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Sur l'existence et la stabilité de solutions du problème de Cauchy dans certains espaces fonctionnels

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Publications

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Abstract

The main objective of this thesis is to study the well-posedness and temporal regularity in Gevrey spaces and anisotropic Gevrey spaces for some partial differential equations. This thesis is divided into two parts:

First one is to study the local and global well-posedness for the Kawahara equation and the m-Korteweg-de Vries system with the initial data in analytical Gevrey spaces. In addition, the Gevrey regularity of the solutions in variable time is provided.

The second part consists in studying the local well-posedness and the time regularity for the Kadomtsev-Petviashvili I equation and the global well-posedness for the Kadomtsev-Petviashvili II equation with initial data in anisotropic Gevrey spaces.

Keywords

Approximate conservation law, Uniform radius of spatial analyticity, Well-posedness , Gevrey spaces, Bourgain spaces, Time regularity.

Résumé

L'objectif principal de cette thèse est d'étudier le problème de Cauchy local et global dans les espaces de Gevrey et les espaces de Gevrey anisotropique pour certaines équations aux dérivées partielles. Cette thèse est divisée en deux parties :

La première consiste á étudier le problème local et global pour l'équation de Kawahara et le système m-Korteweg-de Vries avec des données initiales dans des espaces analytiques de Gevrey . De plus, la régularité des solutions en temps est fournie.

La deuxième partie consiste á étudier le problème local et la régularité temporelle de la solutions de l'équation de Kadomtsev-Petviashvili I et étudier le problème global de l'équation de Kadomtsev-Petviashvili II avec des données initiales dans des espaces de Gevrey anisotropiques.

Mots clés

Loi de conservation , l'analyticité , bien posé, les espaces de Gevrey, les espaces de Bourgain, la régularité temporelle.

List of symbols

We use the following notations throughout this thesis

Acronyms

- ✓ IVP: Initial value problem.
- ✓ $G^{\sigma,\delta,s}$: Analytic Gevrey functions.
- ✓ $G^{\delta,s}$: Analytic function spaces.
- ✓ G^{σ}: Class of Gevrey functions of order σ .
- ✓ G^{δ,s_1,s_2} :Anisotropic Gevrey space.
- ✓ G^{δ_1,δ_2} here $s_1 = s_2 = 0$:Anisotropic Gevrey space.
- ✓ H^s: Sobolev Spaces .
- ✓ H^{s_1,s_2}: Anisotropic Sobolev Spaces .
- ✓ $X_{s,b}$: Bourgain space.
- ✓ $X_{\sigma,\delta,s,b}$: Gevrey Bourgain space.
- ✓ $X_{\delta,s,b}$: Analytic Bourgain space.
- ✓ $X_{s_1,s_2,b}$:Anisotropic Bourgain space.
- ✓ $X^{s_1,s_2}_{\delta,b}$:Anisotropic Gevrey Bourgain space.
- ✓ $C([0,T],G^{\sigma,\delta,s})$: The space of continuous functions from the time interval $[0,T]$ into $G^{\sigma,\delta,s}$.
- ✓ $\mathscr{F}(f), \widehat{f}$:Fourier transform.
- ✓ $\mathscr{F}^{-1}(f)$: Inverse Fourier transform.
- ✓ G^1 : Space of all analytic functions.

✓ $\mathscr{S}'(\mathbb{R}^d)$: The class of tempered distributions.

✓ L^2: Lebesgue spaces.

✓ mKdV: modified Korteweg-de Vries equation.

✓ KP: Kadomtsev-Petviashvili equation.

Notation

✓ $\alpha = (\alpha_1, \dots, \alpha_d)$, $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d$, if $\beta \leqslant \alpha$:

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

✓ Let $\xi = (\xi_1, \dots, \xi_d), x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we denote

$$\begin{array}{ll} \|\xi\| & = |\xi_1| + \cdots + |\xi_d|, \\ |\xi| & = (|\xi_1|^2 + \cdots + |\xi_d|^2)^{\frac{1}{2}}, \\ \langle \xi \rangle & = (1 + |\xi|^2)^{\frac{1}{2}}, \\ (x \cdot \xi) & = x_1 \xi_1 + \cdots + x_n \xi_n. \end{array}$$

✓ Let $f, g \in C^{\|\alpha\|}(\Omega)$, Ω be an open subset of $\mathbb{R}^d$, we may write the Leibniz formula

$$\partial^\alpha (fg) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f \partial^\beta g.$$

✓ The Fourier transform on $\mathbb{R}^d$

$$\widehat{f}(\xi) = \mathscr{F} f(\xi) = (\sqrt{2\pi})^{-d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx, \quad \xi \in \mathbb{R}^d.$$

✓ $\|\cdot\|_{L^p_x}$:

$$\|f\|_{L^p_x} = \left(\int_{\mathbb{R}} |f(x,t)|^p dx \right)^{\frac{1}{p}}.$$

✓ $\|\cdot\|_{L^p_x L^q_t}$:

$$\|f\|_{L^p_x L^q_t} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,t)|^q dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

✓ Plancherel theorem :

$$\forall f \in L^2, \quad \|f\|_{L^2} = \|\hat{f}\|_{L^2}.$$

Contents

Introduction

The Gevrey classes play an important role in the theory of the linear partial differential equations as intermediate spaces between the spaces of the C^∞ and the analytic functions. In particular, whenever the properties of a certain operator differ in the C^∞ and in the analytic framework, it is natural to test its behaviour on the classes of the Gevrey functions .

The basic example, and first source of investigation [26], is the heat operator in $\mathbb{R}^d, d \geqslant 2$

$$L = \frac{\partial}{\partial_{x_d}} - \sum_{j=1}^{d-1} \frac{\partial^2}{\partial^2_{x_j}},$$

whose fundamental solution is given by

$$E(x_1,\dots,x_d) = \begin{cases} (4\pi x_d)^{(1-d)/2} \exp[-(x_1^2+\cdots+x_{d-1}^2)/4x_d] & for \;\; x_d > 0, \\ 0 & for \;\; x_d \leqslant 0. \end{cases}$$

The function E is not analytic for $x_d = 0$, however E is in $C^\infty(\mathbb{R}^d \backslash 0)$, this reflects on the solutions of the homogeneous equation $Lu = 0$ which are not analytic in general, though always C^∞. A precise estimate of the regularity of E can be given by observing that for any fixed compact subset $K \subset \mathbb{R}^d$, $0 \notin K$, we have for all $\alpha = (\alpha_1,\dots,\alpha_d)$

$$|\partial^\alpha E| \leqslant C^{|\alpha|+1}(\alpha!)^2, \quad x \in K, \tag{1}$$

where C depends only on K.

Generalizing (1), one defines $G^\sigma(\Omega)$, Ω open subset of $\mathbb{R}^d$, $1 \leqslant \sigma < \infty$, as the set of all functions $f(x)$ in Ω such that for any $K \subseteq \Omega$.

$$|\partial^\alpha f| \leqslant C^{|\alpha|+1}(\alpha!)^\sigma, \quad x \in K,$$

for a suitable constant C .

The Gevrey classes $G^\sigma, \sigma \geqslant 1$, have numerous applications, a few of the main applications being listed below.

1. Gevrey micro-local analysis see [71].

2. Gevrey solvability see [20, 27].

3. Hyperbolic equations see [11, 13].

4. Divergent series and singular differential equations [60, 70].

5. Dynamical systems see [69, 28].

6. **Evolution partial differential equations** see [21, 22, 55].

Jean Bourgain has been the first to observe the local smoothing effect related to the bilinear estimate and to establish the well-posedness result for low regularity. In [9], he showed global well-posedness for initial data in H^s for $s \geqslant 0$. More precisely, this has been the first well-posedness result in H^s with $s < \frac{3}{2}$ for perdioc KdV. G. Kenig, C. E. Ponce and L. Vega [46] improved this result states local well-posedness in H^s, $s > -\frac{1}{2}$. Well-posedness for the non-periodic gKdV equation in spaces of analytic functions has been proved by Grujić and Kalisch [30]. By using the analytic spaces $G^{\delta,s}$ introduced by Foias and Temam [22] and which are defined by the norm

$$\|f\|^2_{G^{\delta,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta\langle\xi\rangle} \langle\xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty,$$

they showed that for given initial data that are analytic in a symmetric strip $\{z = x+iy : |y| < \delta\}$ in the complex plane of width 2δ there exists a time T such that the corresponding gKdV solution is analytic in the same strip during the time period $[0,T]$. In other words, the uniform radius of spatial analyticity does not shrink as time progresses. Further results on the uniform radius of spatial analyticity have been established by Bona, Grujić and Kalisch [10]. This thesis is structured as follows.

Chapter 1: We introduce the necessary function spaces and Gevrey classes will be used throughout this thesis.

Chapter 2: We study in section 2.4 the local well-posedness for Kawahara equation

$$\begin{cases} \partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u + \mu\partial_x(u^2) = 0, \\ u(x,0) = u_0(x), \end{cases}$$

in analytic Gevrey spaces $G^{\sigma,\delta,s}, s > -\frac{7}{4}, \sigma \geqslant 1$ and $\delta > 0$. We use the local result and a Gevrey approximate conservation law to gradually extend the local solution for all time. The solution to the corresponding Cauchy problem for Kawahara equation belongs to $G^{\sigma,5\sigma}$ i.e, $u(\cdot,t) \in G^\sigma(\mathbb{R})$ in spacial variable and $u(x,\cdot) \in G^{5\sigma}([0,T])$ in time variable .

Chapter 3: In this chapter study the initial value problem associated with a system consisting modified Korteweg-de Vries-type equations

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(uv^2) = 0, \\ \partial_t v + \beta\partial_x^3 v + \partial_x(u^2 v) = 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases}$$

we will prove this initial value problem is locally and globally well-posed. Also, Gevrey regularity of the solution in time variable is provided.

Chapter 4: Is devoted to the local well-posedness for the fifth order Kadomtsev-Petviashvili I equation

$$\begin{cases} \partial_t u + \alpha\partial_x^3 u + \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(x,y,0) = f(x,y). \end{cases}$$

In anisotropic Gevrey spaces $G^{\delta,s_1,s_2}, s_1, s_2 \geqslant 0$, and $\delta > 0$. Furthermore, the solutions belongs to $G^{\sigma}(\mathbb{R})$ in x,y and belongs to $G^{5\sigma}([0,T])$ in time variable .

Chapter 5: The local and global well-posedness of the fifth-order Kadomtsev-Petviashvili II (KPII) equation is discussed in anisotropic Gevrey spaces G^{δ_1,δ_2}.

$$\begin{cases} \partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(x,y,0) = f(x,y). \end{cases}$$

Chapter 1

Preliminary

In this chapter, we introduce the spaces of functions and the Gevrey classes which we shall use in the thesis.

1.1 The Gevrey classes

The Gevrey classes play an important role in various branches of partial and ordinary differential equations. In fact, the name is given in honour of Maurice Gevrey[26]. Since the scale of spaces G^σ starts from the analytic functions (with $\sigma = 1$) and ends in the C^∞- category (with $\sigma = \infty$).

Class of Gevrey functions of order σ

Let $\sigma \geqslant 1$ be a fixed real number. We begin by recalling the definition of $G^\sigma(\Omega)$, class of Gevrey functions of order σ in Ω.

Definition 1.1. The function $f(x)$ is in $G^\sigma(\Omega)$ if $f(x) \in C^\infty(\Omega)$ and for every compact subset K of Ω there exists a positive constant C such that for all α and $x \in K$

$$|\partial^\alpha f(x)| \leqslant C^{\|\alpha\|+1}(\alpha!)^\sigma.$$

It will be sometimes useful to refer to the equivalent estimates

$$|\partial^\alpha f(x)| \leqslant RC^{\|\alpha\|}(\alpha!)^\sigma,$$

where R and C are two positive constants independent of α and $x \in K$.

In particular $G^1(\Omega) = A(\Omega)$ is the space of all analytic functions in Ω. Obviously we have $G^s(\Omega) \subset G^t(\Omega)$ whenever $s \leqslant t$. Take note that both the inclusions

$$A(\Omega) \subset \bigcap_{\sigma>1} G^\sigma(\Omega),$$

and

$$\bigcup_{\sigma\geqslant 1} G^\sigma(\Omega) \subset C^\infty(\Omega),$$

are strict.

1.2 Sobolev and Gevrey spaces

Let us now consider the class of Cauchy problems for nonlinear dispersive PDE on $\mathbb{R}_t \times \mathbb{R}_x^d$, of the form

$$u_t = Lu + N(u), \quad (t \in \mathbb{R}, x \in \mathbb{R}^d) \quad u(0,x) = u_0(x), \tag{1.1}$$

and for which local well-posedness for initial data u_0 in a range of the Sobolev spaces $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$ can be proved using a contraction mapping argument based on estimates for the nonlinear operator $N(\cdot)$ in the Bourgain spaces $X_{s,b}(\mathbb{R} \times \mathbb{R}^d)$ (see subsection 1.3). Here we denote

$$L = ih(D), \;\; D = \frac{\nabla_x}{i},$$

and $h(D)$ is the Fourier multiplier given by

$$h(D)f = \mathscr{F}_x^{-1}[h(\xi)\mathscr{F}_x f(\xi)],$$

where $h(\xi)$ is a given function .

For example, the Kawahara equation is of the form (1.1) with $d = 1$, $h(\xi) = \alpha\xi^5 - \beta\xi^3 + \gamma\xi$ and $N(u) = -\mu\partial_x(u^2)$.

We limit attention to nonlinear operators $N(\cdot)$ containing second order and higher order terms and satisfying the following assumption.
$N(u)$ is a finite linear combination of $k-$linear operators $N_k(u_1,\ldots,u_k)$ for $k \geq 2$, where N_k is of the form

$$\mathscr{F}N_k(u_1,\ldots,u_k)(\xi) = \int_{\xi_1+\cdots+\xi_k=\xi} m_k(\xi_1\ldots\xi_k)\prod_{j=1}^{k}\widehat{u_j}(\xi_j),$$

for a given symbol m_k. Here we use the shorthand

$$\int_{\xi_1+\cdots+\xi_k=\xi} f(\xi_1,\ldots,\xi_k) = \int_{(\mathbb{R}^d)^{k-1}} f\left(\xi_1,\ldots,\xi_{k-1},\xi - \sum_{j=1}^{k-1}\xi_j\right) d\xi_1\ldots d\xi_{k-1}.$$

For example, for the Kawahara equation we have $N(u) = N_2(u,u)$ with $m_2(\xi_1,\xi_2) = -i\mu(\xi_1 + \xi_2)$ and

$$\mathscr{F}N_2(u_1,u_2)(\xi) = -i\mu\xi\int \widehat{u_1}(\xi-\xi_1)\widehat{u_2}(\xi_1)d\xi_1.$$

The Sobolev Spaces H^s

Natural spaces to measure the regularity of the initial data in Cauchy problems are the classical Sobolev spaces $H^s(\mathbb{R}^d), s \in \mathbb{R}$, which are defined as

$$H^s(\mathbb{R}^d) = \left\{ f \in \mathscr{S}'(\mathbb{R}^d); \|f\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^s |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi < \infty \right\},$$

in the line, where $\mathscr{S}'(\mathbb{R}^d)$ denotes the class of tempered distributions. We can write

$$\|f\|_{H^s} = \left\| \langle\xi\rangle^s \widehat{f}(\xi) \right\|_{L^2_\xi}.$$

Sobolev spaces are named after the Russian mathematician Sergei Sobolev. Their importance comes from the fact that weak solutions of some important partial differential equations exist in appropriate Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with the derivatives understood in the classical sense.

Proposition 1.2. *(Proposition 3.1, page 46 in [56]).*

1. *If* $0 \leqslant s < s'$, *then* $H^{s'}(\mathbb{R}^d) \subset H^s(\mathbb{R}^d)$.
2. $H^s(\mathbb{R}^d)$ *is a Hilbert space with respect to the inner product* $\langle \cdot, \cdot \rangle_s$ *defined as follows*

$$\text{If } f, g \in H^s(\mathbb{R}^d), \text{ then } \langle f, g \rangle_s = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \widehat{f}(\xi) \overline{(1 + |\xi|^2)^{\frac{s}{2}} \hat{g}(\xi)} \mathrm{d}\xi \,.$$

3. *For any* $s \in \mathbb{R}$, *the Schwartz space* $\mathscr{S}(\mathbb{R}^d)$ *is dense in* $H^s(\mathbb{R}^d)$.
4. *If* $s_1 \le s \le s_2$, *with* $s = \theta s_1 + (1-\theta)s_2, 0 \le \theta \le 1$, *then*

$$\|f\|_{H^s} \le \|f\|_{H^{s_1}}^{\theta} \|f\|_{H^{s_2}}^{1-\theta}.$$

A really interesting fact is that for positive integer values of s, we can give a description of H^s without using the Fourier transform.

Theorem 1.3. *(Theorem 3.1, page 47 in [56]) If k is a positive integer, then* $H^k(\mathbb{R}^d)$ *coincides with the space of functions* $f \in L^2(\mathbb{R}^d)$ *whose derivatives (in the distribution sense)* $f^{(j)}$ *belongs to* $L^2(\mathbb{R}^d)$ *for every* $j \le k$. *In this case, the norms*

$$\|f\|_{H^k} \quad \text{and} \quad \sum_{j=1}^{k} \left\| f^{(j)} \right\|_{L^2},$$

are equivalent.

Furthermore, the following proposition allows us to relate "weak derivatives" with derivatives in the classical sense.

Theorem 1.4. *(Embedding - Theorem 3.2, page 47 in [56]). If* $s > k + \frac{1}{2}$, *then* $H^s(\mathbb{R}^d)$ *is continuously embedding in* $C^k_\infty(\mathbb{R}^d)$, *the space of functions with k continuous derivatives vanishing at infinity. In other words, if* $f \in H^s(\mathbb{R}^d), s > \frac{1}{2} + k$, *then (after a possible modification of f in a set of measure zero)* $f \in C^k_\infty(\mathbb{R}^d)$ *and*

$$\|f\|_{C^k} \le \|f\|_{H^s}.$$

Proposition 1.5. *(Sobolev Lemma). For* $s > \frac{1}{2}$, *we have*

$$\|u\|_{L^\infty} \le C\|u\|_{H^s},$$

for some positive constant depending only on s.

The Gevrey space $G^{\delta,s}(\mathbb{R}^d)$

A class of analytic functions suitable for our analysis is the analytic class $G^{\delta,s}(\mathbb{R})$ introduced by Foias and Temam [22].

Now consider (1.1) with data u_0 in the Gevrey space $G^{\delta,s}(\mathbb{R}^d)$ defined, for $\delta > 0$ and $s \in \mathbb{R}$, by

$$G^{\delta,s}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{G^{\delta,s}(\mathbb{R}^d)} < \infty \right\},$$

where

$$\|f\|_{G^{\delta,s}(\mathbb{R}^d)} = \left\| e^{\delta\|\xi\|} \langle \xi \rangle \widehat{f}(\xi) \right\|_{L^2(\mathbb{R}^d)}.$$

We record the fact that any $f \in G^{\delta,s}$ has a uniform radius of analyticity δ.

Lemma 1.6. *([79]) Every $f \in G^{\delta,s}(\mathbb{R}^d)$ has a holomorphic extension to the strip*

$$S_\delta = \left\{ x + iy \in \mathbb{C}^d : x, y \in \mathbb{R}^d \text{ and } |y_j| < \delta \text{ for } j = 1, \ldots, d \right\}.$$

Observe that the norm $\|f\|_{G^{\delta,s}}$ is obtained from the standard Sobolev norm

$$\|f\|_{H^s} = \left\| \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L^2_\xi},$$

by the substitution

$$f \to e^{\delta\|D\|} f.$$

Indeed,

$$\|f\|_{G^{\delta,s}} = \left\| e^{\delta\|D\|} f \right\|_{H^s}.$$

1.3 Bourgain spaces

The Bourgain spaces or $X_{s,b}$-space turn out to be an appropriate space to establish a fixed point argument in. In this work, we will mainly use these spaces in order to prove the well-posedness results for the Kawahara, mKdV, KP equations.

For $s, b \in \mathbb{R}$, the Bourgain space $X_{s,b} = X_{s,b}^{\tau=h(\xi)}(\mathbb{R}_t \times \mathbb{R}_x^d)$ associated to the dispersive operator $\partial_t - ih(D)$ is defined to be completion of $\mathscr{S}(\mathbb{R}_t \times \mathbb{R}_x^d)$with respect to the norm

$$\|u\|_{X_{s,b}} = \| \langle \xi \rangle^s \langle \tau - h(\xi) \rangle^b \widehat{u}(t, \xi) \|_{L^2_{\tau,\xi}},$$

where

$$\widehat{u}(\xi, \tau) = (\sqrt{2\pi})^{d+1} \int_{\mathbb{R}\times\mathbb{R}^d} e^{-i(t\tau + x\cdot\xi)} u(x,t)\, dx dt, \quad (\tau \in \mathbb{R}, \xi \in \mathbb{R}^d),$$

is the space-time Fourier transform.

The space $X_{s,b}$ is well-suited for capturing the dispersive smoothing effect of the operator $\partial_t - ih(D)$ away from the characteristic hypersurface $\tau = h(\xi)$ (see section 2.6 of [82]).

By analogy with the relation ship $G^{\delta,s} = e^{-\delta\|D\|}(H^s)$, we define the Gevrey-modified Bourgain space $X_{\delta,s,b}$, for $\delta > 0$ by

$$X_{\delta,s,b} = e^{-\delta\|D\|}(X_{s,b}),$$

with norm

$$\|u\|_{X_{\delta,s,b}} = \left\| e^{\delta\|\xi\|} \langle\xi\rangle^s \langle\tau - h(\xi)\rangle^b \widehat{u}(\xi,\tau) \right\|_{L^2_{\xi,\tau}}.$$

Note that $X_{\delta,s,b}$ is well-defined, since $e^{-\delta\|D\|} = \mathscr{F}^{-1} e^{-\delta\|\cdot\|} \mathscr{F}$ who maps $X_{s,b}$ in to itself, for $\delta \geqslant 0$.

