle = (-1)^{n(p+1)+1} \langle \alpha, *d * \beta \rangle$$

so

$$\delta = (-1)^{n(p+1)+1} * d*$$

and the claim results. ■

Proposition 3.31. *We may consider $0 \leq p \leq n$ as inner product on*

$$A(M) = \bigoplus_{p=0}^{n} A^p(M)$$

with $A^p(M)$ and $A^q(M)$ being orthogonal for $p \neq q$.

Proposition 3.32. *The operator δ define on $A^p(M)$, just as d is defined on $A^p(M)$,the pair δ and d are adjuncts of each other.*

$$\mathrm{A^p}(M) \underset{d}{\overset{\delta}{\rightleftarrows}} \mathrm{A^{p-1}}(M)$$

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle \qquad \forall \alpha \in \mathrm{A}^{p-1}(M), \beta \in \mathrm{A^p}(M)$$

Proof. Linearity and orthogonality of the $A^p(M)$ provides reduction to consideration of the case in which α is $(p-1)$ form and β is a p-form.

$$\begin{aligned} d(\alpha \wedge *\beta) &= d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d * \beta \\ &= d\alpha \wedge *\beta - \alpha \wedge *\delta\beta \end{aligned}$$

by integrating both side over M and applying Stokes' theorem to left-hand side, we get

$$\begin{aligned} \int_M (d\alpha \wedge *\beta - \alpha \wedge *\delta\beta) &= \int_M (d\alpha \wedge *\beta) - (\alpha \wedge *\delta\beta) \\ &= \int_M (d\alpha \wedge *\beta) - \int_M (\alpha \wedge *\delta\beta) \\ &= \langle d\alpha, \beta \rangle - \langle \alpha, \delta\beta \rangle \\ &= 0 \end{aligned}$$

hence

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle$$

■

Theorem 3.33. *Laplacian commutes with $*$, that is*

$$*\Delta = \Delta*$$

Proposition 3.34. $\Delta\alpha = 0$ *if and only if* $d\alpha = 0$ *and* $\delta\alpha = 0$.

Proof. Clearly $\Delta\alpha = 0$ if $d\alpha = 0$ and $\delta\alpha = 0$. Now,

$$\langle \Delta\alpha, \alpha\rangle = \langle (d\delta + \delta d)\alpha, \alpha\rangle = \langle \delta\alpha, \delta\alpha\rangle + \langle d\alpha, d\alpha\rangle$$

Thus if $\Delta\alpha = 0$, then $d\alpha = 0$ and $\delta\alpha = 0$.

■

Corollary 3.35. *The only harmonics function ($\Delta f = 0$) on a compact, connected, oriented, Riemannian manifold are the constant functions.*

Definition 3.36. *A form $\omega \in A^p(M)$ is called harmonic if $\Delta\omega = 0$.*

Definition 3.37. *Let M be an orientable and compact Riemannian manifold of dimension n. For every p, with $0 \leq p \leq n$, let*

$$\mathbb{H}^p(M) = \{\omega \in \mathcal{A}^p(M) \mid \Delta\omega = 0\},$$

be the space of harmonic p-forms.

Proposition 3.38. *Let M be an orientable and compact Riemannian manifold of dimension n, we have a linear map,*

$$* : \mathbb{H}^p(M) \to \mathbb{H}^{n-p}(M).$$

Theorem 3.39. ***(Hodge Decomposition Theorem)*** *Let M be an orientable and compact Riemannian manifold of dimension n. For every p, with $0 \leq p \leq n$, the space, $\mathbb{H}^p(M)$, is finite dimensional and we have the following orthogonal direct sum decomposition of the space of p-forms:*

$$\begin{aligned}\mathcal{A}^p(M) &= \mathbb{H}^p(M) \oplus \Delta\left(\mathcal{A}^p(M)\right)\\ &= \mathbb{H}^p(M) \oplus d\left(\delta\left(\mathcal{A}^p(M)\right)\right) \oplus \delta\left(d\left(\mathcal{A}^p(M)\right)\right)\\ &= \mathbb{H}^p(M) \oplus d\left(\mathcal{A}^{p-1}(M)\right) \oplus \delta\left(\mathcal{A}^{p+1}(M)\right).\end{aligned}$$

Proposition 3.40. *For every $p \geq 0$, the composition $A^p(M) \xrightarrow{d} A^{p+1}(M) \xrightarrow{d} A^{p+2}(M)$ is identically zero, that is,*

$$d \circ d = 0, \ (d \circ d : A^p(M) \to A^{p+2}(M))$$

or, using superscripts, $d^{p+1} \circ d^p = 0$.

Definition 3.41. *A differential form, ω, is* ***closed*** *iff $d\omega = 0$,* ***exact*** *iff $\omega = d\eta$, for some differential form η . For every $p \geq 0$, let*

$$Z^p(M) = \{\omega \in A^p(M) \mid d\omega = 0\} = \operatorname{Ker} d : A^p(M) \to A^{p+1}(M),$$

be the vector space of closed p-forms, also called ***p-cocycles*** *and for every $p \geq 1$, let*

$$B^p(M) = \{\omega \in A^p(M) \mid \exists\, \eta \in A^{p-1}(M), \omega = d\eta\} = \operatorname{Im} d : A^{p-1}(M) \to A^p(M)$$

be the vector space of exact p-forms, also called ***p-coboundaries****. Set $B^0(M) = (0)$. Forms in $A^p(M)$ are also called* ***p-cochains****. As $B^p(M) \subseteq Z^p(M)$ (by Proposition 3.40), for every $p \geq 0$, we define the p^{th}* ***de Rham cohomology group*** *of M as the quotient space*

$$H^p_{DR}(M) = Z^p(M)/B^p(M).$$

An element of $H^p_{DR}(M)$ is called a ***cohomology class*** *and is denoted $[\omega]$, where $\omega \in Z^p(M)$ is a cocycle.*
The real vector space, $H_{DR}(M) = \oplus_{p\geq 0} H^p_{DR}(M)$, is called the ***de Rham cohomology algebra*** *of M.*