The restriction of $X_{s,b}$ to a time-slab $(0,T) \times \mathbb{R}^d$ is denoted $X^T_{s,b}$. This is a Banach space when equipped with the norm

$$\|u\|_{X^T_{s,b}} = \inf\left\{ \|v\|_{X_{s,b}} : v \in X_{s,b} \text{ and } u = v \text{ on } (0,T) \times \mathbb{R}^d \right\}.$$

The restriction $X^T_{\delta,s,b}$ is similarly defined, and then we clearly have

$$X^T_{\delta,s,b} = e^{-\delta\|D\|}(X^T_{s,b}),$$

hence the well-known properties of $X_{s,b}$ and its restrictions carry over to $X_{\delta,s,b}$ simply by the substitution $u \longrightarrow e^{\delta\|D\|}u$.

Chapter 2

Kawahara equation [1]

In this chapter, we study a Cauchy problem for the Kawahara equation with data in analytic Gevrey spaces

$$\begin{cases} \partial_t u + \alpha \partial_x^5 u + \beta \partial_x^3 u + \gamma \partial_x u + \mu \partial_x (u^2) = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

For $x \in \mathbb{R}$ and $t \geq 0$ the parameters $\alpha \neq 0$, β, γ and μ are real numbers.

The model $(2.1)_1$ is also called the fifth order shallow water equation and is also called the modified Kawahara equation for the nonlinear term $\partial_x(u^3)$. It arises in study of the water waves with surface tension, in which the Bond number takes on the critical value, where the Bond number represents a dimensionless magnitude of surface tension in the shallow water regime, cf. [50, 49].

First, by using linear and bilinear estimates in analytic Gevrey Bourgain spaces $X_{\sigma,\delta,s,b}(\mathbb{R}^2)$ and analytic Gevrey spaces $G^{\sigma,\delta,s}(\mathbb{R})$, the local well-posedness of the Cauchy problem for the Kawahara equation on the line is established for analytic initial data $u_0(x)$ that can be extended as holomorphic functions in a strip around the x-axis. Next we use this local result and a Gevrey approximate conservation law to prove that global solutions exist. Furthermore, we obtain explicit lower bounds for the radius of spatial analyticity $r(t)$ given by $r(t) \geqslant ct^{-1}$. Also, Gevrey regularity of the solution in time variable is provided.

2.1 Function spaces

In this section we will present the elementary spaces and lemmas used in this chapter.

Analytic Gevrey spaces

A class of analytic functions suitable for our analysis is the analytic class $G^{\delta,s}(\mathbb{R})$, which may be defined as

$$G^{\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|^2_{G^{\delta,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta|\xi|}(1+|\xi|)^{2s}|\widehat{f}(\xi)|^2 d\xi < \infty \right\}, \tag{2.2}$$

[1] Aissa Boukarou, Kaddour Guerbati and Khaled Zennir , Local wellposedness and time regularity for a fifth-order shallow water equations in analytic Gevrey-Bourgain spaces. Monatsh Math 193, 763-782 (2020). https://doi.org/10.1007/s00605-020-01464-x

for $\delta \geqslant 0$ and $s \in \mathbb{R}$. In the particular case where $\delta = 0$, the space $G^{0,s}(\mathbb{R})$ is reduced to the Sobolev space $H^s(\mathbb{R})$. Our first step is to recall the embedding property of Gevrey spaces , for all $0 < \delta' < \delta$ and $s, s' \in \mathbb{R}$ we have

$$G^{\delta,s}(\mathbb{R}) \subset G^{\delta',s'}(\mathbb{R}), \tag{2.3}$$

i.e.$\|f\|_{G^{\delta',s'}} \leqslant C_{s,s',\delta,\delta'} \|f\|_{G^{\delta,s}}$.
We define also the spaces of analytic Gevrey functions $G^{\sigma,\delta,s}(\mathbb{R})$ for $\sigma \geq 1$ by

$$G^{\sigma,\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta|\xi|^{1/\sigma}} (1+|\xi|)^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty \right\}. \tag{2.4}$$

If $u_0 \in G^{\sigma,\delta,s}(\mathbb{R})$, then u_0 belongs to the Gevrey class $G^{\sigma}(\mathbb{R})$. In the case when $\sigma = 1$, we denote $G^{1,\delta,s}(\mathbb{R}) = G^{\delta,s}(\mathbb{R})$. Thus, if $u_0 \in G^{\delta,s}(\mathbb{R})$ then u_0 is analytic on the line and admits a holomorphic extension $\widetilde{u_0}$ on the strip $S_\delta = \{x - iy \in \mathbb{C}; |y| < \delta\}$. Hence, in this context, we refer to the parameter $\delta > 0$ as the uniform radius of analyticity of the function u_0.

Analytic Gevrey Bourgain spaces

As in Grujić and Kalisch [30] we consider a space that is a hybrid between the analytic space and a space of the analytic Bourgain. More precisely, for $s, b \in \mathbb{R}$ and $\delta > 0$ define $X_{\delta,s,b}(\mathbb{R}^2)$ to be the Banach space equipped with the norm

$$\|u\|^2_{X_{\delta,s,b}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{2\delta|\xi|} (1+|\xi|)^{2s} (1+|\tau+\phi(\xi)|)^{2b} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{2.5}$$

where $\phi(\xi) = \alpha\xi^5 - \beta\xi^3 + \gamma\xi$. For $\delta = 0$, $X_{\delta,s,b}(\mathbb{R}^2)$ coincides with the space $X_{s,b}(\mathbb{R}^2)$ introduced by Bourgain [8], and Kenig, Ponce and Vega [48]. Also, for $T > 0$, $X^T_{\delta,s,b}(\mathbb{R}^2)$ denotes the restricted analytic Bourgain space defined by

$$\|u\|_{X^T_{\delta,s,b}(\mathbb{R}^2)} = \inf \left\{ \|v\|_{X_{\delta,s,b}(\mathbb{R}^2)} : v = u \ on \ (0,T) \times \mathbb{R} \right\}. \tag{2.6}$$

We consider also a space that is a hybrid between the analytic Gevrey space and a space of the analytic Gevrey Bourgain. For $\sigma \geq 1$ define $X_{\sigma,\delta,s,b}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}} (1+|\xi|)^{2s} (1+|\tau+\phi(\xi)|)^{2b} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau, \tag{2.7}$$

The next lemma shows that $X_{\sigma,\delta,s,b}(\mathbb{R}^2)$ is continuously embedded in $C\left([0,T], G^{\sigma,\delta,s}(\mathbb{R})\right)$, provided $b > 1/2$. For $\delta = 0$ the proof can be found, for instance, in [82], Section 2.6.

Lemma 2.1. *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$, $\sigma \geqslant 1$ and $\delta > 0$, Then, for all $u \in X_{\sigma,\delta,s,b}(\mathbb{R}^2)$ and some constant $C_0 > 0$ we have*

$$|u|_{T,\sigma,\delta,s} = \sup_{t \in [0,T]} \|u(\cdot,t)\|_{G^{\sigma,\delta,s}(\mathbb{R})} \leqslant C_0 \|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}.$$

Proof. First, we observe that the operator $A^{\sigma,\delta}$ defined by

$$\widehat{A^{\sigma,\delta}u}^{x}(\xi,t) = e^{\delta|\xi|^{1/\sigma}}\widehat{u}^{x}(\xi,t), \tag{2.8}$$

satisfies

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \|A^{\sigma,\delta}u\|_{X_{s,b}(\mathbb{R}^2)} \quad \textit{and} \quad \|u\|_{G^{\sigma,\delta,s}(\mathbb{R})} = \|A^{\sigma,\delta}u\|_{H^s(\mathbb{R})}, \tag{2.9}$$

where $X_{s,b}(\mathbb{R}^2)$ is introduced in [44]. We observe that $A^{\sigma,\delta}u$ belongs to $C([0,T],H^s(\mathbb{R}))$ and for some $C_0>0$, we have

$$\|A^{\sigma,\delta}u\|_{C([0,T],H^s(\mathbb{R}))} \leq C_0\|A^{\sigma,\delta}u\|_{X_{s,b}(\mathbb{R}^2)}. \tag{2.10}$$

Thus, it follows that $u \in C\left([0,T],G^{\sigma,\delta,s}\right)$ and

$$\|u\|_{C\left([0,T],G^{\sigma,\delta,s}(\mathbb{R})\right)} \leq C_0\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}. \tag{2.11}$$

□

Remark 2.2. This will be of importance due to the fact that we will show that, given an initial data $u_0(x) \in G^{\sigma,\delta,s}(\mathbb{R})$ there is a unique solution $u \in X_{\sigma,\delta,s,b}(\mathbb{R}^2)$to the Cauchy problem (2.1), for a certain $b > \dfrac{1}{2}$ and therefore there is a solution to the Cauchy problem (2.1), $u \in C\left([0,T],G^{\sigma,\delta,s}(\mathbb{R})\right)$.

2.2 Linear estimates

Now we consider the linear Cauchy problem

$$\begin{cases} \partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u = F(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{2.12}$$

by using Duhamel's formula (Taking Fourier transform with respect to x in (2.12), solving the resulting differential equation in t and using inverse Fourier transform reduces the Cauchy problem (2.12) to the following integral equation) we may write the solution

$$u(x,t) = S(t)u_0(x) - \int_0^t S(t-t')F(x,t')dt',$$

where the unit operator related to the corresponding linear equation is

$$S(t) = \mathscr{F}_x^{-1}e^{-it(\alpha\xi^5-\beta\xi^3+\gamma\xi)}\mathscr{F}_x,$$

here, the nonlinear terms F is given by $\mu\partial_x(u^2)$.

Next, we localize it in time variable by using a cut-off function $\psi \in C_0^\infty(\mathbb{R})$, with $\psi = 1$ in $[-\frac{1}{2},\frac{1}{2}]$, supp$\psi \subset [-1,1]$ and define $\psi_T(t) = \psi(\frac{t}{T})$. We consider the operator Φu given by

$$\Phi(u)(x,t) = \psi_1(t)S(t)u_0(x) - \psi_1(t)\int_0^t S(t-t')F(x,t')dt'. \tag{2.13}$$

Our goal is to solve the equation $\Phi(u) = u$. We estimate now the fist part in the right hand side of (2.13).

Lemma 2.3. *(See [44].) Let $s \in \mathbb{R}$ and $\frac{1}{2} < b < 1$. Then*

$$\|\psi_1(t)S(t)u_0\|_{X_{s,b}(\mathbb{R}^2)} \leq C\,\|u_0\|_{H^s(\mathbb{R})}. \tag{2.14}$$

Lemma 2.4. *Let $s \in \mathbb{R}$, $\frac{1}{2} < b < 1$, $\delta > 0$ and $\sigma \geqslant 1$. Then*

$$\|\psi_1(t)S(t)u_0\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C\,\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}. \tag{2.15}$$

Proof. We observe, by considering the operator $A^{\sigma,\delta}$ in (2.8), that

$$\begin{aligned}
\|\ \psi_1(t)S(t)u_0\ \|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &= C\int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}}(1+|\ \xi\ |)^{2s}(1+|\ \tau+\phi(\xi)\ |)^{2b} \\
&\cdot|\ \widehat{\psi}(\tau+\phi(\xi))\ |^2|\ \widehat{u_0}(\xi)\ |^2\ d\xi d\tau \\
&= C\int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}}(1+|\ \xi\ |)^{2s}(1+|\ \tau+\phi(\xi)\ |)^{2b} \\
&\cdot|\ \widehat{\psi}(\tau+\phi(\xi))\ |^2|\ e^{\delta|\xi|^{1/\sigma}}\widehat{u_0}(\xi)\ |^2\ d\xi d\tau \\
&= C\int_{\mathbb{R}^2}(1+|\ \xi\ |)^{2s}(1+|\ \tau+\phi(\xi)\ |)^{2b} \\
&\cdot|\ \widehat{\psi}(\tau+\phi(\xi))\ |^2|\ \widehat{A^{\sigma,\delta}u_0}(\xi)\ |^2\ d\xi d\tau \\
&=\|\ \psi_1(t)S(t)(A^{\sigma,\delta}u_0)\ \|^2_{X_{s,b}(\mathbb{R}^2)}\ .
\end{aligned}$$

Now, by using Lemma 2.3 , there exists $C > 0$ such that

$$\|\psi_1(t)S(t)(A^{\sigma,\delta}u_0)\|_{X_{s,b}(\mathbb{R}^2)} \leq C\,\|A^{\sigma,\delta}u_0\|_{H^s(\mathbb{R})} = C\,\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}.$$

□

We will estimate the integral part of $\Phi(u)$.

Lemma 2.5. *(See [44].) Let $s \in \mathbb{R}$, $\frac{1}{2} < b < 1$, $0 < T \leq 1$. Then*

$$\left\|\psi_1(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'\right\|_{X_{s,b}(\mathbb{R}^2)} \leq C\|F\|_{X_{s,b-1}(\mathbb{R}^2)}. \tag{2.16}$$

Lemma 2.6. *Let $s \in \mathbb{R}$, $\frac{1}{2} < b < 1$, $\delta > 0$ and $\sigma \geqslant 1$, then for some constant $C > 0$, we have*

$$\left\|\psi_1(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'\right\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C\|F\|_{X_{\sigma,\delta,s,b-1}(\mathbb{R}^2)}. \tag{2.17}$$

Proof. Define $U = \psi_1(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'$. Let us consider the operator $A^{\sigma,\delta}$ given by (2.8), then we have

$$\begin{aligned}
\widehat{A^{\sigma,\delta}U}^x(\xi,t) &= \psi_1(t)\int_0^t \left(e^{-i(t-t')\phi(\xi)}\right)e^{\delta|\xi|^{1/\sigma}}\widehat{F}^x(\xi,t')\mathrm{d}t' \\
&= \psi_1(t)\int_0^t \widehat{[S(t-t')(A^{\sigma,\delta}F)]}^x(\xi,t')\mathrm{d}t'.
\end{aligned}$$

Thus,

$$\|U\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \|A^{\sigma,\delta}U\|_{X_{s,b}(\mathbb{R}^2)} = \left\| \psi_1(t)\int_0^t S(t-t')A^{\sigma,\delta}F(x,t')dt' \right\|_{X_{s,b}(\mathbb{R}^2)}.$$

Using Lemma 2.5, we have

$$\left\| \psi_1(t)\int_0^t S(t-t')A^{\sigma,\delta}F(x,t')dt' \right\|_{X_{s,b}(\mathbb{R}^2)} \leq C\|A^{\sigma,\delta}F\|_{X_{s,b-1}(\mathbb{R}^2)} = C\|F\|_{X_{\sigma,\delta,s,b-1}(\mathbb{R}^2)}.$$

□

Lemma 2.7. *Let $s \in \mathbb{R}$, $\sigma \geqslant 1$ and $\delta \geq 0$. If $\frac{1}{2} < b_1 \leq b_1' < 1$, then for any $T > 0$, we have,*

$$\|\psi_T(t)F\|_{X_{\sigma,\delta,s,b-1}(\mathbb{R}^2)} \leq CT^{b_1'-b_1}\|F\|_{X_{\sigma,\delta,s,b'-1}(\mathbb{R}^2)}, \tag{2.18}$$

where C depends only on b and b'.

Proof. The proof of the Lemma 2.7 for $\delta = 0$ can be found in Lemma 2.11 of [82], for $\delta > 0$ as one merely has to replace F by $A^{\sigma,\delta}F$, where the operator define in (2.8).

□

2.3 Bilinear estimate

The next result provides the essential bilinear estimate needed for the proof of Theorem 2.12 and Theorem 2.18.

Corollary 2.8. *(See [44].) If $s > -\frac{7}{4}$, let $b_1 > \frac{1}{2}$ be close enough to $\frac{1}{2}$ and $b_1' > \frac{1}{2}$. Then*

$$\|\partial_x(u_1u_2)\|_{X_{s,b_1-1}(\mathbb{R}^2)} \leqslant C\|u_1\|_{X_{s,b_1'}(\mathbb{R}^2)}\|u_2\|_{X_{s,b_1'}(\mathbb{R}^2)}. \tag{2.19}$$

Remark 2.9. Setting

$$f_i(\xi,\tau) = (1+|\xi|)^s(1+|\tau+\phi(\xi)|)^{b_1'}\widehat{u_i}(\xi,\tau), \ \ i = 1,2,$$

the estimate of Corollary 2.8 can be rewritten as

$$
\begin{aligned}
&\|\partial_x(u_1u_2)\|_{X_{s,b_1-1}(\mathbb{R}^2)}\\
&=\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau+\phi(\xi)|)^{b_1-1}}\widehat{u_1u_2}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=C\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau+\phi(\xi)|)^{b_1-1}}\int_{\mathbb{R}^2}\widehat{u_1}(\xi_1,\tau_1)\widehat{u_2}(\xi-\xi_1,\tau-\tau_1)d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=C\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau+\phi(\xi)|)^{b_1-1}}\int_{\mathbb{R}^2}\frac{\widehat{f_1}(\xi_1,\tau_1)}{(1+|\xi|)^s(1+|\tau_1+\phi(\xi_1)|)^{b_1'}}\frac{\widehat{f_2}(\xi-\xi_1,\tau-\tau_1)}{(1+|\xi-\xi_1|)^s(1+|\tau-\tau_1+\phi(\xi-\xi_1)|)^{b_1'}}\right.\\
&\left.\cdot\widehat{u_2}(\xi-\xi_1,\tau-\tau_1)d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&\leqslant C\|f_1\|_{L^2_{\xi,\tau}}\|f_2\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.
\end{aligned}
\tag{2.20}
$$

Lemma 2.10. *If $s>-\frac{7}{4}$, let $\sigma\geqslant 1$, $\delta>0$ and $b_1>\frac{1}{2}$ be close enough to $\frac{1}{2}$ and $b_1'>\frac{1}{2}$. Then*

$$\|\partial_x(u_1u_2)\|_{X_{\sigma,\delta,s,b_1-1}(\mathbb{R}^2)}\leqslant C\|u_1\|_{X_{\sigma,\delta,s,b_1'}(\mathbb{R}^2)}\|u_2\|_{X_{\sigma,\delta,s,b_1'}(\mathbb{R}^2)}. \tag{2.21}$$

Proof. We observe, by considering the operator $A^{\sigma,\delta}$ in (2.8), that

$$\begin{aligned}
&\|\partial_x(u_1u_2)\|_{X_{\sigma,\delta,s,b_1-1}(\mathbb{R}^2)}\\
&=\left\|e^{\delta|\xi|^{1/\sigma}}(1+|\xi|)^s(1+|\tau+\phi(\xi)|)^{b_1-1}\widehat{\partial_x(u_1u_2)}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=\left\|(1+|\xi|)^s(1+|\tau+\phi(\xi)|)^{b_1-1}\xi e^{\delta|\xi|^{1/\sigma}}\widehat{u_1u_2}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=\left\|(1+|\xi|)^s(1+|\tau+\phi(\xi)|)^{b_1-1}\xi(\sqrt{2\pi})^{-2}e^{\delta|\xi|^{1/\sigma}}\widehat{u_1}*\widehat{u_2}\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&\leqslant\left\|(1+|\xi|)^s(1+|\tau+\phi(\xi)|)^{b_1-1}\xi(\sqrt{2\pi})^{-2}\int_{\mathbb{R}^2}e^{\delta|\xi-\xi_1|^{1/\sigma}}\widehat{u_1}(\xi-\xi_1,\tau-\tau_1)e^{\delta|\xi_1|^{1/\sigma}}\widehat{u_2}(\xi_1,\tau_1)d\xi_1d\tau_1\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=\left\|(1+|\xi|)^s(1+|\tau+\phi(\xi)|)^{b_1-1}(\widehat{\partial_xA^{\sigma,\delta}u_1A^{\sigma,\delta}}u_2)(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=\left\|\partial_x(A^{\sigma,\delta}u_1A^{\sigma,\delta}u_2)\right\|_{X_{s,b_1-1}(\mathbb{R}^2)},
\end{aligned}$$

where $\delta|\xi|^{1/\sigma}\leqslant\delta|\xi-\xi_1|^{1/\sigma}+\delta|\xi_1|^{1/\sigma}, \quad \forall\sigma\geqslant 1.$
Now, by using Corollary 2.8 , there exists $C>0$ such that

$$\begin{aligned}
\|\partial_x(A^{\sigma,\delta}u_1A^{\sigma,\delta}u_2)\|_{Xs,b_1-1(\mathbb{R}^2)} &\leqslant C\|A^{\sigma,\delta}u_1\|_{X_{s,b_1'}(\mathbb{R}^2)}\parallel A^{\sigma,\delta}u_2\parallel_{X_{s,b_1'}(\mathbb{R}^2)}\\
&=C\parallel u_1\parallel_{X_{\sigma,\delta,s,b_1'}(\mathbb{R}^2)}\parallel u_2\parallel_{X_{\sigma,\delta,s,b_1'}(\mathbb{R}^2)}.
\end{aligned}$$

□

Finally, for $\delta>0$ and $\sigma=1$ we will need the restrictions of $X_{1,\delta,s,b}(\mathbb{R}^2)=X_{\delta,s,b}(\mathbb{R}^2)$ to a time slab $(0,T)\times\mathbb{R}$. This space is denoted by $X^T_{1,\delta,s,b}(\mathbb{R}^2)=X^T_{\delta,s,b}(\mathbb{R}^2)$, and is a Banach space when equipped with the norm

$$\|u\|_{X^T_{\delta,s,b}(\mathbb{R}^2)}=\inf\{\|v\|_{X_{\delta,s,b}(\mathbb{R}^2)}:v=u \text{ on } [0,T]\times\mathbb{R}\}.$$

Lemma 2.11. *[75] Let $s\in\mathbb{R}$, $\delta\geq 0$, $-\frac{1}{2}<b<\frac{1}{2}$ and $T>0$. Then, for any time interval $I\subset[0,T]$, we have*

$$\|\chi_I(t)u\|_{X_{\delta,s,b}(\mathbb{R}^2)}\leq C\|u\|_{X^T_{\delta,s,b}(\mathbb{R}^2)}, \tag{2.22}$$

where $\chi_I(t)$ is the characteristic function of I, and C depends only on b.

2.4 Local well-posedness

The previous sections provide all necessary statements to conclude the contraction argument and to show local well-posedness. Y.Jia and Z.Huo [44] proved that the Cauchy problem of Kawahara equation is locally well-posed for data in $H^s, s > -\frac{7}{4}$. We improved this result, states local well-posedness in $G^{\sigma,\delta,s}, s > -\frac{7}{4}, \sigma \geqslant 1$ and $\delta > 0$.

Theorem 2.12. ***(Local well-posedness in*** $G^{\sigma,\delta,s}(\mathbb{R})$***)*** *Let* $s > -\frac{7}{4}, \sigma \geqslant 1, \delta > 0$ *and* $u_0 \in G^{\sigma,\delta,s}(\mathbb{R})$. *Then there exist a real number* $b > \frac{1}{2}$, *which is near enough to* $\frac{1}{2}$ *and a positive time* T *depending only on* σ, δ, s *and* u_0, *such that the Cauchy problem (2.1) is locally well-posed in* $C\left([0,T], G^{\sigma,\delta,s}(\mathbb{R})\right)$.

From Theorems (2.12), we have the following Corollary (2.13).

Corollary 2.13. ***(Local well-posedness in*** $G^{\delta,s}(\mathbb{R})$***))*** *Let* $s > -\frac{7}{4}, \delta > 0$ *and* $u_0 \in G^{\delta,s}$. *Then there exist a real number* $b > \frac{1}{2}$, *which is near enough to* $\frac{1}{2}$ *and a positive time* T *depending only on* δ, s *and* u_0, *such that the Cauchy problem (2.1) is locally well-posed in* $C\left([0,T], G^{\delta,s}\right)$. *Furthermore, the solution* u *satisfies the bound*

$$\|u(t)\|_{G^{\delta,s}} \leq c\|u\|_{X_{\delta,s,b}} \leq 2C\|u_0\|_{G^{\delta,s}},\ t \in [0,T], \tag{2.23}$$

and

$$T = \frac{c_0}{(1+\|u_0\|_{G^{\delta,s}(\mathbb{R})})^a}, \tag{2.24}$$

for certain constants $c_0 = (8C^2 2^{b'-b}|\mu|)^{-a}, c_0, C > 0$ *and* $a > 1$.

2.4.1 Existence of solution

For the proof of local well-posedness in the analytic Gevrey spaces, we will use the standard Banach contraction principle for functions $X_{\sigma,\delta,s,b}(\mathbb{R}^2)$ in a given closed ball $\mathbb{B}$. we define again the integral operator

$$\Phi(u)(x,t) = \psi_1(t)S(t)u_0(x) - \psi_1(t)\int_0^t S(t-t')\mu\partial_x u^2(x,t')dt'. \tag{2.25}$$

If that is useful for the nonlinear estimates, as will be, one can also introduce additional cut-off in $\partial_x u^2$ and consider the equation

$$\Phi(u)(x,t) = \psi_1(t)S(t)u_0(x) - \psi_1(t)\int_0^t S(t-t')\psi_{2T}(t')\mu\partial_x u^2(x,t')dt', \tag{2.26}$$

which is actually identical with (2.25) since $\psi_{2T} = 1$ on support of ψ_T.

Lemma 2.14. *Let* $s > -\frac{7}{4}$, $\sigma \geq 1$, $\delta > 0$ *and* $b > \frac{1}{2}$. *Then, for all* $u_0 \in G^{\sigma,\delta,s}(\mathbb{R})$ *and* $0 < T < 1$, *with some constant* $C > 0$, *we have*

$$\|\Phi(u)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R})} + C|\mu|(2T)^{b'-b}\|u\|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}, \tag{2.27}$$

and

$$\|\Phi(u)-\Phi(v)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C|\mu|(2T)^{b'-b}\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\|u+v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}, \tag{2.28}$$

for all $u, v \in X_{\sigma,\delta,s,b}(\mathbb{R}^2)$.

Proof. To prove estimate (2.27), we follow

$$\begin{aligned}
\|\Phi(u)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &\leq \|\psi_1(t)S(t)u_0\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&+\left\|\psi_1(t)\int_0^t S(t-t')\psi_{2T}(t')\mu\partial_x u^2 dt'\right\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&\leq C\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R}^2)} + C|\mu|\left\|\psi_{2T}(t)\partial_x u^2\right\|_{X_{\sigma,\delta,s,b-1}(\mathbb{R}^2)} \\
&\leq C\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R}^2)} + C|\mu|(2T)^{b'-b}\left\|\partial_x u^2\right\|_{X_{\sigma,\delta,s,b'-1}(\mathbb{R}^2)} \\
&\leq C\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R}^2)} + C|\mu|(2T)^{b'-b}\|u\|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}.
\end{aligned}$$

Here $b_1 = b'$ and $b_1' = b$, for the estimate (2.28), we observe that

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &= \left\|\psi_1(t)\int_0^t S(t-t')\psi_{2T}(t')\mu\left(\partial_x u^2 - \partial_x v^2\right)(x,t')dt'\right\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&\leqslant C|\mu|\left\|\psi_{2T}(t')\left(\partial_x u^2 - \partial_x v^2\right)\right\|_{X_{\sigma,\delta,s,b-1}(\mathbb{R}^2)} \\
&\leqslant C|\mu|(2T)^{b'-b}\left\|\left(\partial_x u^2 - \partial_x v^2\right)\right\|_{X_{\sigma,\delta,s,b'-1}(\mathbb{R}^2)} \\
&\leqslant C|\mu|(2T)^{b'-b}\|u+v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}.
\end{aligned}$$

Thus, from the previous results, we obtain (2.28).

□

We will show that the map Φ is a contraction on the ball $\mathbb{B}(0,r)$ to $\mathbb{B}(0,r)$.

Proposition 2.15. *Let $s > -\frac{7}{4}$, $\sigma \geq 1$, $\delta > 0$ and $b > \frac{1}{2}$. Then, for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{R})$, such that*

$$T = \frac{c_0}{(1+\|u_0\|_{G^{\delta,s}(\mathbb{R})})^a}, \tag{2.29}$$

the map $\Phi : \mathbb{B}(0,r) \to \mathbb{B}(0,r)$ is a contraction, where $\mathbb{B}(0,r)$ is given by

$$\mathbb{B}(0,r) = \{u \in X_{\sigma,\delta,s,b}(\mathbb{R}^2); \|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq r\} \text{ with } r = 2C\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}.$$

Proof. From Lemma 2.14, for all $u \in \mathbb{B}(0,r)$, we have

$$\begin{aligned}
\|\Phi(u)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &\leq C\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R})} + C|\mu|(2T)^{b'-b}\|u\|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&\leq \frac{r}{2} + C|\mu|(2T)^{b'-b}r^2.
\end{aligned}$$

If we define $a = \frac{1}{b'-b}$ and $c_0 = (8C^2 2^{b'-b}|\mu|)^{-a}$ then for T given as in (2.29), we have that

$$T \leq \frac{1}{(4C|\mu|2^{b'-b}r)^{\frac{1}{b'-b}}},$$

and hence,

$$\| \Phi(u) \|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq r, \quad \forall u \in \mathbb{B}(0,r).$$

Thus, Φ maps $\mathbb{B}(0,r)$ into $\mathbb{B}(0,r)$, which is a contraction, since

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &\leq C|\mu|(2T)^{b'-b}\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\|u+v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}, \\
&\leq C|\mu|(2T)^{b'-b}\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\left(\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}+\|v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\right), \\
&\leq C|\mu|(2T)^{b'-b}2r\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&\leq \frac{1}{2} \| u-v \|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}, \quad \forall u,v \in \mathbb{B}(0,r).
\end{aligned}$$

The proof is now complete.

□

2.4.2 The uniqueness

Uniqueness of the solution in $C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))$ can be proved by the following standard argument.

Lemma 2.16. *Suppose u and v are solutions to (2.1) in $C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))$ with $u(\cdot,0) = v(\cdot,0)$ in $G^{\sigma,\delta,s}(\mathbb{R})$, where $\sigma \geqslant 1, \delta > 0$ and $s > -\frac{7}{4}$. Then $u = v$.*

Proof.

Setting $w = u - v$, we see that w solves the Cauchy problem

$$\partial_t w + \alpha\partial_x^5 w + \beta\partial_x^3 w + \gamma\partial_x w + \mu\partial_x w(u+v) = 0, \quad w(0) = 0.$$

Multiplying both sides by w and integrating in space yield

$$w\partial_t w + \alpha w\partial_x^5 w + \beta w\partial_x^3 w + \gamma w\partial_x + \mu w\partial_x w(u+v) = 0,$$

Thus, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t,\cdot)\|_{L^2(\mathbb{R})}^2 &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} w^2(t,x)dx = \int_{\mathbb{R}} w(t,x)\partial_t w(t,x)dx \\
&= -\mu\int_{\mathbb{R}} w(t,x)\partial_x(u^2-v^2)dx,
\end{aligned} \tag{2.30}$$

since we have

$$\int_{\mathbb{R}} w(t,x)\partial_x^5 w(t,x)dx = \int_{\mathbb{R}} w(t,x)\partial_x^3 w(t,x)dx = \int_{\mathbb{R}} w(t,x)\partial_x w(t,x)dx = 0.$$

Thanks to Equation (2.30) we have

$$\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})} = -2\mu\int_{\mathbb{R}} w(t,x)\partial_x(u^2-v^2)dx = -2\mu\int_{\mathbb{R}} w(t,x)\partial_x\left[f(t,x)w(t,x)\right]dx,$$

where $f = u+v$. Integrating by parts the last integral we obtain

$$\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})} = -\mu\int_{\mathbb{R}} \partial_x f(t,x)w^2(t,x)dx,$$

from which we deduce the inequality

$$\left|\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}\right| \leqslant |\mu|\|\partial_x f\|_{L^\infty([0,T]\times\mathbb{R})}\|w(t)\|^2_{L^2(\mathbb{R})}. \tag{2.31}$$

Since $u, v \in C([0,T], G^{\sigma,\delta,s}(\mathbb{R}))$ we have that u and v are continuous in t on the compact set $[0,T]$ and are $G^{\sigma,\delta,s}(\mathbb{R})$ in x. Thus, we can conclude that

$$|\mu|\|\partial_x f\|_{L^\infty([0,T]\times\mathbb{R})} \le c < \infty. \tag{2.32}$$

Therefore, from (2.31) and (2.32) we obtain the differential inequality

$$\left|\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}\right| \le c\|w(t)\|^2_{L^2(\mathbb{R})}, \quad 0 \le t \le T.$$

Solving it gives

$$\|w(t)\|^2_{L^2(\mathbb{R})} \le e^c\|w(0)\|^2_{L^2(\mathbb{R})}, \quad 0 \le t \le T. \tag{2.33}$$

Since $\|w(0)\|^2_{L^2(\mathbb{R})} = 0$, from (2.33) we obtain that $w(t) = 0$, $0 \le t \le T$ or $u = v$.

□

2.4.3 Continuous dependence of the initial data

To prove continuous dependence of the initial data we will prove the following.

Lemma 2.17. *Let $s > -\frac{7}{4}$, $\sigma \ge 1$, $\delta > 0$ and $b > \frac{1}{2}$. Then, for all $u_0, v_0 \in G^{\sigma,\delta,s}(\mathbb{R})$, if u and v are two solutions to (2.1) corresponding to initial data u_0 and v_0. We have*

$$|u-v|_{T,\sigma,\delta,s} \le 2C_0C\|u_0-v_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}. \tag{2.34}$$

Proof. If u and v are two solutions to (2.1), corresponding to initial data u_0 and v_0, we have from Lemma 2.1

$$|u-v|_{T,\sigma,\delta,s} = \sup_{t\in[0,T]}\|u(\cdot,t)-v(\cdot,t)\|_{G^{\sigma,\delta,s}(\mathbb{R})} \leqslant C_0\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} = C_0\|\Phi(u)-\Phi(v)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}.$$

By taking $u, v \in \mathbb{B}(0,r)$ and $T \leq \frac{1}{(4C|\mu|2^{b'-b}r)^{\frac{1}{b'-b}}}$,

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &\leq \|\psi_1(t)S(t)(u_0-v_0)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&+\|\psi_1(t)\int_0^t S(t-t')\psi_{2T}(t')\left(\partial_x u^2-\partial_x v^2\right)dt'\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}, \\
&\leq C\|u_0-v_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}+C(2T)^{b'-b}\|u+v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}, \\
&\leq C\|u_0-v_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}+\frac{1}{2}\|u-v\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}.
\end{aligned}$$

Thus

$$\|\Phi(u)-\Phi(v)\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq 2C\|u_0-v_0\|_{G^{\sigma,\delta,s}(\mathbb{R})},$$

then

$$|u-v|_{T,\sigma,\delta,s} \leq 2C_0C\|u_0-v_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}.$$

□

This completes the prove of Theorem 2.12.

2.5 Lower bound for radius of spatial analyticity

We shall state our second main result. For this, we need to recall an important property of the space $G^{\delta,s}(\mathbb{R})$. For $\delta>0$ and $s\in\mathbb{R}$, it is straightforward to show that if a function f belongs to $G^{\delta,s}(\mathbb{R})$, then it is the restriction to the real line of a holomorphic function $f(x+iy)$ in the strip

$$S_\delta = \{x+iy\in\mathbb{C}, |y|<\delta\}.$$

This $\delta>0$ is called the (uniform) radius of spatial analyticity of f.

In fact, the following Paley-Wiener theorem provides an alternative description of $G^{\delta,s}(\mathbb{R})$ (see [45]).

Paley-Wiener Theorem. $f\in G^{\delta,s}$ if and only if $f(x)$ is the restriction to the real line of a holomorphic function $f(x+iy)$ in the strip

$$S_\delta = \{x+iy\in\mathbb{C}, |y|<\delta\},$$

and satisfies the bound

$$\sup_{|y|<\delta}\|f(x+iy)\|_{H^s_x}<\infty.$$

In the view of the Paley-Wiener theorem, it is natural to take initial data in $G^{\delta,s}(\mathbb{R})$ and obtain a better understanding of the behavior of solution as we try to extend it globally in time. It means that given $u_0\in G^{\delta,s}(\mathbb{R})$ for some initial radius $\delta>0$ we want to estimate the behavior of the radius of analyticity $\delta(T)$ as time T growth. This is our second novelty and main goal in this chapter.

In section 2.4 we have prove local well-posedness in the space $G^{\delta,s}(\mathbb{R})$ with $\delta > 0$ and $s > -\frac{7}{4}$ (Corollary 2.13) i.e., the local solution is analytic in the spatial variable. In this section, we use the local result and a Gevrey approximate conservation law to gradually extend the local solution for all time. Furthermore, we obtain explicit lower bounds on the radius of spatial analyticity $r(t)$ at any time $t \geqslant 0$, which is given by $r(t) \geqslant ct^{-1}$, where $\delta > 0$ can be taken arbitrarily small and c is a positive constant, that will be described more precisely later.

Our second main result for the Kawahara equation yields an estimate on how the width of the strip of the radius of the spatial analyticity decay with time.

Theorem 2.18. *(Lower bound for radius of spatial analyticity) Let $s > -\frac{7}{4}$ and $\delta_0 > 0$, and assume $u_0 \in G^{\delta_0,s}(\mathbb{R})$, then the solution given by Corollary 2.13 extends globally in time and for any $T > 0$, we have*

$$u \in C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right) \text{ with } \delta(T) = \min\left\{\delta_0, \frac{C_1}{T}\right\},$$

where $C_1 > 0$ is a constant depending on u_0, δ_0 and s .

The method used here for proving lower bounds on the radius of analyticity was introduced in [76] in the context of the 1D Dirac-Klein-Gordon equations. It was applied to the modified Kawahara equation [66] and the non-periodic KdV equation in [77] improving an earlier result of Bona et al. [10], to the dispersion-generalized periodic KdV equation in [35] and to the quartic generalized KdV equation on the line in [78].

2.5.1 Approximate Conservation Law

We start by recalling that

$$\mathscr{I}(u) = \int u^2(x,t)dx,$$

is conserved for a solution u of (2.1), since by using Riemman-Lebesgue's lemma and integration by parts we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R})} &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2(x,t)dx = \int_{\mathbb{R}} u(x,t)\partial_t u(x,t)dx \\
&= \int_{\mathbb{R}} u(x,t)\left[-\alpha\partial_x^5 u(x,t) - \beta\partial_x^3 u(x,t) - \gamma\partial_x u(x,t) - \mu\partial_x(u^2)(x,t)\right] dx \\
&= -\mu\int_{\mathbb{R}} u(x,t)\partial_x(u^2)(x,t) \\
&= \mu\int_{\mathbb{R}} u^2(x,t)\partial_x u(x,t) \\
&= \frac{\mu}{3}\int_{\mathbb{R}} \partial_x(u^3)(x,t)dx = 0.
\end{aligned}$$

Our goal in this section is to establish an approximate conservation law for a solution to (2.1) based on the conservation $L^2(\mathbb{R})$ norm of solutions of the equation. Explicitly, we aim at proving Theorem 2.19.

Theorem 2.19. *Let $\kappa \in [0,1]$ and $0 < T < T_1 < 1$, T_1 be as in Corollary 2.13, there exist $b \in (1/2,1)$ and $C > 0$, such that for any $\delta > 0$ and any solution $u \in X^T_{\delta,0,b}(\mathbb{R}^2)$ to the Cauchy problem (2.1) on the time interval $[0,T]$, we have the estimate*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + C\delta^\kappa \|u\|^3_{X^T_{\delta,0,b}(\mathbb{R}^2)}.$$

Moreover, we have

$$\sup_{t\in[0,T]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + C\delta^\kappa \|u(0)\|^3_{G^{\delta,0}(\mathbb{R})}.$$

Theorem 2.19 is of fundamental importance as it guarantees, by combining it with Corollary 2.8 and applying them repeatedly,we can glue intervals in a way to gradually extend the local solution in time. This will lead to the global well-posedness of solutions in Gevrey spaces, as in Theorem 2.18.

For the proof of Theorem 2.19, we require the following preliminary estimate.

Lemma 2.20. *Given $\kappa \in [0,1]$, there exist $b \in (1/2,1)$ and $C > 0$, such that for all $T > 0$ and $u \in X_{\delta,0,b}(\mathbb{R}^2)$, we have*

$$\|G\|_{X_{0,b-1}(\mathbb{R}^2)} \leq C\delta^\kappa \|u\|^2_{X_{\delta,0,b}(\mathbb{R}^2)}, \tag{2.35}$$

where $G = \partial_x \left[(A^{1,\delta}u)^2 - A^{1,\delta}(u)^2\right]$ and the operator $A^{1,\delta}$ given by (2.8).