The Hodge Decomposition Theorem has a number of important corollaries, one of which is Hodge Theorem:

Theorem 3.42. ***(Hodge Theorem)*** *Let M be an orientable and compact Riemannian manifold of dimension n. For every p, with $0 \leq p \leq n$, there is an isomorphism between $\mathbb{H}^p(M)$ and the de Rham cohomology vector space, $H^p_{\mathrm{DR}}(M)$:*

$$H^p_{\mathrm{DR}}(M) \cong \mathbb{H}^p(M).$$

Proof. Since by Proposition 3.34, every harmonic form, $\omega \in \mathbb{H}^p(M)$, is closed, we get a linear map from $\mathbb{H}^p(M)$ to $H^p_{\mathrm{DR}}(M)$ by assigning its cohomology class, $[\omega]$, to ω. This map is injective. Indeed if $[\omega] = 0$ for some $\omega \in \mathbb{H}^p(M)$, then $\omega = d\eta$, for some $\eta \in \mathcal{A}^{p-1}(M)$ so

$$(\omega, \omega) = (d\eta, \omega) = (\eta, \delta\omega).$$

But, as $\omega \in \mathbb{H}^p(M)$ we have $\delta\omega = 0$ by Proposition3.34, so $(\omega, \omega) = 0$, that is, $\omega = 0$. Our map is also surjective, this is the hard part of Hodge Theorem. By the Hodge Decomposition Theorem, for every closed form, $\omega \in \mathcal{A}^p(M)$, we can write

$$\omega = \omega_H + d\eta + \delta\theta$$

with $\omega_H \in \mathbb{H}^p(M), \eta \in \mathcal{A}^{p-1}(M)$ and $\theta \in \mathcal{A}^{p+1}(M)$. Since ω is closed and $\omega_H \in \mathbb{H}^p(M)$, we have $d\omega = 0$ and $d\omega_H = 0$, thus

$$d\delta\theta = 0$$

and so

$$0 = (d\delta\theta, \theta) = (\delta\theta, \delta\theta),$$

that is, $\delta\theta = 0$. Therefore, $\omega = \omega_H + d\eta$, which implies $[\omega] = [\omega_H]$, with $\omega_H \in \mathbb{H}^p(M)$, proving the surjectivity of our map.

■

Theorem 3.43. *Let M^n be a compact Riemannian manifold. Then every cohomology class in $\mathbb{H}^p(M)$ $\quad (0 \leq p \leq n)$ contains precisely one harmonic form.*

Proof. Uniqueness : Let $\omega_1, \omega_2 \in A^P(M)$ be cohomologous and both harmonic. Then either $p = 0$ (in which case $\omega_1 = \omega_2$ anyway) or

$$(\omega_1 - \omega_2, \omega_1 - \omega_2) = (\omega_1 - \omega_2, d\eta)$$

for some $\eta \in A^{p-1}(M)$, since ω_1 and ω_2 are cohomologous

$$(\omega_1 - \omega_2, \omega_1 - \omega_2) = (\delta\left(\omega_1 - \omega_2\right), \eta) = \quad 0,$$

since ω_1 and ω_2 are harmonic,
hence satisfy $\delta\omega_1 = 0 = \delta\omega_2$.

Since $(\cdot, \cdot)$ is positive definite, we conclude $\omega_1 = \omega_2$, hence uniqueness.
For the proof of existence, which is much harder, we shall use Dirichlet's principle (Dirichlet's principle states that, if the function $u(x)$ is the solution to $\Delta u + f = 0$). Let ω_0 be a (closed) differential form, representing the given cohomology class in $\mathbb{H}^p(M)$. All forms cohomologous to ω_0 then are of the form

$$\omega = \omega_0 + d\alpha \quad \left(\alpha \in \Omega^{p-1}(M)\right).$$

We now minimize the L^2-norm

$$D(\omega) := (\omega, \omega)$$

in the class of all such forms. The essential step consists in showing that the infimum is achieved by a smooth form η. Such an η then has to satisfy the Euler-Lagrange equations for D, i.e.

$$\begin{aligned} 0 &= \frac{d}{dt}(\eta + td\beta, \eta + td\beta)_{|t=0} \quad \text{for all } \beta \in \Omega^{p-1}(M) \\ &= 2(\eta, d\beta) \end{aligned}$$

This implies $\delta\eta = 0$. Since $d\eta = 0$ anyway, η is harmonic.

■

Conclusion

This thesis has proposed a proof of Hodge theorem by starting from the notion of Riemannian manifold with give some examples and some information of the connection , curvature and the tensor . Afterwards , the Laplacian has been cited with the generalization of the Laplacian on Riemannian manifolds. Eventually, Hodge theorem has been proved and Hodge Decomposition Theorem has been stated as a consequence of Hodge theorem. Hodge theorem is an important bridge that filling the gap between the field of partial differential equations and Algebraic topology. Some problems that are hard in the nature of Partial differential equations can easily approached via Algebraic topology and vice versa, thus a smart use of Hodge can be helpful to open up many new possibilities in both fields

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