Proof. Let $L = \left[(A^{1,\delta}u)^2 - A^{1,\delta}(u)^2\right]$. Then

$$\|G\|_{X_{0,b-1}(\mathbb{R}^2)} = \left\| \frac{\xi}{(1+|\tau+\phi(\xi)|)^{1-b}} \widehat{L}(\xi,\tau) \right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \frac{|\xi|^2}{(1+|\tau+\phi(\xi)|)^{2(1-b)}} |\widehat{L}(\xi,\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}. \tag{2.36}$$

We shall calculate the Fourier transform of L:

$$\begin{aligned}
|\widehat{L}(\xi,\tau)| &= \left| \widehat{(A^{1,\delta}u)^2} - \widehat{A^{1,\delta}(u)^2} \right| \\
&= C\left| (e^{\delta|\xi|}\widehat{u} * e^{\delta|\xi|}\widehat{u})(\xi,\tau) - e^{\delta|\xi|}(\widehat{u}*\widehat{u})(\xi,\tau) \right| \\
&= C\left| \int_{\mathbb{R}^2} \left(e^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1) e^{\delta|\xi-\xi_1|}\widehat{u}(\xi-\xi_1,\tau-\tau_1) - e^{\delta|\xi|}\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi-\xi_1,\tau-\tau_1) \right) d\xi_1 d\tau_1 \right| \\
&\leqslant C \int_{\mathbb{R}^2} \left(e^{\delta|\xi_1|} e^{\delta|\xi-\xi_1|} - e^{\delta|\xi|} \right) |\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi-\xi_1,\tau-\tau_1)| d\xi_1 d\tau_1.
\end{aligned}$$

We need the following estimate

Lemma 2.21. *For $\delta > 0$, $\theta \in [0,1]$ and $\xi, \xi_1 \in \mathbb{R}$ we have*

$$e^{\delta|\xi-\xi_1|}e^{\delta|\xi_1|} - e^{\delta|\xi|} \leq [2\delta \min(|\xi-\xi_1|, |\xi_1|)]^{\theta} e^{\delta|\xi-\xi_1|}e^{\delta|\xi_1|}. \tag{2.37}$$

Also, we have

$$\min(|\xi-\xi_1|, |\xi_1|) \leq 2\frac{(1+|\xi-\xi_1|)(1+|\xi_1|)}{(1+|\xi|)}. \tag{2.38}$$

Proof. If $\xi-\xi_1$ and ξ_1 have the same sign, then the left side of (2.37) equals zero and the inequality holds trivially, therefore, we assume that $\xi-\xi_1$ and ξ_1 have opposite signs. Without loss of generality, assume $\xi-\xi_1 \geqslant 0$ and $\xi_1 \leqslant 0$. If $|\xi_1| \leqslant |\xi-\xi_1|$, then $\xi-\xi_1+\xi_1 \geqslant 0$, and so the left side of (2.37) becomes

$$\begin{aligned}
e^{\delta|\xi-\xi_1|}e^{\delta|\xi_1|} - e^{\delta|\xi|} &= e^{\delta(\xi-\xi_1)}e^{-\delta\xi_1} - e^{\delta((\xi-\xi_1)+\xi_1)} \\
&= e^{\delta((\xi-\xi_1)+\xi_1)}\left(e^{-2\delta\xi_1} - 1\right) \\
&\leqslant (2\delta|\xi_1|)^{\theta} e^{-2\delta\xi_1} e^{\delta((\xi-\xi_1)+\xi_1)} \\
&= (2\delta|\xi_1|)^{\theta} e^{\delta((\xi-\xi_1)-\xi_1)} \\
&= (2\delta|\xi_1|)^{\theta} e^{\delta|\xi-\xi_1|}e^{|\xi_1|}.
\end{aligned}$$

Here, we have used the fact that, for $x \geqslant 0$, the inequalities $e^x - 1 \leqslant e^x$ and $e^x - 1 \leqslant xe^x$ both hold, hence also $e^x - 1 \leqslant x^{\theta}e^x$ for $x \geqslant 0$ and $\theta \in [0,1]$.
On the other hand, if $|\xi_1| \geqslant |\xi-\xi_1|$, then $\xi-\xi_1+\xi_1 \leqslant 0$, so the left side of (2.37) becomes

$$\begin{aligned}
e^{\delta|\xi-\xi_1|}e^{\delta|\xi_1|} - e^{\delta|\xi|} &= e^{\delta(\xi-\xi_1)}e^{-\delta\xi_1} - e^{-\delta((\xi-\xi_1)+\xi_1)} \\
&= e^{-\delta((\xi-\xi_1)+\xi_1)}\left(e^{2\delta(\xi-\xi_1)} - 1\right) \\
&\leqslant (2\delta|\xi-\xi_1|)^{\theta} e^{2\delta(\xi-\xi_1)} e^{-\delta((\xi-\xi_1)+\xi_1)} \\
&= (2\delta|\xi-\xi_1|)^{\theta} e^{\delta((\xi-\xi_1)-\xi_1)} \\
&= (2\delta|\xi-\xi_1|)^{\theta} e^{\delta|\xi-\xi_1|}e^{|\xi_1|}.
\end{aligned}$$

□

For $\kappa \in [0,1]$, from Lemma 2.21, we write

$$\begin{aligned}
|\widehat{L}(\xi,\tau)| &\leqslant C\int_{\mathbb{R}^2}\left(e^{\delta|\xi_1|}e^{\delta|\xi-\xi_1|}-e^{\delta|\xi|}\right)|\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi-\xi_1,\tau-\tau_1)|d\xi_1 d\tau_1\\
&\leqslant C\int_{\mathbb{R}^2}[2\delta\min(|\xi-\xi_1|,|\xi_1|)]^{\kappa}e^{\delta|\xi_1|}|\widehat{u}(\xi_1,\tau_1)|e^{\delta|\xi-\xi_1|}|\widehat{u}(\xi-\xi_1,\tau-\tau_1)|d\xi_1 d\tau_1\\
&\leqslant C\int_{\mathbb{R}^2}\left[4\delta\frac{(1+|\xi-\xi_1|)(1+|\xi_1|)}{(1+|\xi|)}\right]^{\kappa}e^{\delta|\xi_1|}|\widehat{u}(\xi_1,\tau_1)|e^{\delta|\xi-\xi_1|}|\widehat{u}(\xi-\xi_1,\tau-\tau_1)|d\xi_1 d\tau_1\\
&\leqslant C(4\delta)^{\kappa}\int_{\mathbb{R}^2}\frac{(1+|\xi-\xi_1|)^{\kappa}(1+|\xi_1|)^{\kappa}}{(1+|\xi|)^{\kappa}}e^{\delta|\xi_1|}|\widehat{u}(\xi_1,\tau_1)|e^{\delta|\xi-\xi_1|}|\widehat{u}(\xi-\xi_1,\tau-\tau_1)|d\xi_1 d\tau_1.
\end{aligned} \tag{2.39}$$

Setting $v=(A^{1,\delta}u)$ and $f(\xi,\tau)=(1+|\tau+\phi(\xi)|)^b\widehat{v}(\xi,\tau)$ we have $e^{\delta|\xi|}\widehat{u}(\xi,\tau)=\widehat{v}(\xi,\tau)=f(\xi,\tau)(1+|\tau+\phi(\xi)|)^{-b}$ and therefore we can write (2.39) as

$$\begin{aligned}
|\widehat{L}(\xi,\tau)| &\leqslant C(4\delta)^{\kappa}\int_{\mathbb{R}^2}\frac{(1+|\xi-\xi_1|)^{\kappa}(1+|\xi_1|)^{\kappa}}{(1+|\xi|)^{\kappa}}\frac{|\widehat{f}(\xi_1,\tau_1)|}{(1+|\tau_1+\phi(\xi_1)|)^b}\\
&\quad\cdot\frac{|\widehat{f}(\xi-\xi_1,\tau-\tau_1)|}{(1+|\tau-\tau_1+\phi(\xi-\xi_1)|)^b}d\xi_1 d\tau_1.
\end{aligned} \tag{2.40}$$

It follows from (2.36) and (2.40) that

$$\begin{aligned}
\|G\|_{X_{0,b-1}(\mathbb{R}^2)} &= \left\|\frac{\xi}{(1+|\tau+\phi(\xi)|)^{1-b}}\widehat{L}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&\leqslant C(4\delta)^{\kappa}\left[\int_{\mathbb{R}^2}\frac{|\xi|^2}{(1+|\tau+\phi(\xi)|)^{2(1-b)}}\left(\int_{\mathbb{R}^2}\frac{(1+|\xi-\xi_1|)^{\kappa}(1+|\xi_1|)^{\kappa}}{(1+|\xi|)^{\kappa}}\right.\right.\\
&\quad\left.\left.\cdot\frac{|\widehat{f}(\xi_1,\tau_1)|}{(1+|\tau_1+\phi(\xi_1)|)^b}\frac{|\widehat{f}(\xi-\xi_1,\tau-\tau_1)|}{(1+|\tau-\tau_1+\phi(\xi-\xi_1)|)^b}d\xi_1 d\tau_1\right)^2 d\xi d\tau\right]^{\frac{1}{2}}\\
&= C(4\delta)^{\kappa}\left\|\frac{\xi}{(1+|\tau+\phi(\xi)|)^{1-b}}\int_{\mathbb{R}^2}\frac{(1+|\xi-\xi_1|)^{\kappa}(1+|\xi_1|)^{\kappa}}{(1+|\xi|)^{\kappa}}\right.\\
&\quad\left.\cdot\frac{|\widehat{f}(\xi_1,\tau_1)|}{(1+|\tau_1+\phi(\xi_1)|)^b}\frac{|\widehat{f}(\xi-\xi_1,\tau-\tau_1)|}{(1+|\tau-\tau_1+\phi(\xi-\xi_1)|)^b}d\xi_1 d\tau_1\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.
\end{aligned}$$

Now by taking $s=-\kappa\in[-1,0]$ we obtain

$$\|G\|_{X_{0,b-1}(\mathbb{R}^2)} \leqslant C(4\delta)^{\kappa}\Bigg\|\frac{\xi(1+|\xi|)^s}{(1+|\tau+\phi(\xi)|)^{1-b}}\int_{\mathbb{R}^2}\frac{|\widehat{f}(\xi_1,\tau_1)|}{(1+|\xi_1|)^s(1+|\tau_1+\phi(\xi_1)|)^b} \cdot\frac{|\widehat{f}(\xi-\xi_1,\tau-\tau_1)|}{(1+|\xi-\xi_1|)^s(1+|\tau-\tau_1+\phi(\xi-\xi_1)|)^b}d\xi_1 d\tau_1\Bigg\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.$$

By Remark 2.9 we get

$$\begin{aligned}\|G\|_{X_{0,b-1}(\mathbb{R}^2)} &\leqslant C(4\delta)^{\kappa}\|f\|^3_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\ &=C(4\delta)^{\kappa}\|v\|^3_{X_{0,b}(\mathbb{R}^2)}\\ &=C(4\delta)^{\kappa}\|A^{1,\delta}u\|^3_{X_{0,b}(\mathbb{R}^2)}\\ &=C(4\delta)^{\kappa}\|u\|^3_{X_{\delta,0,b}(\mathbb{R}^2)},\end{aligned} \tag{2.41}$$

concluding the proof.

□

Proof of Theorem 2.19. Let κ,δ,T,b and u be as in the statement of Theorem 2.19. We start by defining the auxiliary function $V(t,x)=A^{1,\delta}u(t,x)$. Since u is real-valued we also have V real-valued. Applying the exponential $A^{1,\delta}$ to Equation (2.1), it is easily seen that we obtain

$$\partial_t V+\alpha\partial_x^5 V+\beta\partial_x^3 V+\gamma\partial_x V+\mu A^{1,\delta}\partial_x u^2=0,$$

which is equivalent to

$$\partial_t V+\alpha\partial_x^5 V+\beta\partial_x^3 V+\gamma\partial_x V+2\mu V\partial_x V=\mu(\partial_x V^2-\partial_x(A^{1,\delta}u^2)).$$

Therefore, setting

$$G=\partial_x\left[(A^{1,\delta}u)^2-A^{1,\delta}(u)^2\right],$$

we obtain

$$\partial_t V+\alpha\partial_x^5 V+\beta\partial_x^3 V+\gamma\partial_x V+2\mu V\partial_x V=\mu G. \tag{2.42}$$

Multiplying (2.42) by V and integrating in space we obtain

$$\int_{\mathbb{R}} V\partial_t V dx+\alpha\int_{\mathbb{R}} V\partial_x^5 V dx+\beta\int_{\mathbb{R}} V\partial_x^3 V dx+\gamma\int_{\mathbb{R}} V\partial_x V dx+2\mu\int_{\mathbb{R}} V^2\partial_x V dx=\mu\int_{\mathbb{R}} VG dx. \tag{2.43}$$

Integration by parts is justified, since we may assume that $V(t,x)$ decays to zero as $|x| \longrightarrow \infty$, and the same holds for all spatial derivatives.[2] Thus, (2.43) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} V^2 dx + \frac{\alpha}{2}\int_{\mathbb{R}} \partial_x(\partial_x^2 V \partial_x^2 V)dx + \frac{\beta}{2}\int_{\mathbb{R}} \partial_x(\partial_x V \partial_x V)dx + \frac{\gamma}{2}\int_{\mathbb{R}} \partial_x V^2 dx + \frac{2\mu}{3}\int_{\mathbb{R}} \partial_x V^3 dx = \mu \int_{\mathbb{R}} VG dx, \tag{2.44}$$

and furthermore, the second, third, forth and fifth terms on the left side vanish

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} V^2 dx = \mu \int_{\mathbb{R}} VG\, dx.$$

Now integrating the last equality with respect to $t \in [0,T]$ we obtain

$$\frac{1}{2}\left[\int_{\mathbb{R}} V^2(x,T)dx - \int_{\mathbb{R}} V^2(x,0)dx\right] = \mu \int_0^T \int_{\mathbb{R}} VG\, dxdt. \tag{2.45}$$

Recall that

$$\|u(t)\|_{G^{\delta,0}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta|\xi|}|\widehat{u}(\xi,t)|^2 dx = \int_{\mathbb{R}} |\widehat{V}(\xi,t)|^2 dx = \int_{\mathbb{R}} V^2(x,t)dx,$$

where in the last equality we used Plancherel theorem and the fact that we are assuming that the solution u is real valued.
It follows from the last equality and from (2.45) that

$$\int_{\mathbb{R}} V^2(x,T)dx = \int_{\mathbb{R}} V^2(x,0)dx + 2\mu \int_{\mathbb{R}^2} \chi_{[0,T]}(t) VG\, dxdt.$$

Thus,

$$\begin{aligned}\|u(T)\|^2_{G^{\delta,0}} &= \|u(0)\|^2_{G^{\delta,0}} + 2\mu \int_{\mathbb{R}^2} \chi_{[0,T]}(t) VG\, dxdt, \\ &\leqslant \|u(0)\|^2_{G^{\delta,0}} + 2|\mu| \left| \int_{\mathbb{R}^2} \chi_{[0,T]}(t) VG\, dxdt \right|. \end{aligned} \tag{2.46}$$

Recalling that V is real-valued, it follows from Plancherel's identity and Cauchy-Schwarz inequality that

$$\begin{aligned}\int_{\mathbb{R}^2} \chi_{[0,T]}(t) VG\, dxdt &= \int_{\mathbb{R}^2} (\chi_{[0,T]}(\cdot)V)(\xi,\tau)\overline{(\chi_{[0,T]}(\cdot)G)}(\xi,\tau) d\xi d\tau \\ &= \int_{\mathbb{R}^2} \widehat{(\chi_{[0,T]}(\cdot)V)}(\xi,\tau)\overline{\widehat{(\chi_{[0,T]}(\cdot)G)}}(\xi,\tau) d\xi d\tau.\end{aligned}$$

[2] Indeed, we are aiming to prove (2.35) for a given $\delta > 0$, but by the monotone convergence theorem, it suffices to prove it for all $\delta' < \delta$ (the constant C being uniform). For $U = A^{1,\delta'}u$, we get by Cauchy-Schwarz and by the assumption that $u \in X_{\delta,0,b} \subset L_t^\infty G_{\delta,0}$,

$$\int |\widehat{\partial_x^j U}(\xi,t)| d\xi = \int |e^{(\delta'-\delta)|\xi|} e^{\delta|\xi|} \xi^j \widehat{u}(\xi,t)| d\xi \leqslant \left(\int \xi^{2j} e^{-2\rho|\xi|} d\xi\right)^{\frac{1}{2}} \|u(t)\|_{G_{\delta,0}} < \infty,$$

where $\rho = \delta - \delta' > 0$ and $j \in \{0,1,\cdots\}$. Therefore, by Riemann-Lebesgue, $\partial_x^j U \longrightarrow 0$ as $|x| \longrightarrow \infty$.

Then, Hölder's inequality yields

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\chi_{[0,T]}(t)VG\,dxdt\right| &= \left|\int_{\mathbb{R}^2}(1+|\tau+\phi(\xi)|)^{1-b}(\widehat{\chi_{[0,T]}(\cdot)V})(\xi,\tau)\right.\\ &\quad\left.\cdot(1+|\tau+\phi(\xi)|)^{b-1}\overline{(\widehat{\chi_{[0,T]}(\cdot)G})}(\xi,\tau)d\xi d\tau\right|\\ &\leqslant \left\|(1+|\tau+\phi(\xi)|)^{1-b}(\widehat{\chi_{[0,T]}(\cdot)V})(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\ &\quad\cdot\left\|(1+|\tau+\phi(\xi)|)^{b-1}(\widehat{\chi_{[0,T]}(\cdot)G})(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\ &\leqslant \|\chi_{[0,T]}(\cdot)V\|_{X_{0,1-b}(\mathbb{R}^2)}\|\chi_{[0,T]}(\cdot)G\|_{X_{0,b-1}(\mathbb{R}^2)},\end{aligned} \tag{2.47}$$

we have both $-\frac{1}{2}<b-1<\frac{1}{2}$ and $\frac{1}{2}<1-b<\frac{1}{2}$. Therefore, one can use Lemma 2.11 to obtain

$$\|\chi_{[0,T]}(\cdot)V\|_{X_{0,1-b}(\mathbb{R}^2)}\leqslant C\|V\|_{X^T_{0,1-b}(\mathbb{R}^2)},$$

and

$$\|\chi_{[0,T]}(\cdot)G\|_{X_{0,b-1}(\mathbb{R}^2)}\leqslant C\|G\|_{X^T_{0,b-1}(\mathbb{R}^2)}.$$

Since $0<T<1$ and using the fact that $\psi_T=1$ for $t\in[0,T]$and the definition of $\|\cdot\|_{X^T_{\delta,s,b}(\mathbb{R}^2)}$, (see (2.6)), it follows from (2.47) and from the last relation that

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\chi_{[0,T]}(t)VG\,dxdt\right| &\leqslant C\|V\|_{X^T_{0,1-b}(\mathbb{R}^2)}\|G\|_{X^T_{0,b-1}(\mathbb{R}^2)}\\ &\leqslant C\|\psi_T V\|_{X_{0,1-b}(\mathbb{R}^2)}\|\psi_T G\|_{X_{0,b-1}(\mathbb{R}^2)}.\end{aligned} \tag{2.48}$$

Since $-\frac{1}{2}<b-1<\frac{1}{2}$ and $\frac{1}{2}<1-b<\frac{1}{2}$ it follows from Lemma 2.7 that

$$\|\psi_T V\|_{X^T_{0,1-b}(\mathbb{R}^2)}\leqslant C\|V\|_{X_{0,1-b}(\mathbb{R}^2)}, \tag{2.49}$$

and

$$\|\psi_T G\|_{X^T_{0,b-1}(\mathbb{R}^2)}\leqslant C\|G\|_{X_{0,b-1}(\mathbb{R}^2)}. \tag{2.50}$$

Noticing that

$$\|V\|_{X_{0,1-b}(\mathbb{R}^2)}=\|u\|_{X_{\delta,0,1-b}(\mathbb{R}^2)}\leqslant\|u\|_{X_{\delta,0,b}(\mathbb{R}^2)}, \tag{2.51}$$

since we have $1-b<b$, we can conclude from it and Lemma 2.20 that for any $\kappa\in[0,1]$ there exists a constant C such that

$$\|G\|_{X_{0,b-1}(\mathbb{R}^2)}\leq C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}(\mathbb{R}^2)}. \tag{2.52}$$

Therefore, we conclude from (2.46) and (2.48)-(2.52) that

$$\sup_{t\in[0,T]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + C\delta^{\kappa}\|u\|^3_{X_{\delta,0,b}(\mathbb{R}^2)}.$$

Finally, by using the condition (2.23) we conclude that

$$\sup_{t\in[0,T]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + C\delta^{\kappa}\|u(0)\|^3_{G^{\delta,0}(\mathbb{R})}.$$

The proof is now complete.

2.5.2 Global extension and radius analyticity −Proof of Theorem 2.18

Fix $\delta_0 > 0$, $s > -\frac{7}{4}$, $\kappa \in (0,1)$, and $u_0 \in G^{\delta_0,s}(\mathbb{R})$. It suffices to prove that the solution $u(t)$ to (2.1) satisfies

$$u \in C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right) \quad for\ all\ T' > 0,$$

where

$$\delta(T) = \min\left\{\delta_0, C_1 T^{-(\frac{1}{\kappa})}\right\},$$

and $C_1 > 0$ is a constant depending on u_0, δ_0, s, and κ.
By Corollary 2.13, there is a maximal time $T^* = T^*(u_0,\delta_0,s) \in (0,\infty]$, such that

$$u \in C\left([0,T^*), G^{\delta_0,s}(\mathbb{R})\right).$$

If $T^* = \infty$ then $r(t) = \delta_0$ and we are done.
If $T^* < \infty$, as we assume henceforth, it remains to prove

$$u \in C\left([0,T], G^{C_1 T^{-(\frac{1}{\kappa})},s}(\mathbb{R})\right), \quad for\ all\ T \geq T^*. \tag{2.53}$$

We first prove this in the case $s = 0$. Then, at the end of this section, we do the general case, which essentially reduces to $s = 0$.

The case s=0

Fix $T \geq T^*$, we will show that, for $\delta > 0$ sufficiently small

$$\|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq 2\|u(0)\|^2_{G^{\delta_0,0}(\mathbb{R})}, \quad for\ t \in [0,T]. \tag{2.54}$$

We need to set the following time-step

$$T_0 = \frac{c_0}{(1+2\|u(0)\|_{G^{\delta_0,s}(\mathbb{R})})^a} < \frac{c_0}{(1+\|u(0)\|_{G^{\delta_0,s}(\mathbb{R})})^a} = T_{\delta_0}, \tag{2.55}$$

where $c_0 > 0$ and $a > 1$ are as in Corollary 2.13 (with $s = 0$). The smallness conditions on δ will be

$$\delta \leq \delta_0 \ \ and \ \ \frac{2T}{T_0} C\delta^{\kappa} 2^{\frac{3}{2}} \parallel u(0) \parallel_{G^{\delta_0,0}(\mathbb{R})} \leq 1, \tag{2.56}$$

where $C > 0$ is the constant in Theorems 2.19. Proceeding by induction, we will verify that

$$\sup_{t\in[0,nT_0]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + nC\delta^{\kappa}2^{\frac{3}{2}}\|u(0)\|^3_{G^{\delta_0,0}(\mathbb{R})}, \tag{2.57}$$

$$\sup_{t\in[0,nT_0]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} \leq 2\|u(0)\|^2_{G^{\delta_0,0}(\mathbb{R})}, \tag{2.58}$$

for $n \in \{1,\cdots,m+1\}$, where $m \in \mathbb{N}$ is chosen, so that $T \in [mT_0,(m+1)T_0)$. This m does exist, since by Corollary 2.13 and the definition of T^*, we have

$$T_0 < \frac{c_0}{(1+\|u(0)\|_{G^{\delta_0,0}(\mathbb{R})})^a} < T^*, \quad hence \ \ T_0 < T.$$

In the first step, we cover the interval $[0,T_0]$, and by Theorem 2.19, we have

$$\begin{aligned}\sup_{t\in[0,T_0]} \|u(t)\|^2_{G^{\delta,0}(\mathbb{R})} &\leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + C\delta^{\kappa}\|u(0)\|^3_{G^{\delta,0}(\mathbb{R})} \\ &\leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + C\delta^{\kappa}\|u(0)\|^3_{G^{\delta_0,0}(\mathbb{R})},\end{aligned}$$

where we used that

$$\|u(0)\|_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|_{G^{\delta_0,0}(\mathbb{R})}, \tag{2.59}$$

since $\delta \leq \delta_0$. This verifies (2.57) for $n = 1$ and now, (2.58) follows using again (2.59) as well as

$$C\delta^{\kappa}\|u(0)\|_{G^{\delta_0,0}(\mathbb{R})} \leq 1.$$

The latter follows from (2.56), since $T_0 < T$.

Next, assuming that (2.57) and (2.58) hold for some $n \in \{1,\cdots,m\}$, we will prove that they hold for $n+1$. We estimate

$$\begin{aligned}&\sup_{t\in[nT_0,(n+1)T_0]} \|u(t)\|^2_{G^{\delta,0}} \\ &\leq \|u(nT_0)\|^2_{G^{\delta,0}} + C\delta^{\kappa}\|u(nT)\|^3_{G^{\delta,0}} && by\ Theorem\ 2.19 \\ &\leq \|u(nT_0)\|^2_{G^{\delta,0}} + C\delta^{\kappa}2^{\frac{3}{2}}\|u(0)\|^3_{G^{\delta_0,0}} && by\ (2.58) \\ &\leq \|u(0)\|^2_{G^{\delta,0}} + nC\delta^{\kappa}2^{\frac{3}{2}}\|u(0)\|^3_{G^{\delta_0,0}} + C\delta^{\kappa}2^{\frac{3}{2}}\|u(0)\|^3_{G^{\delta_0,0}}, && by\ (2.57)\end{aligned}$$

verifying (2.57) with n replaced by $n+1$. To get (2.58) with n replaced by $n+1$, it is then enough to have

$$(n+1)C\delta^{\kappa}2^{\frac{3}{2}}\|u(0)\|_{G^{\delta_0,0}} \leq 1,$$

but this holds by (2.56), since

$$n+1 \leq m+1 \leq \frac{T}{T_0} + 1 < \frac{2T}{T_0}.$$

We have thus proved (2.54) under the smallness assumptions (2.56)on δ. Since $T \geq T^*$, the condition (2.56) must fail for $\delta = \delta_0$, that is, the left side must be strictly larger than 1, since otherwise we would be able to continue the solution in $G^{\delta_0,0}(\mathbb{R})$ beyond the time T, contradicting the maximality of T^*. Therefore, there must be some $\delta \in (0,\delta_0)$for which equality holds in (2.56),and using (2.55), we get

$$2T\frac{(1+2\|u(0)\|_{G^{\delta_0,0}(\mathbb{R})})^a}{c_0}C\delta^{\kappa}2^{\frac{3}{2}}\|u(0)\|_{G^{\delta_0,0}(\mathbb{R})} = 1,$$

hence,

$$\delta = C_1 T^{-(\frac{1}{\kappa})},$$

where

$$C_1 = \left(\frac{c_0}{C2^{\frac{5}{2}}\|u(0)\|_{G^{\delta_0,0}(\mathbb{R})}(1+2\|u(0)\|_{G^{\delta_0,0}(\mathbb{R})})^{\beta}}\right)^{\frac{1}{\kappa}}.$$

We have proved that (2.54) holds for this δ, hence $\|u(t)\|_{G^{\delta,0}(\mathbb{R})} < \infty$ for $t \in [0,T]$, and this completes the proof of (2.53) for the case $s = 0$.

The General Case

For general s, we use the embedding (5.3) to get

$$u_0 \in G^{\delta_0,s}(\mathbb{R}) \subset G^{\delta_0/2,0}(\mathbb{R}).$$

The case $s = 0$ already being proved, we know that there is a $T_1 > 0$, such hat

$$u \in C\left([0,T_1), G^{\delta_0/2,0}\right),$$

and

$$u \in C\left([0,T], G^{2\rho T^{-1},0}\right), \quad for\ T \geq T_1,$$

where $\rho > 0$ depends on u_0, δ_0 and κ. Applying again the embedding (5.3), we now conclude that

$$u \in C\left([0,T_1), G^{\delta_0/4,s}\right),$$

and

$$u \in C\left([0,T], G^{\rho T^{-1},s}\right), \quad for\ T \geq T_1,$$

and these together imply (2.53). The proof of Theorem 2.18 is now completed.

2.6 Time regularity

Our next goal is to study Gevrey's temporal regularity of the unique solution obtained in the Theorem 2.12. A non-periodic function $f(x)$ is the Gevrey class of order σ i.e, $f(x) \in G^{\sigma}$, if there exists a constant $C > 0$ such that

$$|\partial_x^l f(x)| \leq C^{l+1}(l!)^{\sigma} \quad l = 0,1,2,\dots. \tag{2.60}$$

Here we will show that for $x \in \mathbb{R}$, for every $t \in [0,T]$ and $j,l \in \{0,1,2,\dots\}$, there exist $C>0$ such that,

$$|\partial_t^j \partial_x^l u(x,t)| \leq C^{j+l+1}(j!)^{5\sigma}(l!)^{\sigma}. \tag{2.61}$$

i.e, $u(\cdot,t) \in G^{\sigma}(\mathbb{R})$ in spacial variable and $u(x,\cdot) \in G^{5\sigma}([0,T])$ in time variable .

Theorem 2.22. *Let $s > -\frac{7}{4}$, $\sigma \geqslant 1, \delta > 0$ and $\beta = \gamma = \mu = \alpha = 1$. If $u_0 \in G^{\sigma,\delta,s}(\mathbb{R})$, then the solution $u \in C\left([0,T], G^{\sigma,\delta,s}(\mathbb{R})\right)$ given by Theorem 2.12 belongs to the Gevrey class $G^{5\sigma}([0,T])$ in time variable.*

Now the regularity of the unique solution obtained in the Theorem 2.18. A non-periodic function $f(x)$ is the Gevrey class of order 1 i.e,$f(x) \in G^1$ here $\sigma = 1$, if there exists a constant $C>0$ such that

$$|\partial_x^l f(x)| \leq C^{l+1}(l!) \quad l = 0,1,2,\dots. \tag{2.62}$$

Here we will show that for $x \in \mathbb{R}$, for every $t \in [0,T]$ and $j,l \in \{0,1,2,\dots\}$, there exist $C>0$ such that,

$$|\partial_t^j \partial_x^l u(x,t)| \leq C^{j+l+1}(j!)^{5}(l!). \tag{2.63}$$

i.e, $u(\cdot,t) \in G^{1}(\mathbb{R})$ in spacial variable and $u(x,\cdot) \in G^{5}([0,T])$ in time variable .

Corollary 2.23. *Let $s > -\frac{7}{4}$ and $\delta > 0$. If $u_0 \in G^{\delta,s}(\mathbb{R})$, then the solution $u \in C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right)$ given by Theorem 2.18 belongs to the Gevrey class $G^5([0,T])$ in time variable.*

2.6.1 G^{σ}-regularity in the spacial variable

Proposition 2.24. *Let $s > -\frac{7}{4}$ and let $\delta > 0$, $\sigma \geq 1$, $u \in C\left([0,T]; G^{\sigma,\delta,s}(\mathbb{R})\right)$ be the solution of (2.1). Then*

$$u \in G^{\sigma} \text{ in } x, \forall t \in [0,T],$$

i.e., for some $C>0$, we have

$$|\partial_x^l u(x,t)| \leqslant C^{l+1}(l!)^{\sigma}, l \in \{0,1,\cdots\}, \quad \forall x \in \mathbb{R},\ t \in [0,T]. \tag{2.64}$$

Proof. $\forall t \in [0,T]$, we have

$$\begin{aligned}\|\partial_x^l u(\cdot,t)\|^2_{H^s(\mathbb{R})} &= \int_{\mathbb{R}} |\xi|^{2l}(1+|\xi|)^{2s}|\widehat{u}(\xi,t)|^2 d\xi \\ &= \int_{\mathbb{R}} |\xi|^{2l} e^{-2\delta|\xi|^{1/\sigma}}(1+|\xi|)^{2s} e^{2\delta|\xi|^{1/\sigma}}|\widehat{u}(\xi,t)|^2 d\xi.\end{aligned}$$

We observe that

$$e^{\frac{2\delta}{\sigma}|\xi|^{1/\sigma}} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{2\delta}{\sigma}|\xi|^{\frac{1}{\sigma}}\right)^j \geq \frac{1}{(2l)!}\left(\frac{2\delta}{\sigma}\right)^{2l}|\xi|^{\frac{2l}{\sigma}}, \quad \forall l \in \{0,1,...\}, \xi \in \mathbb{R}.$$

This implies that

$$|\xi|^{2l}e^{-2\delta|\xi|^{1/\sigma}} \leq C_{\delta,\sigma}^{2l}(2l)!^{\sigma}.$$

Thus,

$$\begin{aligned}\|\partial_x^l u(\cdot,t)\|_{H^s(\mathbb{R})}^2 &\leq C_{\delta,\sigma}^{2l}(2l)!^{\sigma}\int_{\mathbb{R}} e^{2\delta|\xi|^{1/\sigma}}(1+|\xi|)^{2s}|\widehat{u}(\xi,t)|^2 d\xi \\ &= C_{\delta,\sigma}^{2l}(2l)!^{\sigma}\|u(\cdot,t)\|_{G^{\sigma,\delta,s}(\mathbb{R})}^2.\end{aligned}$$

Since $(2l)! \leq A_1^{2l}(l!)^2$, for some $A_1 > 0$, one conclude that, if $s \geq 0$,

$$|\partial_x^l u(x,t)| \leq \|\partial_x^l u(\cdot,t)\|_{L^2(\mathbb{R})} \leq \|\partial_x^l u(\cdot,t)\|_{H^s(\mathbb{R})} \leq C_0 C_1^l (l)!^{\sigma} \quad \forall\, t \in [0,T],$$

here $C_0 = \|u(\cdot,t)\|_{G^{\sigma,\delta,s}(\mathbb{R})}$ and $C_1 = A_1 C_{\sigma,\delta}$, this implies that u is Gevrey of order σ in x, for $s \geq 0$.
Now, for $-\frac{7}{4} < s < 0$, and for any $0 < \varepsilon < \delta$, there exists a positive constant $C = C_{s,\varepsilon} > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}} e^{2(\delta-\varepsilon)|\xi|^{1/\sigma}}|\widehat{u}(\xi,t)|^2 d\xi \ &\leq C\int_{\mathbb{R}} \frac{e^{2\varepsilon|\xi|^{1/\sigma}}}{(1+|\xi|)^{-2s}} e^{2(\delta-\varepsilon)|\xi|^{1/\sigma}}|\widehat{u}(\xi,t)|^2 d\xi \\ &= C\int_{\mathbb{R}} e^{2\delta|\xi|^{1/\sigma}}(1+|\xi|)^{2s}|\widehat{u}(\xi,t)|^2 d\xi.\end{aligned} \tag{2.65}$$

This implies that if $u \in C\left([0,T];G^{\sigma,\delta,s}(\mathbb{R})\right)$ and $s < 0$, then $u \in C\left([0,T];G^{\sigma,\delta-\varepsilon,0}(\mathbb{R})\right)$, which allows us to conclude that u is in G^{σ} in x, for all $s > -\frac{7}{4}$.

□

2.6.2 $G^{5\sigma}$-regularity in the time variable

We will now prove the temporal regularity of solution.

Let us consider as in [39], for $\varepsilon > 0$, the sequences

$$m_q = \frac{c(q!)^{\sigma}}{(q+1)^2}, (q = 0,1,2,...), \tag{2.66}$$

and

$$M_q = \varepsilon^{1-q} m_q, \varepsilon > 0 \ \textit{and} \ (q = 1,2,3,...), \tag{2.67}$$

where c will be chosen (see [2]) so that the next inequality holds

$$\sum_{0\leq l\leq k} \binom{k}{l} m_l m_{k-l} \leqslant m_k. \tag{2.68}$$

Removing the endpoints 0 and k in the left hand side of (2.68) and using the sequence M_q, we obtain

$$\sum_{0<l<k} \binom{k}{l} M_l M_{k-l} \leq M_k, \forall \varepsilon > 0. \tag{2.69}$$

Next, one can check that for any $\varepsilon > 0$ the sequence M_q satisfies the following inequality

$$M_j \leqslant \varepsilon M_{j+1},\ for\ j \geq 2. \tag{2.70}$$

Also, one can check that for a given $C > 1$, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$ we have

$$C^{j+1} j! \leqslant M_j,\ for\ j \geq 2. \tag{2.71}$$

For some $C > 0$, we define the following constants

$$M_0 = \frac{c}{8}\ and\ M = \max\left\{2, \frac{8C}{c}, \frac{4C^2}{c}\right\}. \tag{2.72}$$

The next lemma is the main idea for the proof of Theorem 2.22.

Lemma 2.25. *Let u be the solution of (2.1) satisfying (2.64), then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$ we have*

$$|\partial_t^j \partial_x^l u(x,t)| \leqslant M^{2j+1} M_{l+5j}, j \in \{0,1,2,...\}, l \in \{0,1,2,...\}, \tag{2.73}$$

for all $x \in \mathbb{R}$, $t \in [0,T]$.

In order to prove Lemma 2.25 we shall need the following key result.

Lemma 2.26. *Given $n, k \in \{0,1,2,...\}$ we have*

$$\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q} L_{(n-p)+5(k-q)} L_{p+5q+1} \leqslant \sum_{r=1}^{m}\binom{m}{r} L_r L_{m-r}, \tag{2.74}$$

where $L_j, j = 0,1,...,m$ positive real numbers with $m = n+5k+1$

Proof. For $n,k \in \{0,1,2,\ldots\}$ given let $m = n+5k+1$. For $k = n = 0$ inequality (2.74) reads $L_0 L_1 \leqslant L_0 L_1$, which is true. Therefore, we assume that either $k \geqslant 1$ or $n \geqslant 1$. Then, changing the order of the summations and making a change of variables gives

$$\begin{aligned}
&\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q} L_{(n-p)+5(k-q)} L_{p+1+5q} \\
&= \sum_{q=0}^{k}\sum_{p=0}^{n}\binom{n}{p}\binom{k}{q} L_{(n-p)+5(k-q)} L_{p+1+5q} \\
&= \sum_{q=0}^{k}\sum_{r=1+5q}^{n+1+5q}\binom{n}{r-1-5q}\binom{k}{q} L_{m-r} L_r \\
&= \sum_{r=1}^{m}\sum_{q=i_0(r)}^{i_1(r)}\binom{n}{r-1-5q}\binom{k}{q} L_{m-r} L_r,
\end{aligned}$$

with $i_0(r) = \max\left\{0, \left[\left[\frac{r-n-1}{5}\right]\right]\right\}, i_1(r) = \min\left\{\left[\frac{r-1}{5}\right], 5k\right\}$, where $[x]$ is the integer part of a number x and $[[x]]$ is the lesser integer that is greater than or equal to x. To complete the proof of inequality (2.74) we must to show that

$$\sum_{q=i_0(r)}^{i_1(r)}\binom{n}{r-1-5q}\binom{k}{q} \leqslant \binom{m}{r}.$$

This is a consequence of the following result.

Lemma 2.27.
$$\sum_{q=i_0(r)}^{\theta} \binom{n}{r-1-5q}\binom{k}{q} \leqslant \binom{m-4k+4\theta}{r}, \tag{2.75}$$
for all $i_0(r) \leqslant \theta \leqslant i_1(r)$.

In fact, using (2.75) with $\theta = i_1(r)$ it suffices to show that
$$\binom{m-4k+4i_1(r)}{r} \leqslant \binom{m}{r}. \tag{2.76}$$

For $i_1(r) = k$ relation (2.76) holds as an equality. If $0 \leqslant i_1(r) < k$ then $1 \leqslant 4\,(k - i_1(r))$ and therefore $m - 4k + 4i_1(r) \leqslant m - 1 < m$, which shows that (2.76) holds as a strict inequality. This completes the proof of Lemma 2.26.

□

Proof. (**Of Lemma** 2.27)
We prove it by induction on θ. For this, we use the following elementary inequality: If $a,b,c \in \mathbb{N}, b \leqslant a$ then
$$\binom{a}{b} \leqslant \binom{a+c}{b+c}.$$
With $a = n$, $b = r-1-5i_0(r), c = 1+2i_0(r)$ and using the definition of m gives
$$\binom{n}{r-1-5i_0(r)} = \binom{m-1-5k}{r-1-5i_0(r)} \leqslant \binom{m-5k+4i_0(r)}{r-i_0(r)}.$$
Now, since for $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ with $\alpha \leqslant \beta$ and $\gamma \leqslant \delta$ we have that
$$\binom{\beta}{\alpha}\binom{\delta}{\gamma} \leqslant \binom{\beta+\delta}{\alpha+\gamma},$$
we get
$$\binom{n}{r-1-5i_0(r)}\binom{k}{i_0(r)} \leqslant \binom{m-5k+4i_0(r)}{r-i_0(r)}\binom{k}{i_0(r)} \leqslant \binom{m-4k+4i_0(r)}{r},$$
which proves (2.75) for $\theta = i_0(r)$. Next, we assume that (2.76) holds for $i_0(r) \leqslant \theta < i_1(r)$ and we will prove it for $(\theta+1)$. By using the induction hypotheses we obtain
$$\begin{aligned}\sum_{q=i_0(r)}^{\theta+1} \binom{n}{r-1-5q}\binom{k}{q} &= \sum_{q=i_0(r)}^{\theta} \binom{n}{r-1-5q}\binom{k}{q} + \binom{n}{r-1-5(\theta+1)}\binom{k}{\theta+1} \\ &\leqslant \binom{m-4k+4\theta}{r} + \binom{n}{r-1-5(\theta+1)}\binom{k}{\theta+1}.\end{aligned}$$
Now with $a = n, b = r-1-5(\theta+1), c = 4(\theta+1)$ we get
$$\binom{n}{r-1-5(\theta+1)} \leqslant \binom{n+4(\theta+1)}{r-\theta-2}.$$

Now, using the last inequality together with

$$\binom{\nu}{\mu}+\binom{\nu}{\mu+1}=\binom{\nu+1}{\mu+1},$$

we have

$$\begin{aligned}\sum_{q=i_0(r)}^{\theta+1}\binom{n}{r-1-5q}\binom{k}{q} &\leqslant \binom{m-4k+4\theta}{r}+\binom{n+4(\theta+1)}{r-\theta-2}\binom{k}{\theta+1}\\ &\leqslant \binom{m-4k+4\theta}{r}+\binom{n+k+4\theta+4}{r-1}\\ &= \binom{m-4k+4\theta}{r}+\binom{m-1-4k+4\theta+4}{r-1}\\ &\leqslant \binom{m-4k+4\theta}{r}+\binom{m-4k+4\theta+3}{r-1}+\binom{m-4k+4\theta}{r-1}\\ &\leqslant \binom{m-4k+4\theta+1}{r}+\binom{m-4k+4\theta+3}{r-1}\\ &\leqslant \binom{m-4k+4\theta+3}{r}+\binom{m-4k+4\theta+3}{r-1}\\ &\leqslant \binom{m-4k+4(\theta+1)}{r},\end{aligned}$$

which completes the proof of Lemma 2.27.

□

Now we are ready to complete the proof of the estimates (2.73) for $\partial_t^j\partial_x^l u(x,t)$.

Proof. (Of Lemma 2.25)
We will prove (2.73) by induction. Let $j=0$, for $l=0$, it follows from (2.64) and the definition of M in (2.72) that

$$|u(x,t)|\le C\le MM_0,\quad \forall x\in\mathbb{R},\ t\in[0,T].$$

Similarly, for $l=1$, we have

$$|\partial_x u(x,t)|\le C^2\leqslant MM_1,\ \forall x\in\mathbb{R},\ t\in[0,T].$$

By (2.64) and (2.70), for $l\ge 2$, there exists $\varepsilon_0>0$ such that for any $0<\varepsilon\leqslant\varepsilon_0$, we have

$$|\partial_x^l u(x,t)|\le C^{l+1}(l!)^\sigma\le M_l\le MM_l,\ \forall x\in\mathbb{R},\ t\in[0,T].$$

This completes the proof of (2.73) for $j=0$ and $l\in\{0,1,...\}$.
Next, we will assume that (2.73) is true for $0\le q\le j$ and $l\in\{0,1,...\}$ and we will prove it for $q=j+1$ and $l\in\{0,1,...\}$.
We begin by noting that

$$\begin{aligned}|\partial_t^{j+1}\partial_x^l u(x,t)| &= |\partial_t^j\partial_x^l(\partial_t u(x,t))|\\ &\leqslant |\partial_t^j\partial_x^{l+5}u(x,t)|+|\partial_t^j\partial_x^{l+3}u(x,t)|+|\partial_t^j\partial_x^{l+1}u(x,t)|+|\partial_t^j\partial_x^l(\partial_x u^2(x,t))|.\end{aligned}$$

Using the induction hypotheses and the condition $M \geqslant 2$, we estimate the second term $\partial_t^j \partial_x^{l+5} u(x,t)$, $\partial_t^j \partial_x^{l+3} u(x,t)$ and $\partial_t^j \partial_x^{l+1} u(x,t)$ as follows

$$\begin{aligned} |\partial_t^j \partial_x^{l+5} u(x,t)| &\leqslant M^{2j+1} M_{l+5+5j} = M^{-2} M^{2(j+1)+1} M_{l+5(j+1)} \\ &\leqslant \frac{1}{4} M^{2(j+1)+1} M_{l+5(j+1)}, \end{aligned} \tag{2.77}$$

and

$$\begin{aligned} |\partial_t^j \partial_x^{l+3} u(x,t)| &\leqslant M^{2j+1} M_{l+3+5j} = M^{-2} M^{2(j+1)+1} M_{l+5j+3)} \\ &\leqslant \frac{\varepsilon^2}{4} M^{2(j+1)+1} M_{l+5(j+1)}, \end{aligned} \tag{2.78}$$

and

$$|\partial_t^j \partial_x^{l+1} u(x,t)| \leqslant M^{2j+1} M_{l+1+5j} \leqslant \frac{\varepsilon^4}{4} M^{2(j+1)+1} M_{l+5(j+1)}. \tag{2.79}$$

Next, we estimate the non-linear term , using Leibniz's formula we write $\partial_t^j \partial_x^l (\partial_x u^2(x,t))$ as

$$\begin{aligned} |\partial_t^j \partial_x^{l+1} (u^2)| &\leqslant \sum_{p=0}^{l+1} \sum_{q=0}^{j} \binom{l+1}{p} \binom{j}{p} |\partial_t^{j-q} \partial_x^{l+1-p} u| |\partial_t^q \partial_x^p u| \\ &\leqslant \sum_{p=0}^{l+1} \sum_{q=0}^{j} \binom{l+1}{p} \binom{j}{p} M^{2(j-q)+1} M_{l+1-p+5(j-q)} M^{2q+1} M_{p+5q} \\ &= M^{2(j+1)} \sum_{p=0}^{l+1} \sum_{q=0}^{j} \binom{l+1}{p} \binom{j}{p} M_{l+1-p+5(j-q)} M_{p+5q}. \end{aligned}$$

Next, using Lemma 2.26, with $n = l+1, k = j, L_j = M_j, m = l+1+5j$, we obtain

$$\begin{aligned} &\sum_{p=0}^{l+1} \sum_{q=0}^{j} \binom{l+1}{p} \binom{j}{p} M_{l+1-p+5(j-q)} M_{p+5q} \\ &\leqslant \sum_{r=1}^{m} \binom{m}{r} L_r L_{m-r} \leq (M_0 + \varepsilon) M_m \\ &= (M_0 + \varepsilon) M_{l+5j+1}, \end{aligned}$$

then

$$\begin{aligned} |\partial_t^j \partial_x^{l+1} (u^2)| &\leqslant M^{2(j+1)} (M_0 + \varepsilon) M_{l+5j+1} \\ &\leqslant M^{-2} M^{2(j+1)+1} \varepsilon^4 (M_0 + \varepsilon) M_{l+5(j+1)} \\ &\leqslant \frac{\varepsilon^4 (M_0 + \varepsilon)}{4} M^{2(j+1)+1} M_{l+5(j+1)}. \end{aligned}$$

Noting that in the last inequality we have used the fact that $l+5j+1 \geq 2$, since we are assuming that either $j \neq 0$ or $l \neq 0$.

Now, choosing $\varepsilon \leq \varepsilon_0 = \left(\frac{1}{(M_0+\varepsilon)}\right)^{\frac{1}{4}} < 1$ to obtain

$$\varepsilon^4(M_0+\varepsilon) \leq \varepsilon^4(M_0+1) \leq (M_0+1)\left(\frac{1}{(M_0+1)}\right) = 1.$$

Hence,

$$|\partial_t^j \partial_x^{l+1}(u^2)| \leq \frac{1}{4} M^{2(j+1)+1} M_{l+5(j+1)}. \tag{2.80}$$

Combining this estimate with the estimates (2.77),(2.78) and (2.79) yields

$$|\partial_t^{j+1} \partial_x^l u(x,t)| \quad = M^{2(j+1)+1} M_{l+5(j+1)}.$$

Which completes the proof of Lemma 2.25.

□

2.6.3 Proof of Theorem 2.22

By Lemma 2.25, we have

$$|\partial_t^j \partial_x^l u(x,t)| \leqslant M^{2j+1} M_{l+5j},\ j \in \{0,1,2,...\},\ l \in \{0,1,2,...\},$$

where

$$M_q = \varepsilon^{1-q} \frac{c(q!)^\sigma}{(q+1)^2},\ q = 1,2,....$$

Applying this inequality for $j \in \{1,2,...\}$ and $l = 0$ gives

$$\begin{aligned} |\partial_t^j u(x,t)| &\leqslant M^{2j+1} M_{5j} = M M^{2j} \varepsilon^{1-5j} \frac{c((5j)!)^\sigma}{(5j+1)^2} \\ &\leqslant M\varepsilon c \left(\frac{M^2}{\varepsilon^5}\right)^j ((5j)!)^\sigma \\ &\leqslant L_0 L^j ((5j)!)^\sigma \\ &\leqslant L_0 L^j A^{5\sigma j} ((j!)^5)^\sigma \\ &\leqslant A_0^{j+1} (j!)^{5\sigma}, \end{aligned} \tag{2.81}$$

where $L_0 = M\varepsilon c,\ L = \frac{M^2}{\varepsilon^5}$ since $(5j)! \leqslant A^{5j}(j!)^5$ for $A > 0$ and $A_0 = max\{L_0, LA^{5\sigma}\}$. We also have from (2.73) for $l = j = 0$, that

$$|u(x,t)| \leq MM_0 = M\frac{c}{8}, \quad \forall x \in \mathbb{R},\ t \in [0,T]. \tag{2.82}$$

Setting $C = max\{M\frac{c}{8}, A_0\}$, it follows from (2.81) and (2.82) that for $j \in \{0,1,2,...\}$, we have

$$|\partial_t^j u(x,t)| \leqslant C^{j+1}(j!)^{5\sigma}, \quad \forall x \in \mathbb{R},\ t \in [0,T].$$

Hence, $u \in G^{5\sigma}$ in t.
Which completes the proof of Theorem 2.22.

Chapter 3

Coupled system of m-KdV equations[1]

3.1 Introduction

In this chapter, we study the initial value problem associated with a system consisting modified Korteweg-de Vries-type equations

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(uv^2) = 0, \\ \partial_t v + \beta \partial_x^3 v + \partial_x(u^2 v) = 0, \\ u(x,0) = u_0(x), \;\; v(x,0) = v_0(x), \end{cases} \tag{3.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions and $0 < \beta < 1$ is a constant.

This system was derived by Gear and Grimshaw [23] as a model to describe the strong interaction of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of Korteweg-de Vries equations with both linear and nonlinear coupling terms.

For $\beta = 1$, the system (3.1) reduces to a special case of a broad class of nonlinear evolution equations considered by Ablowiz et al. [1] in the inverse scattering context. In this case, the well-posedness issues along with the existence and stability of solitary waves for this system are widely studied in the literature.

3.2 Function spaces

The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)$, for $s, b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, subjected to the norm

$$\|w\|^2_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}} (1+|\xi|)^{2s} (1+|\tau - \beta\xi^3|)^{2b} \mid \widehat{w}(\xi,\tau) \mid^2 d\xi d\tau,$$

and

$$\|w\|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{2\delta|\xi|^{1/\sigma}} (1+|\xi|)^{2s} (1+|\tau - \xi^3|)^{2b} \mid \widehat{w}(\xi,\tau) \mid^2 d\xi d\tau.$$

[1] A. Boukarou, K. Guerbati, Kh. Zennir, S. Alodhaibi, S. Alkhalaf.WellPosedness and Time Regularity for a System of Modified Kortewegde Vries-Type Equations in Analytic Gevrey Spaces. Mathematics 2020, 8, 809,doi.org/10.3390/math8050809.

For any interval I, we define the localized spaces $X^{\beta,I}_{\sigma,\delta,s,b} = X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}\times I)$ with norm

$$\|w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}\times I)} = inf\left\{\|W\|_{X^{\beta}_{\delta,s,b}}; W|_{\mathbb{R}\times I} = w\right\},$$

and $X^{I}_{\sigma,\delta,s,b}$ with norm

$$\|w\|_{X_{\delta,s,b}(\mathbb{R}\times I)} = inf\left\{\|W\|_{X_{\delta,s,b}}; W|_{\mathbb{R}\times I} = w\right\}.$$

It is well known that the spaces $X^{\beta}_{\sigma,\delta,s,b}$ is continuously embedded in $C\left([0,T], G^{\sigma,\delta,s}(\mathbb{R})\right)$, provided $b > 1/2$.

3.3 Linear Estimates

By using Duhamel's formula of the Cauchy problems (3.1), we define the following application with the use of cutoff functions satisfying $\psi \in C_0^{\infty}$, $\psi = 1$ in $[-1,1]$ and supp$\psi \subset [-2,2]$, $\psi_T(t) = \psi(\frac{t}{T})$, we consider the operator Λ, Γ given by

$$\begin{cases} \Lambda[u,v](t) = \psi(t)S(t)u_0 - \psi_T(t)\displaystyle\int_0^t S(t-t')F_1(t')dt', \\ \\ \Gamma[u,v](t) = \psi(t)S_{\beta}(t)v_0 - \psi_T(t)\displaystyle\int_0^t S_{\beta}(t-t')F_2(t')dt', \end{cases} \tag{3.2}$$

where $S(t) = e^{-t\partial_x^3}$ and $S_{\beta}(t) = e^{-t\beta\partial_x^3}$ are the unitary groups associated with the linear problems. The nonlinear terms define by $F_1 = \partial_x(uv^2)$, $F_2 = \partial_x(u^2v)$.

Remark 3.1. The evidence for linear estimates is the same as in Chapter 2.

Lemma 3.2. *([18]). Let $s,b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$. For some constant $C > 0$, we have*

$$\|\psi(t)S(t)u_0\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C\,\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{R})},$$

$$\|\psi(t)S_{\beta}(t)v_0\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C\,\|v_0\|_{G^{\sigma,\delta,s}(\mathbb{R})}.$$

Lemma 3.3. *([18]). Let $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leq 0 \leq b < b'+1$, $0 \leq T \leq 1$, $\delta > 0$ and $\sigma \geq 1$, then for some constant $C > 0$, we have*

$$\left\|\psi_T(t)\int_0^t S(t-t')F_1(x,t')\mathrm{d}t'\right\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq CT^{1-b+b'}\|F_1\|_{X_{\sigma,\delta,s,b'}(\mathbb{R}^2)},$$

and

$$\left\|\psi_T(t)\int_0^t S_{\beta}(t-t')F_2(x,t')\mathrm{d}t'\right\|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq CT^{1-b+b'}\|F_2\|_{X^{\beta}_{\sigma,\delta,s,b'}(\mathbb{R}^2)}.$$

Lemma 3.4. *Let $\Theta \in S(\mathbb{R})$ be a Schwartz function in time, $s \in \mathbb{R}$ and $\delta \geq 0$. If $-\frac{1}{2} < b \leq b' < \frac{1}{2}$, then for any $T > 0$, we have,*

$$\|\Theta_T(t)w\|_{X_{\delta,s,b}(\mathbb{R}^2)} \leq CT^{b'-b}\|w\|_{X_{\delta,s,b'}(\mathbb{R}^2)},$$

and

$$\|\Theta_T(t)w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}^2)} \leq CT^{b'-b}\|w\|_{X^{\beta}_{\delta,s,b'}(\mathbb{R}^2)},$$

where C depends only on b and b'.

Lemma 3.5. *([77]) Let $s \in \mathbb{R}$, $\delta \geq 0$, $-\frac{1}{2} < b < \frac{1}{2}$ and $T > 0$. Then, for any time interval $I \subset [0,T]$, we have*

$$\|\chi_I(t)w\|_{X_{\delta,s,b}(\mathbb{R}^2)} \leq C\|w\|_{X^T_{\delta,s,b}},$$

and

$$\|\chi_I(t)w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}^2)} \leq C\|w\|_{X^{\beta,T}_{\delta,s,b}},$$

where $\chi_I(t)$ is the characteristic function of I, and C depends only on b.

3.4 Trilinear estimates

The following Lemma states the desired trilinear estimate.

Lemma 3.6. *([18]). Let $s > -\frac{1}{2}$, $b > \frac{1}{2}$ and b' be as in Lemma 3.3. Then*

$$\| \partial_x(uv^2) \|_{X_{s,b'}(\mathbb{R}^2)} \leq C \| u \|_{X_{s,b}} \| v \|^2_{X^{\beta}_{s,b}(\mathbb{R}^2)},$$

and

$$\| \partial_x(u^2v) \|_{X^{\beta}_{s,b'}(\mathbb{R}^2)} \leq C \| u \|^2_{X_{s,b}} \| v \|_{X^{\beta}_{s,b}(\mathbb{R}^2)} .$$

Remark 3.7. Setting

$$f_i(\xi,\tau) = (1+|\xi|)^s(1+|\tau-\xi^3|)^{b_1}\widehat{u_i}(\xi,\tau), \;\; i = 1,2,$$

and

$$g_i(\xi,\tau) = (1+|\xi|)^s(1+|\tau-\beta\xi^3|)^{b_1}\widehat{v_i}(\xi,\tau), \;\; i = 1,2,$$

the estimate of Lemma 3.6 can be rewritten as

$$\|\partial_x(u_1v_1v_2)\|_{X_{s,b'}(\mathbb{R}^2)}$$
$$=\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-\xi^3|)^{b'}}\widehat{u_1v_1v_2}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}$$
$$=C\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-\xi^3|)^{b'}}\int_{\mathbb{R}^4}\widehat{u_1}(\xi_1,\tau_1)\widehat{v_1}(\xi_2,\tau_2)\widehat{v_2}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}$$
$$=C\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-\xi^3|)^{b'}}\int_{\mathbb{R}^4}\frac{\widehat{f_1}(\xi_1,\tau_1)}{(1+|\xi|)^s(1+|\tau_1-\xi_1^3|)^{b_1}}\frac{\widehat{g_1}(\xi_2,\tau_2)}{(1+|\xi|)^s(1+|\tau_2-\beta\xi_2^3|)^{b_1}}\right.$$
$$\left.\cdot\frac{\widehat{g_2}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^{b_1}}d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}$$
$$\leqslant C\|f_1\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\|g_1\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\|g_2\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.$$

and

$$\|\partial_x(u_1u_2v_1)\|_{X^\beta_{s,b'}(\mathbb{R}^2)}$$
$$=\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-\beta\xi^3|)^{b'}}\widehat{u_1u_2v_1}(\xi,\tau)\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}$$
$$=C\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-\beta\xi^3|)^{b'}}\int_{\mathbb{R}^4}\widehat{u_1}(\xi_1,\tau_1)\widehat{u_2}(\xi_2,\tau_2)\widehat{v_1}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}$$
$$=C\left\|\frac{\xi(1+|\xi|)^s}{(1+|\tau-\beta\xi^3|)^{b'}}\int_{\mathbb{R}^4}\frac{\widehat{f_1}(\xi_1,\tau_1)}{(1+|\xi|)^s(1+|\tau_1-\xi_1^3|)^{b_1}}\frac{\widehat{f_2}(\xi_2,\tau_2)}{(1+|\xi|)^s(1+|\tau_2-\xi_2^3|)^{b_1}}\right.$$
$$\left.\cdot\frac{\widehat{g_1}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^{b_1}}d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}$$
$$\leqslant C\|f_1\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\|f_1\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\|g_1\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.$$

Lemma 3.8. *Let $s>-\frac{1}{2}$, $\sigma\geq 1$, $\delta>0$, $b>\frac{1}{2}$ and b' be as in Lemma 3.3. Then*

$$\|\,\partial_x(uv^2)\,\|_{X_{\sigma,\delta,s,b'}(\mathbb{R}^2)}\leq C\,\|\,u\,\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\|\,v\,\|^2_{X^\beta_{\sigma,\delta,s,b}(\mathbb{R}^2)},$$

and

$$\|\,\partial_x(u^2v)\,\|_{X^\beta_{\sigma,\delta,s,b'}(\mathbb{R}^2)}\leq C\,\|\,u\,\|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)}\|\,v\,\|_{X^\beta_{\sigma,\delta,s,b}(\mathbb{R}^2)}\,.$$

Proof.

$$\widehat{A^{\delta,\sigma}w}^{x}(\xi,t)=e^{\delta|\xi|^{1/\sigma}}\widehat{w}^{x}(\xi,t), \tag{3.3}$$

$$\begin{aligned}
e^{\delta|\xi|^{1/\sigma}}\widehat{uvv} &= (\sqrt{2\pi})^{-2}e^{\delta|\xi|^{1/\sigma}}\widehat{u}*\widehat{v}*\widehat{v}\\
&\leq (\sqrt{2\pi})^{-2}\int_{\mathbb{R}^4} e^{\delta|\xi-\xi_1|^{1/\sigma}}\widehat{u}(\xi-\xi_1,\tau-\tau_1)e^{\delta|\xi_1-\xi_2|^{1/\sigma}}\widehat{v}(\xi_1-\xi_2,\tau_1-\tau_2)\\
&e^{\delta|\xi_2|^{1/\sigma}}\widehat{v}(\xi_2,\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&= \widehat{A^{\delta,\sigma}uA^{\delta,\sigma}vA^{\delta,\sigma}v},
\end{aligned}$$

and

$$\begin{aligned}
e^{\delta|\xi|^{1/\sigma}}\widehat{uuv} &= (\sqrt{2\pi})^{-2}e^{\delta|\xi|^{1/\sigma}}\widehat{u}*\widehat{u}*\widehat{v}\\
&\leq (\sqrt{2\pi})^{-2}\int_{\mathbb{R}^4} e^{\delta|\xi-\xi_1|^{1/\sigma}}\widehat{u}(\xi-\xi_1,\tau-\tau_1)e^{\delta|\xi_1-\xi_2|^{1/\sigma}}\widehat{u}(\xi_1-\xi_2,\tau_1-\tau_2)\\
&e^{\delta|\xi_2|^{1/\sigma}}\widehat{v}(\xi_2,\tau_2)d\xi_1 d\xi_2 d\tau_1 d\tau_2\\
&= \widehat{A^{\delta,\sigma}uA^{\delta,\sigma}uA^{\delta,\sigma}v},
\end{aligned}$$

since $\delta \mid \xi \mid^{1/\sigma} \leq \delta \mid \xi-\xi_1 \mid^{1/\sigma} + \delta \mid \xi_1-\xi_2 \mid^{1/\sigma} + \delta \mid \xi_2 \mid^{1/\sigma}, \quad \forall \sigma \geq 1$. Then

$$\begin{aligned}
\| \partial_x(uv^2) \|_{X_{\sigma,\delta,s,b'}(\mathbb{R}^2)} &= \| e^{\delta|\xi|^{1/\sigma}}(1+|\xi|)^s(1+|\tau-\xi^3|)^b\widehat{\partial_x(uvv)}(\xi,\tau) \|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&\leq \| \partial_x(A^{\delta,\sigma}uA^{\delta,\sigma}vA^{\delta,\sigma}v) \|_{X_{s,b'}(\mathbb{R}^2)},
\end{aligned}$$

and

$$\begin{aligned}
\| \partial_x(u^2v) \|_{X^{\beta}_{\sigma,\delta,s,b'}(\mathbb{R}^2)} &= \| e^{\delta|\xi|^{1/\sigma}}(1+|\xi|)^s(1+|\tau-\beta\xi^3|)^b\widehat{\partial_x(uuv)}(\xi,\tau) \|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&\leq \| \partial_x(A^{\delta,\sigma}uA^{\delta,\sigma}uA^{\delta,\sigma}v) \|_{X^{\beta}_{s,b'}(\mathbb{R}^2)}.
\end{aligned}$$

Now, by using Lemma 3.6, there exists $C>0$ such that

$$\begin{aligned}
\| \partial_x(A^{\delta,\sigma}uA^{\delta,\sigma}vA^{\delta,\sigma}v) \|_{X_{s,b'}(\mathbb{R}^2)} &\leq C \| A^{\delta,\sigma}u \|_{X_{s,b}} \| A^{\delta,\sigma}v \|^2_{X^{\beta}_{s,b}(\mathbb{R}^2)}\\
&= C \| u \|_{X_{\sigma,\delta,s,b}} \| v \|^2_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)},
\end{aligned}$$

and

$$\begin{aligned}
\| \partial_x(A^{\delta,\sigma}uA^{\delta,\sigma}uA^{\delta,\sigma}v) \|_{Xs,b'(\mathbb{R}^2)} &\leq C \| A^{\delta,\sigma}u \|^2_{X_{s,b}} \| A^{\delta,\sigma}v \|_{X^{\beta}_{s,b}(\mathbb{R}^2)}\\
&= C \| u \|^2_{X_{\sigma,\delta,s,b}} \| v \|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)}.
\end{aligned}$$

□

3.5 Local well-posedness

Xavier Carvajal and Mahendra Panthee [18] proved the IVP (3.1) is locally well-posed for data in $(u_0,v_0)\in H^s(\mathbb{R})\times H^s(\mathbb{R}), s>-\frac{1}{2}$. We improved this result, states local well-posedness in $G^{\sigma,\delta,s}(\mathbb{R})\times G^{\sigma,\delta,s}(\mathbb{R}), s>-\frac{1}{2}, \sigma\geq 1$ and $\delta>0$.

Theorem 3.9. *Let $s>-\frac{1}{2}, 0<\beta<1, \sigma\geq 1, \delta>0$ and $(u_0,v_0)\in G^{\sigma,\delta,s}(\mathbb{R})\times G^{\sigma,\delta,s}(\mathbb{R})$, Then for some real number $b>\frac{1}{2}$ and a constant $T=T(\|(u_0,v_0)\|_{G^{\sigma,\delta,s}(\mathbb{R})\times G^{\sigma,\delta,s}(\mathbb{R})})$, such that (3.1) admits a unique local solution $(u,v)\in C\left([0,T],G^{\sigma,\delta,s}(\mathbb{R})\right)\times C\left([0,T],G^{\sigma,\delta,s}(\mathbb{R})\right)$. Moreover, the map $(u_0,v_0)\to(u,v)$ is Lipschitz continuous from $G^{\sigma,\delta,s}(\mathbb{R})\times G^{\sigma,\delta,s}(\mathbb{R})$ to $C\left([0,T],G^{\sigma,\delta,s}(\mathbb{R})\right)\times C\left([0,T],G^{\sigma,\delta,s}(\mathbb{R})\right)$.*

Corollary 3.10. *Let $\delta>0$ and $s>-\frac{1}{2}$. Then for any $(u_0,v_0)\in G^{\delta,s}(\mathbb{R})\times G^{\delta,s}(\mathbb{R})$, there exists $T=T(\|(u_0,v_0)\|_{G^{\delta,s}(\mathbb{R})\times G^{\delta,s}}(\mathbb{R}))$ and unique solution (u,v) of (3.1) on $[0,T]$ such that*

$$(u,v)\in C\left([0,T],G^{\delta,s}(\mathbb{R})\right)\times C\left([0,T],G^{\delta,s}(\mathbb{R})\right).$$

Moreover the solution depends on (u_0,v_0), where

$$T=\frac{c_0}{(1+\parallel(u_0,v_0)\parallel^2_{G^{\delta,s}(\mathbb{R})\times G^{\delta,s}(\mathbb{R})})^{\frac{1}{\varepsilon}}}. \tag{3.4}$$

Furthermore, the solution satisfies

$$\|(u,v)\|_{X_{\delta,s,b}(\mathbb{R}^2)\times X^\beta_{\delta,s,b}(\mathbb{R}^2)}\leq 2C\|(u_0,v_0)\|_{G^{\delta,s}(\mathbb{R})\times G^{\delta,s}(\mathbb{R})},\quad b=\frac{1}{2}+\varepsilon, \tag{3.5}$$

with constant c_0, and $C>0$ depending only on s and b.

3.5.1 Existence of solution

We are now ready to estimate all the terms in (3.2) by using the trilinear estimates in the above Lemmas. We define spaces

$$B_{\sigma,\delta,s,b}=X_{\sigma,\delta,s,b}(\mathbb{R}^2)\times X^\beta_{\sigma,\delta,s,b}(\mathbb{R}^2)\quad\text{and}\quad N^{\sigma,\delta,s}=G^{\sigma,\delta,s}(\mathbb{R})\times G^{\sigma,\delta,s}(\mathbb{R})$$

with norms

$$\|(u,v)\|_{B_{\sigma,\delta,s,b}}=max\left\{\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)},\|v\|_{X^\beta_{\sigma,\delta,s,b}(\mathbb{R}^2)}\right\},$$

and similar for $N^{\sigma,\delta,s}$.

Lemma 3.11. *Let $s>-\frac{1}{2}$, $\sigma\geq 1$ and $\delta>0$, $b>\frac{1}{2}$. Then, for all $(u_0,v_0)\in N^{\sigma,\delta,s}$ and $0<T<1$, with some constant $C>0$, we have*

$$\parallel(\Lambda[u,v],\Gamma[u,v])\parallel_{B_{\sigma,\delta,s,b}}\leq C\left(\parallel(u_0,v_0)\parallel_{N^{\sigma,\delta,s}}+T^\varepsilon\parallel(u,v)\parallel^3_{B_{\sigma,\delta,s,b}}\right), \tag{3.6}$$

and

$$\begin{aligned}
&\|(\Lambda[u,v]-\Lambda[u^*,v^*],\Gamma[u,v]-\Gamma[u^*,v^*])\|_{B_{\sigma,\delta,s,b}} \\
&\leq CT^{\varepsilon}\|(u-u^*,v-v^*)\|_{B_{\sigma,\delta,s,b}}\Big(\|(u,v)\|^2_{B_{\sigma,\delta,s,b}} \\
&+\|(u,v)\|_{B_{\sigma,\delta,s,b}}\|(u^*,v^*)\|_{B_{\sigma,\delta,s,b}}+\|(v^*,v^*)\|^2_{B_{\sigma,\delta,s,b}}\Big).
\end{aligned} \tag{3.7}$$

for all $(u,v),(u^*,v^*)\in B_{\sigma,\delta,s,b}$.

Proof. To prove estimate (3.6), we follow

$$\begin{aligned}
\| \Lambda[u,v] \|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} &\leq C \| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{R})} + CT^{\varepsilon} \| u \|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \| v \|^2_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&\leq C \| (u_0,v_0) \|_{N^{\sigma,\delta,s}} + CT^{\varepsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}},
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\| \Gamma[u,v] \|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)} &\leq C \| v_0 \|_{G^{\sigma,\delta,s}(\mathbb{R})} + CT^{\varepsilon} \| u \|^2_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \| v \|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)} \\
&\leq C \| (u_0,v_0) \|_{N^{\sigma,\delta,s}} + CT^{\varepsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}} .
\end{aligned} \tag{3.9}$$

Therefore, from (3.8) and (3.9), we obtain

$$\| (\Lambda[u,v],\Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq C\Big(\| (u_0,v_0) \|_{N^{\sigma,\delta,s}} + T^{\varepsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}}\Big).$$

For the estimate (3.7), we observe that

$$\Lambda[u,v]-\Lambda[u^*,v^*] = \psi_T(t)\int_0^t S(t-t')\partial_x\left(uv^2-u^*v^{*2}\right)(x,t')dt',$$

and

$$\Gamma[u,v]-\Gamma[u^*,v^*] = \psi_T(t)\int_0^t S_{\beta}(t-t')\partial_x\left(u^2v-u^{*2}v^*\right)(x,t')dt',$$

where

$$\omega = \partial_x(u^2v-u^{*2}v^*) = \partial_x\left[v(u+u^*)(u-u^*)+u^{*2}(v-v^*)\right],$$

and

$$\omega' = \partial_x(uv^2-u^*v^{*2}) = \partial_x\left[u(v+v^*)(v-v^*)+v^{*2}(u-u^*)\right].$$

□

We will show that $\Lambda\times\Gamma$ is a contraction on the ball $\mathbb{B}(0,R)$ to $\mathbb{B}(0,R)$.

Lemma 3.12. *Let* $s\geq -\frac{1}{4}$, $\sigma\geq 1$, $\delta>0$ *and* $b>\frac{1}{2}$. *Then, for all* $(u_0,v_0)\in N^{\sigma,\delta,s}$, *such that the map* $\Lambda\times\Gamma:\mathbb{B}(0,R)\to\mathbb{B}(0,R)$ *is a contraction, where* $\mathbb{B}(0,R)$ *is given by*

$$\mathbb{B}(0,R) = \{(u,v)\in B_{\sigma,\delta,s,b};\|(u,v)\|_{B_{\sigma,\delta,s,b}}\leq R\},$$

with $R=2C\|(u_0,v_0)\|_{N^{\sigma,\delta,s}}$.

Proof. From Lemma 3.11, for all $(u,v) \in \mathbb{B}(0,R)$, we have

$$\| (\Lambda[u,v],\Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq C \| (u_0,v_0) \|_{N^{\sigma,\delta,s}} + CT^{\varepsilon} \| (u,v) \|^3_{B_{\sigma,\delta,s,b}} \leq \frac{R}{2} + CT^{\varepsilon}R^3.$$

We choose T sufficiently small such that $T^{\varepsilon} \leq \frac{1}{4CR^2}$, hence,

$$\| (\Lambda[u,v],\Gamma[u,v]) \|_{B_{\sigma,\delta,s,b}} \leq R, \quad \forall (u,v) \in \mathbb{B}(0,R).$$

Thus, $\Lambda \times \Gamma$ maps $\mathbb{B}(0,R)$ into $\mathbb{B}(0,R)$, which is a contraction, since

$$\begin{aligned}
&\| (\Lambda[u,v]-\Lambda[u^*,v^*],\Gamma[u,v]-\Gamma[u^*,v^*]) \|_{B_{\sigma,\delta,s,b}} \\
&\leq CT^{\varepsilon} \| (u-u^*,v-v^*) \|_{B_{\sigma,\delta,s,b}} \Big(\| (u,v) \|^2_{B_{\sigma,\delta,s,b}} + \| (u,v) \|_{B_{\sigma,\delta,s,b}} \| (u^*,v^*) \|_{B_{\sigma,\delta,s,b}} \\
&+ \| (v^*,v^*) \|^2_{B_{\sigma,\delta,s,b}} \Big), \\
&\leq 3CT^{\varepsilon}R^2 \| (u-u^*,v-v^*) \|_{B_{\sigma,\delta,s,b}} \\
&\leq \frac{3}{4} \| (u-u^*,v-v^*) \|_{B_{\sigma,\delta,s,b}},
\end{aligned}$$

for all $(u,v) \in \mathbb{B}(0,R)$.

□

To proof of the uniqueness see [6].

3.5.2 Continuous dependence of the initial data

To prove continuous dependence of the initial data we will prove the following.

Lemma 3.13. *Let $s > -\frac{1}{2}$ and $\sigma \geq 1$, $\delta > 0$, $b > \frac{1}{2}$. Then, for all $(u_0,v_0),(u_0^*,v_0^*) \in N^{\sigma,\delta,s}$, if (u,v) and (u^*,v^*) are two solutions to (3.1) corresponding to initial data (u_0,v_0) and (u_0^*,v_0^*), We have*

$$\|(u-u^*,v-v^*)\|_{C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))^2} \leq 2C_0C\|(u_0-u_0^*,v_0-v_0^*)\|_{N^{\sigma,\delta,s}}.$$

Proof. If (u,v) and (u^*,v^*) are two solutions to (3.1), corresponding to initial data (u_0,v_0) and (u_0^*,v_0^*), we have

$$\|u-u^*\|_{C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))} \leq C_0\|u-u^*\|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)},$$

and

$$\|v-v^*\|_{C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))} \leq C_0\|v-v^*\|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)}.$$

By taking $(u,v),(u^*,v^*) \in \mathbb{B}(0,R)$ and $T^{\varepsilon} \leq \frac{1}{4CR}$,

$$\| u-u^* \|_{X_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C \| (u_0-u_0^*,v_0-v_0^*) \|_{N^{\sigma,\delta,s}} + \frac{3}{4} \| (u-u^*,v-v^*) \|_{B_{\sigma,\delta,s,b}}.$$

And

$$\| v-v^* \|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}^2)} \leq C \| (u_0-u_0^*, v_0-v_0^*) \|_{N^{\sigma,\delta,s}} + \frac{3}{4} \| (u-u^*, v-v^*) \|_{B_{\sigma,\delta,s,b}} .$$

Thus

$$\| (u-u^*, v-v^*) \|_{B_{\sigma,\delta,s,b}} \leq 4C \| (u_0-u_0^*, v_0-v_0^*) \|_{N^{\sigma,\delta,s}},$$

then

$$\|(u-u^*, v-v^*)\|_{C([0,T],G^{\sigma,\delta,s}(\mathbb{R}))^2} \leq 4C_0C\|(u_0-u_0^*, v_0-v_0^*)\|_{N^{\sigma,\delta,s}}.$$

□

This complete the prove of Theorem 3.9.

3.6 Lower bound for radius of spatial analyticity

3.6.1 Approximate Conservation Law

In the view of the Paley-Wiener Theorem, it is natural to take initial data in $G^{\delta,s}(\mathbb{R}) \times G^{\delta,s}(\mathbb{R})$ to obtain the best behavior of solution and may be extended to be globally in time. It means that given $(u_0, v_0) \in G^{\delta,s}(\mathbb{R}) \times G^{\delta,s}(\mathbb{R})$ for some initial radius $\delta > 0$, we then estimate the behavior of the radius of analyticity $\delta(T)$ over time.

The second result for problem (3.1) is given in the next Theorem.

Theorem 3.14. *Let $s > -\frac{1}{2}, 0 < \beta < 1$ and $\delta_0 > 0$. Assume that $(u_0, v_0) \in G^{\delta,s}(\mathbb{R}) \times G^{\delta,s}(\mathbb{R})$, then the solution in Corollary 3.10 can be extended to be global in time and for any $T > 0$, we have*

$$(u,v) \in C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right) \times C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right),$$

with

$$\delta(T) = \min\left\{\delta_0, C_1 T^{-(2+\sigma_0)}\right\},$$

where $\sigma_0 > 0$ can be taken arbitrarily small and $C_1 > 0$ is a constant depending on w_0, δ_0, s and σ_0.

We start by recalling that

$$\mathscr{I}(u,v) = \int_{\mathbb{R}} (u^2+v^2)dx,$$

is conserved for a solution (u,v) of (3.1). Our goal in this section is to show an approximate conservation law for a solution to (3.1) based on the conservation the $L^2(\mathbb{R})$ norm of solution.

Theorem 3.15. *Let $\kappa \in [0, \frac{1}{2})$ and $0 < T_1 < T < 1$, T be as in Corollary 3.10 with $s = 0$, there exist $b = \frac{1}{2} + \varepsilon$ and $C > 0$, such that for any $\delta > 0$ and any solution $(u,v) \in B^{T_1}_{\delta,0,b}$ to the Cauchy problem (3.1) on the time interval $[0, T_1]$, we have the estimate*

$$\sup_{t\in[0,T_1]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.$$

Moreover, we have

$$\sup_{t\in[0,T_1]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^{\kappa}\|(u(0),v(0))\|^4_{N^{\delta,0}}.$$

We need the following estimate.

Lemma 3.16. *Given* $\kappa \in [0, -\frac{1}{2})$*, there exist* $b = \frac{1}{2} + \varepsilon$*,* $C > 0$ *and* $(u,v) \in B_{\delta,0,b}$*, we have*

$$\|(G_1, G_2)\|_{B_{0,b-1}} \leq C\delta^{\kappa} \|(u,v)\|^3_{B_{\delta,0,b}},$$

where

$$G_1 = \partial_x \left[(A^{\delta,1}uA^{\delta,1}vA^{\delta,1}v) - A^{\delta,1}(uv^2) \right],$$

and

$$G_2 = \partial_x \left[(A^{\delta,1}uA^{\delta,1}uA^{\delta,1}v) - A^{\delta,1}(u^2v) \right].$$

Proof. Let $L_1 = (A^{\delta,1}vA^{\delta,1}uA^{\delta,1}v) - A^{\delta,1}(uv^2)$. Then

$$\|G_1\|_{X_{0,b-1}(\mathbb{R}^2)} = \left\| \frac{\xi}{(1+|\tau - \xi^3|)^{1-b}} \widehat{L_1}(\xi,\tau) \right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \frac{|\xi|^2}{(1+|\tau-\xi^3|)^{2(1-b)}} \,|\, \widehat{L_1}(\xi,\tau)\,|^2 \, d\xi d\tau \right).$$

We shall calculate the Fourier transform of L_1

$$\begin{aligned}
\left|\widehat{L_1}(\xi,\tau)\right| &= \left|\widehat{(A^{\delta,1}vA^{\delta,1}uA^{\delta,1}v)} - \widehat{A^{\delta,1}(uv^2)}\right| = C\Big|(e^{\delta|\xi|}\widehat{u} * e^{\delta|\xi|}\widehat{v} * e^{\delta|\xi|}\widehat{v}*)(\xi,\tau) \\
&\quad - e^{\delta|\xi|}(\widehat{u} * \widehat{v} * \widehat{v}*)(\xi,\tau)\Big| \\
&= C\Big| \int_{\mathbb{R}^4} \Big(e^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1) e^{\delta|\xi_2|}\widehat{v}(\xi_2,\tau_2) e^{\delta|\xi-\xi_1-\xi_2|}\widehat{v}(\xi-\xi_1-\xi_2, \tau-\tau_1-\tau_2) \\
&\quad - e^{\delta|\xi|}\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi_2,\tau_2)\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2) \Big) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \Big| \\
&\leq C \int_{\mathbb{R}^4} \left(e^{\delta|\xi_1|} e^{\delta|\xi_2|} e^{\delta|\xi-\xi_1-\xi_2|} - e^{\delta|\xi|} \right) \\
&\quad \times |\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi_2,\tau_2)\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)| d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\end{aligned}$$

We will use Lemma 2.21 to prove the following corollary.

Corollary 3.17. *For* $\delta > 0$*,* $\theta \in [0,1]$ *and* $\xi, \xi_1, \xi_2 \in \mathbb{R}$ *we have*

$$e^{\delta|\xi_1|} e^{\delta|\xi_2|} e^{\delta|\xi-\xi_1-\xi_2|} - e^{\delta|\xi|} \leq \left[4\delta \frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{(1+|\xi|)} \right]^{\theta} e^{\delta|\xi_1|} e^{\delta|\xi_2|} e^{\delta|\xi-\xi_1-\xi_2|}.$$

Proof. If ξ_1, ξ_2 and $\xi - \xi_1 - \xi_2$ have the same sign, there is nothing to prove. Without any loss of generality, suppose $\xi_1 \geqslant 0$ and $\xi_2 \leqslant 0$. If $\xi_1 \leqslant 0$ and $\xi_2 \geqslant 0$, the change $\widetilde{\xi_1} = \xi_2$ and $\widetilde{\xi_2} = \xi_1$ will reduce the result to the previous case. If $\xi - \xi_1 - \xi_2 \geqslant 0$, writing $\alpha = \xi_1 + (\xi - \xi_1 - \xi_2) = \xi - \xi_2$ we have $\alpha \geqslant 0$ since $\xi - \xi_1 - \xi_2 \geqslant 0$ implies that $\xi - \xi_2 \geqslant \xi_1 \geqslant 0$. Using Lemma 2.21, we obtain

$$\begin{aligned} e^{\delta|\xi_1|}e^{\delta|\xi_2|}e^{\delta|\xi-\xi_1-\xi_2|}-e^{\delta|\xi|} &= e^{\delta|\alpha|}e^{\delta|\xi_2|}-e^{\delta|\alpha+\xi_2|} \\ &\leqslant [2\delta\min\{|\xi-\xi_2|,|\xi_2|\}]^{\theta}\, e^{\delta|\xi_1|}e^{\delta|\xi_2|}e^{\delta|\xi-\xi_1-\xi_2|}. \end{aligned}$$

Analogously, if $\xi-\xi_1-\xi_2\leqslant 0$, then taking $\lambda=\xi_2+(\xi-\xi_1-\xi_2)=\xi-\xi_1\leqslant 0$ we have

$$\begin{aligned} e^{\delta|\xi_1|}e^{\delta|\xi_2|}e^{\delta|\xi-\xi_1-\xi_2|}-e^{\delta|\xi|} &= e^{\delta|\lambda|}e^{\delta|\xi_1|}-e^{\delta|\lambda+\xi_1|} \\ &\leqslant [2\delta\min\{|\xi-\xi_1|,|\xi_1|\}]^{\theta}\, e^{\delta|\xi_1|}e^{\delta|\xi_2|}e^{\delta|\xi-\xi_1-\xi_2|}. \end{aligned}$$

Therefore, for

$$A=\begin{cases} \min\{|\xi-\xi_2|,|\xi_2|\},\ if\ \xi-\xi_1-\xi_2\geqslant 0, \\ \min\{|\xi-\xi_1|,|\xi_1|\},\ if\ \xi-\xi_1-\xi_2\leqslant 0, \end{cases}$$

we can write

$$e^{\delta|\xi_1|}e^{\delta|\xi_2|}e^{\delta|\xi-\xi_1-\xi_2|}-e^{\delta|\xi|} \leqslant [2\delta A]^{\theta}\, e^{\delta|\xi_1|}e^{\delta|\xi_2|}e^{\delta|\xi-\xi_1-\xi_2|}.$$

Using this inequality (see Lemma 2.21)

$$\min(|\xi-\xi_1|,|\xi_1|)\leq 2\frac{(1+|\xi-\xi_1|)(1+|\xi_1|)}{(1+|\xi|)}.$$

And now we can estimate A in the following way. If $\xi-\xi_1-\xi_2\geqslant 0$, then

$$A=\min\{|\xi-\xi_2|,|\xi_2|\}\leq 2\frac{(1+|\xi-\xi_2|)(1+|\xi_2|)}{(1+|\xi|)}.$$

Now observe that

$$\begin{aligned} 1+|\xi-\xi_2| &= 1+|\xi-\xi_1-\xi_2+\xi_1|\leqslant 1+|\xi-\xi_1-\xi_2|+|\xi_1| \\ &= (1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)-|\xi-\xi_1-\xi_2||\xi_1| \\ &\leqslant (1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|), \end{aligned}$$

which implies that

$$A\leq 2\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{(1+|\xi|)}.$$

On the other hand, if $\xi-\xi_1-\xi_2\leqslant 0$, we have

$$A=\min\{|\xi-\xi_1|,|\xi_1|\}\leq 2\frac{(1+|\xi-\xi_1|)(1+|\xi_1|)}{(1+|\xi|)}.$$

And the same procedure as above tells us that $1+|\xi-\xi_1| \leqslant (1+|\xi-\xi_1-\xi_2|)(1+|\xi_2|)$ and we can write

$$A \leq 2\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{(1+|\xi|)}.$$

In other words, we conclude that

$$A \leq 2\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{(1+|\xi|)},$$

and the result is proven.

□

For $\kappa \in [0,\frac{1}{2}) \subset [0,1]$, one can see that

$$\begin{aligned}
\|G_1\|^2_{X_{0,b-1}(\mathbb{R}^2)} &= \left\| \frac{\xi}{(1+|\tau-\xi^3|)^{1-b}} \widehat{L_1}(\xi,\tau) \right\|^2_{L^2_{\xi,\tau}(\mathbb{R}^2)} \\
&\leq (C4\delta)^{2\kappa} \int_{\mathbb{R}^2} \frac{|\ \xi\ |^2}{(1+|\tau-\xi^3|)^{2(1-b)}} \left[\int_{\mathbb{R}^4} \left(\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{(1+|\xi|)} \right)^{\kappa} \right. \\
&\times e^{\delta|\xi_1|} e^{\delta|\xi_2|} e^{\delta|\xi-\xi_1-\xi_2|} \\
&\left. \times |\widehat{u}(\xi_1,\tau_1)\widehat{v}(\xi_2,\tau_2)\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)| d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right]^2 d\xi d\tau \\
&= C(4\delta)^{2\kappa} \left\| \frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\xi^3|)^{1-b}} \int_{\mathbb{R}^4} e^{\delta|\xi_1|}(1+|\xi_1|)^{\kappa}\widehat{u}(\xi_1,\tau_1) e^{\delta|\xi_2|}(1+|\xi_2|)^{\kappa}\widehat{v}(\xi_2,\tau_2) \right. \\
&\left. \times e^{\delta|\xi-\xi_1-\xi_2|}(1+|\xi-\xi_1-\xi_2|)^{\kappa}\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right\|^2_{L^2_{\xi,\tau}(\mathbb{R}^2)}.
\end{aligned}$$

Now by taking $s=-\kappa\in(-\frac{1}{2},0]$, we obtain

$$\begin{aligned}
\|G_1\|_{X_{0,b-1}(\mathbb{R}^2)} &\le C(4\delta)^{\kappa}\left\|\frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\xi^3|)^{1-b}}\int_{\mathbb{R}^4}\frac{e^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1)}{(1+|\xi_1|)^s}\frac{e^{\delta|\xi_2|}\widehat{v}(\xi_2,\tau_2)}{(1+|\xi_2|)^s}\right.\\
&\quad\left.\cdot\frac{e^{\delta|\xi-\xi_1-\xi_2|}\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s}d\xi_1 d\xi_2 d\tau_1 d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&= C(4\delta)^{\kappa}\left\|\frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\xi^3|)^{1-b}}\int_{\mathbb{R}^4}\frac{e^{\delta|\xi_1|}(1+|\tau_1-\xi_1^3|)^b\widehat{u}(\xi_1,\tau_1)}{(1+|\xi_1|)^s(1+|\tau_1-\xi_1^3|)^b}\right.\\
&\quad\cdot\frac{e^{\delta|\xi_2|}(1+|\tau_2-\beta\xi_2^3|)^b\widehat{v}(\xi_2,\tau_2)}{(1+|\xi_2|)^s(1+|\tau_2-\beta\xi_2^3|)^b}\\
&\quad\cdot\frac{e^{\delta|\xi-\xi_1-\xi_2|}(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^b\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^b}\\
&\quad\left.\cdot d\xi_1 d\xi_2 d\tau_1 d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.
\end{aligned}$$

Setting
$w=(A^{1,\delta}u)$, $f(\xi,\tau)=(1+|\xi-\tau^3|)^b\widehat{w}(\xi,\tau)$ we have $e^{\delta|\xi|}\widehat{u}(\xi,\tau)=\widehat{w}(\xi,\tau)=f(\xi,\tau)(1+|\xi-\tau^3|)^{-b}$,
and
$\omega=(A^{1,\delta}v)$, $g(\xi,\tau)=(1+|\xi-\beta\tau^3|)^b\widehat{\omega}(\xi,\tau)$ we have $e^{\delta|\xi|}\widehat{v}(\xi,\tau)=\widehat{\omega}(\xi,\tau)=g(\xi,\tau)(1+|\xi-\beta\tau^3|)^{-b}$ therefore we get

$$\begin{aligned}
\|G_1\|_{X_{0,b-1}(\mathbb{R}^2)} &\le C(4\delta)^{\kappa}\left\|\frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\xi^3|)^{1-b}}\int_{\mathbb{R}^4}\frac{\widehat{f}(\xi_1,\tau_1)}{(1+|\xi_1|)^s(1+|\tau_1-\xi_1^3|)^b}\right.\\
&\quad\cdot\frac{\widehat{g}(\xi_2,\tau_2)}{(1+|\xi_2|)^s(1+|\tau_2-\beta\xi_2^3|)^b}\\
&\quad\cdot\frac{\widehat{g}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^b}\\
&\quad\left.\cdot d\xi_1 d\xi_2 d\tau_1 d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.
\end{aligned}$$

By Remark 3.7 we get

$$\begin{aligned}
\|G_1\|_{X_{0,b-1}(\mathbb{R}^2)} &\leq C\delta^\kappa \|f\|_{L^2_{\xi,\tau}(\mathbb{R}^2)} \|g\|^2_{L^2_{\xi,\tau}(\mathbb{R}^2)} \\
&= C\delta^\kappa \|w\|_{X_{0,b}(\mathbb{R}^2)} \|\omega\|^2_{X^\beta_{0,b}(\mathbb{R}^2)} \\
&= C\delta^\kappa \|A^{\delta,1}u\|_{X_{0,b}(\mathbb{R}^2)} \|A^{\delta,1}v\|^2_{X^\beta_{0,b}(\mathbb{R}^2)} \qquad (3.10) \\
&= C\delta^\kappa \|u\|_{X_{\delta,0,b}(\mathbb{R}^2)} \|v\|^2_{X^\beta_{\delta,0,b}(\mathbb{R}^2)} \\
&\leqslant C\delta^\kappa \|(u,v)\|^3_{B_{\delta,0,b}}.
\end{aligned}$$

Now let $L_2 = (A^{\delta,1}uA^{\delta,1}uA^{\delta,1}v) - A^{\delta,1}(u^2v)$. Then

$$\|G_2\|_{X^\beta_{0,b-1}(\mathbb{R}^2)} = \left\| \frac{\xi}{(1+|\tau-\beta\xi^3|)^{1-b}} \widehat{L_2}(\xi,\tau) \right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} \frac{|\ \xi\ |^2}{(1+|\tau-\beta\xi^3|)^{2(1-b)}} \ |\ \widehat{L_1}(\xi,\tau)\ |^2 \, d\xi d\tau \right).$$

We shall calculate the Fourier transform of L_2

$$\begin{aligned}
\left|\widehat{L_2}(\xi,\tau)\right| &= \left|\widehat{(A^{\delta,1}uA^{\delta,1}uA^{\delta,1}v)} - \widehat{A^{\delta,1}(u^2v)}\right| = C\Big|(e^{\delta|\xi|}\widehat{u} * e^{\delta|\xi|}\widehat{u} * e^{\delta|\xi|}\widehat{v} *)(\xi,\tau) \\
&\quad - e^{\delta|\xi|}(\widehat{u} * \widehat{u} * \widehat{v} *)(\xi,\tau)\Big| \\
&= C\left| \int_{\mathbb{R}^4} \Big(e^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1) e^{\delta|\xi_2|}\widehat{u}(\xi_2,\tau_2) e^{\delta|\xi-\xi_1-\xi_2|}\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2) \right. \\
&\quad \left. - e^{\delta|\xi|}\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi_2,\tau_2)\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2) \Big) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right| \\
&\leq C \int_{\mathbb{R}^4} \left(e^{\delta|\xi_1|} e^{\delta|\xi_2|} e^{\delta|\xi-\xi_1-\xi_2|} - e^{\delta|\xi|} \right) \\
&\quad \times |\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi_2,\tau_2)\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)| d\xi_1 d\xi_2 d\tau_1 d\tau_2.
\end{aligned}$$

For $\kappa \in [0, \frac{1}{2}) \subset [0,1]$, one can see that

$$\begin{aligned}
\|G_2\|^2_{X^{\beta}_{0,b-1}(\mathbb{R}^2)} &= \left\| \frac{\xi}{(1+|\tau-\beta\xi^3|)^{1-b}} \widehat{L_2}(\xi,\tau) \right\|^2_{L^2_{\xi,\tau}(\mathbb{R}^2)} \\
&\leq (C4\delta)^{2\kappa} \int_{\mathbb{R}^2} \frac{|\ \xi\ |^2}{(1+|\tau-\beta\xi^3|)^{2(1-b)}} \left[\int_{\mathbb{R}^4} \left(\frac{(1+|\xi-\xi_1-\xi_2|)(1+|\xi_1|)(1+|\xi_2|)}{(1+|\xi|)} \right)^{\kappa} \right. \\
&\times e^{\delta|\xi_1|} e^{\delta|\xi_2|} e^{\delta|\xi-\xi_1-\xi_2|} \\
&\left. \times |\widehat{u}(\xi_1,\tau_1)\widehat{u}(\xi_2,\tau_2)\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)| d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right]^2 d\xi d\tau \\
&= C(4\delta)^{2\kappa} \left\| \frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\beta\xi^3|)^{1-b}} \int_{\mathbb{R}^4} e^{\delta|\xi_1|}(1+|\xi_1|)^{\kappa}\widehat{u}(\xi_1,\tau_1) e^{\delta|\xi_2|}(1+|\xi_2|)^{\kappa}\widehat{u}(\xi_2,\tau_2) \right. \\
&\left. \times e^{\delta|\xi-\xi_1-\xi_2|}(1+|\xi-\xi_1-\xi_2|)^{\kappa}\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2) d\xi_1 d\xi_2 d\tau_1 d\tau_2 \right\|^2_{L^2_{\xi,\tau}(\mathbb{R}^2)}.
\end{aligned}$$

Now by taking $s=-\kappa\in(-\frac{1}{2},0]$, we obtain

$$\begin{aligned}\|G_2\|_{X^{\beta}_{0,b-1}(\mathbb{R}^2)} &\leq C(4\delta)^{\kappa}\left\|\frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\beta\xi^3|)^{1-b}}\int_{\mathbb{R}^4}\frac{e^{\delta|\xi_1|}\widehat{u}(\xi_1,\tau_1)}{(1+|\xi_1|)^s}\frac{e^{\delta|\xi_2|}\widehat{u}(\xi_2,\tau_2)}{(1+|\xi_2|)^s}\right.\\
&\left.\cdot\frac{e^{\delta|\xi-\xi_1-\xi_2|}\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s}d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=C(4\delta)^{\kappa}\left\|\frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\beta\xi^3|)^{1-b}}\int_{\mathbb{R}^4}\frac{e^{\delta|\xi_1|}(1+|\tau_1-\xi_1^3|)^b\widehat{u}(\xi_1,\tau_1)}{(1+|\xi_1|)^s(1+|\tau_1-\xi_1^3|)^b}\right.\\
&\cdot\frac{e^{\delta|\xi_2|}(1+|\tau_2-\xi_2^3|)^b\widehat{u}(\xi_2,\tau_2)}{(1+|\xi_2|)^s(1+|\tau_2-\xi_2^3|)^b}\\
&\cdot\frac{e^{\delta|\xi-\xi_1-\xi_2|}(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^b\widehat{v}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^b}\\
&\left.\cdot d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=C(4\delta)^{\kappa}\left\|\frac{\xi(1+|\xi|)^{-\kappa}}{(1+|\tau-\beta\xi^3|)^{1-b}}\int_{\mathbb{R}^4}\frac{\widehat{f}(\xi_1,\tau_1)}{(1+|\xi_1|)^s(1+|\tau_1-\xi_1^3|)^b}\right.\\
&\cdot\frac{\widehat{f}(\xi_2,\tau_2)}{(1+|\xi_2|)^s(1+|\tau_2-\xi_2^3|)^b}\\
&\cdot\frac{\widehat{g}(\xi-\xi_1-\xi_2,\tau-\tau_1-\tau_2)}{(1+|\xi-\xi_1-\xi_2|)^s(1+|\tau-\tau_1-\tau_2-\beta(\xi-\xi_1-\xi_2)^3|)^b}\\
&\left.\cdot d\xi_1d\xi_2d\tau_1d\tau_2\right\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}.\end{aligned}$$

By Remark 3.7 we get

$$\begin{aligned}\|G_2\|_{X^{\beta}_{0,b-1}(\mathbb{R}^2)} &\leq C\delta^{\kappa}\|f\|^2_{L^2_{\xi,\tau}(\mathbb{R}^2)}\|g\|_{L^2_{\xi,\tau}(\mathbb{R}^2)}\\
&=C\delta^{\kappa}\|w\|^2_{X_{0,b}(\mathbb{R}^2)}\|\omega\|_{X^{\beta}_{0,b}(\mathbb{R}^2)}\\
&=C\delta^{\kappa}\|A^{\delta,1}u\|^2_{X_{0,b}(\mathbb{R}^2)}\|A^{\delta,1}v\|_{X^{\beta}_{0,b}(\mathbb{R}^2)} \qquad (3.11)\\
&=C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}(\mathbb{R}^2)}\|v\|_{X^{\beta}_{\delta,0,b}(\mathbb{R}^2)}\\
&\leqslant C\delta^{\kappa}\|(u,v)\|^3_{B_{\delta,0,b}}.\end{aligned}$$

By (3.10) and (3.11) we have

$$\|(G_1,G_2)\|_{B_{0,b-1}} \le C\delta^{\kappa}\|(u,v)\|^3_{B_{\delta,0,b}}.$$

□

Proof. (Of Theorem 3.15) Let $U(t,x)=A^{\delta,1}u(t,x), V(t,x)=A^{\delta,1}v(t,x)$ which are real-valued since the multiplier $A^{\delta,1}$ is even and u,v are real-valued. Applying $A^{\delta,1}$ to (3.1), we obtain

$$\partial_t U+\partial_x^3 U+\partial_x(UV^2)=G_1, \tag{3.12}$$

$$\partial_t U+\partial_x^3 U+\partial_x(U^2V)=G_2, \tag{3.13}$$

where $G_1=\partial_x\left[(A^{\delta,1}uA^{\delta,1}vA^{\delta,1}v)-A^{\delta,1}(uv^2)\right]$, $G_2=\partial_x\left[(A^{\delta,1}uA^{\delta,1}uA^{\delta,1}v)-A^{\delta,1}(u^2v)\right]$. We multiply both sides of (3.12) by U, (3.13) by V and integrate with respect to space variable, we get

$$\int_{\mathbb{R}} U\partial_t U dx+\int_{\mathbb{R}} U\partial_x^3 U dx+\int_{\mathbb{R}} U\partial_x(UV^2)dx=\int_{\mathbb{R}} UG_1 dx,$$

$$\int_{\mathbb{R}} V\partial_t V dx+\int_{\mathbb{R}} V\partial_x^3 V dx+\int_{\mathbb{R}} V\partial_x(U^2V)dx=\int_{\mathbb{R}} VG_2 dx.$$

Next, we have

$$\begin{aligned}&\int_{\mathbb{R}}(U\partial_t U+V\partial_t V)dx+\int_{\mathbb{R}}(U\partial_x^3 U+V\partial_x^3 V)dx+\int_{\mathbb{R}}\left[U\partial_x(UV^2)+V\partial_x(U^2V)\right]dx\\ &=\int_{\mathbb{R}}(UG_1+VG_2)dx.\end{aligned}$$

Then, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}}(U\partial_t U+V\partial_t V)dx+\int_{\mathbb{R}}\partial_x(\partial_x U\partial_x U+\partial_x V\partial_x V)dx+\int_{\mathbb{R}}\partial_x(U^2V^2)dx\\ &=\int_{\mathbb{R}}(UG_1+VG_2)dx.\end{aligned}$$

Noting that $\partial_x^j U(x,t)\longrightarrow 0$ as $|x|\longrightarrow\infty$, we use integration by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(U^2+V^2)dx=\int_{\mathbb{R}}(UG_1+VG_2)dx.$$

Integrating the last equality with respect to $t\in[0,T_1]$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}(U^2(T_1,x)+V^2(T_1,x))dx &= \int_{\mathbb{R}}(U^2(0,x)+V^2(0,x))dx\\ &+ 2\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1+VG_2)dxdt.\end{aligned}$$

Thus

$$\|u(T_1)\|^2_{G^{\delta,0}(\mathbb{R})}+\|v(T_1)\|^2_{G^{\delta,0}(\mathbb{R})}=\|u(0)\|^2_{G^{\delta,0}(\mathbb{R})}+\|v(0)\|^2_{G^{\delta,0}(\mathbb{R})}+2\left|\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1+VG_2)dxdt\right|.$$

By using Holder's inequality, Lemma 3.5, Lemma 3.4 and the fact that

$$-\frac{1}{2} < 1-b < \frac{1}{2}, -\frac{1}{2} < b-1 < \frac{1}{2}.$$

Since $b > \frac{1}{2}$, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \chi_{[0,T_1]}(t)(UG_1 + VG_2)dxdt \right| \\
\leq\ & \|\chi_{[0,T_1]}(t)U\|_{X_{0,1-b}(\mathbb{R}^2)} \|\chi_{[0,T_1]}(t)G_1\|_{X_{0,b-1}(\mathbb{R}^2)} \\
+\ & \|\chi_{[0,T_1]}(t)V\|_{X^{\beta}_{0,1-b}(\mathbb{R}^2)} \|\chi_{[0,T_1]}(t)G_2\|_{X^{\beta}_{0,b-1}(\mathbb{R}^2)} \\
\leq\ & C\|U\|_{X^{T_1}_{0,1-b}(\mathbb{R}^2)} \|G_1\|_{X^{T_1}_{0,b-1}(\mathbb{R}^2)} + C\|V\|_{X^{\beta,T_1}_{0,1-b}(\mathbb{R}^2)} \|G_2\|_{X^{\beta,T_1}_{0,b-1}(\mathbb{R}^2)} \\
\leq\ & C\|\Theta_{T_1}U\|_{X_{0,1-b}(\mathbb{R}^2)} \|\Theta_{T_1}G_1\|_{X_{0,b-1}(\mathbb{R}^2)} + C\|\Theta_{T_1}V\|_{X^{\beta}_{0,1-b}(\mathbb{R}^2)} \|\Theta_{T_1}G_2\|_{X^{\beta}_{0,b-1}(\mathbb{R}^2)} \\
\leq\ & C\|U\|_{X_{0,1-b}(\mathbb{R}^2)} \|G_1\|_{X_{0,b-1}(\mathbb{R}^2)} + C\|V\|_{X^{\beta}_{0,1-b}(\mathbb{R}^2)} \|G_2\|_{X^{\beta}_{0,b-1}(\mathbb{R}^2)},
\end{aligned}$$

where $\Theta_{T_1} = 1$ for $t \in [0,T_1]$, we can conclude from Lemma 3.16

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \chi_{[0,T_1]}(t)(UG_1 + VG_2)dxdt \right| \\
\leq\ & C\|U\|_{X_{0,1-b}(\mathbb{R}^2)} \|G_1\|_{X_{0,b-1}(\mathbb{R}^2)} + C\|V\|_{X^{\beta}_{0,1-b}(\mathbb{R}^2)} \|G_2\|_{X^{\beta}_{0,b-1}(\mathbb{R}^2)} \\
\leq\ & C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}(\mathbb{R}^2)} \|v\|^2_{X^{\beta}_{\delta,0,b}(\mathbb{R}^2)} + C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}(\mathbb{R}^2)} \|v\|^2_{X^{\beta}_{\delta,0,b}(\mathbb{R}^2)} \\
=\ & 2C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}(\mathbb{R}^2)} \|v\|^2_{X^{\beta}_{\delta,0,b}(\mathbb{R}^2)} \\
\leq\ & 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.
\end{aligned}$$

Therefore,

$$\|u(T_1)\|^2_{G^{\delta,0}(\mathbb{R})} + \|v(T_1)\|^2_{G^{\delta,0}(\mathbb{R})} \leq \|u(0)\|^2_{G^{\delta,0}(\mathbb{R})} + \|v(0)\|^2_{G^{\delta,0}(\mathbb{R})} + 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}},$$

and

$$2\|(u(T_1),v(T_1))\|^2_{N^{\delta,0}} \leq 2\|(u(0),v(0))\|^2_{N^{\delta,0}} + 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}},$$

and

$$\sup_{t\in[0,T_1]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.$$

Finally, by using condition (3.5), we conclude that

$$\sup_{t\in[0,T_1]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^{\kappa}\|(u(0),v(0))\|^4_{N^{\delta,0}}.$$

□

3.6.2 Global extension and radius analyticity −Proof of Theorem 3.14

Let $\delta_0 > 0$, $s > -\frac{1}{2}$, $\kappa \in (0,\frac{1}{2})$ be fixed, and $(u_0,v_0) \in N^{\delta_0,s}$. Then, we have to prove that the solution (u,v) of (3.1) satisfies

$$(u,v) \in C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right) \times C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right),$$

where

$$\delta(T) = \min\left\{\delta_0, C_1 T^{-\frac{1}{\kappa}}\right\}, \quad for\ all\ T > 0,$$

and $C_1 > 0$ is a constant depending on u_0, v_0, δ_0, s and κ. By Corollary 3.10, there is a maximal time $T^* = T^*(u_0,v_0,\delta_0,s) \in (0,\infty]$, such that

$$(u,v) \in C\left([0,T^*), G^{\delta_0,s}(\mathbb{R})\right) \times C\left([0,T^*), G^{\delta_0,s}(\mathbb{R})\right).$$

If $T^* = \infty$, then we are done since the solution is defined for $t \in [0,\infty)$.
If $T^* < \infty$, as we assume henceforth, it remains to prove

$$(u,v) \in C\left([0,T], G^{C_1 T^{-\frac{1}{\kappa}},s}(\mathbb{R})\right) \times C\left([0,T], G^{C_1 T^{-\frac{1}{\kappa}},s}(\mathbb{R})\right), \quad for\ all\ T \geq T^*. \tag{3.14}$$

The case s=0

Fixed $T \geq T^*$, we will show that, for $\delta > 0$ sufficiently small

$$\sup_{t\in[0,T]} \|(u(0),v(0))\|^2_{N^{\delta,0}} \leq 2\|(u(0),v(0))\|^2_{N^{\delta_0,0}}.$$

In this case, by Corollary 3.10 and Theorem 3.15 with

$$T_0 = \frac{1}{(16C^3 + 32C^3 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}})^{\frac{1}{\varepsilon}}},$$

the smallness conditions on δ will be

$$\delta < \delta_0 \ \ and \ \ \frac{2T}{T_0} C\delta^{\kappa} 2^2 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}} \leq 1,\ C > 0. \tag{3.15}$$

Here C is the constant in Theorems 3.15.
By induction, we check that

$$\sup_{t\in[0,nT_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + nC\delta^{\kappa}2^2\|(u(0),v(0))\|^4_{N^{\delta_0,0}}. \tag{3.16}$$

And

$$\sup_{t\in[0,nT_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq 2\|(u(0),v(0))\|^2_{N^{\delta_0,0}}, \tag{3.17}$$

for $n \in \{1,\ldots,m+1\}$, where $m \in \mathbb{N}$ is chosen so that $T \in [mT_0,(m+1)T_0)$. This m does exist, by Corollary 3.10 and the definition of T^*, we have

$$T_0 < \frac{1}{(16C^3+16C^3 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}})^{1/\varepsilon}} < T^*, \ \ hence \ \ T_0 < T.$$

In the first step, we cover the interval $[0,T_0]$, and by Theorem 3.15, we have

$$\begin{aligned}\sup_{t\in[0,T_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} &\leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa\|(u(0),v(0))\|^4_{N^{\delta,0}} \\ &\leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa\|(u(0),v(0))\|^4_{N^{\delta_0,0}},\end{aligned}$$

since $\delta \leq \delta_0$, we used

$$\|(u(0),v(0))\|_{N^{\delta,0}} \leq \|(u(0),v(0))\|_{N^{\delta_0,0}}.$$

This satisfies (3.16) for $n=1$ and (3.17) follows using again $\|(u(0),v(0))\|_{N^{\delta,0}} \leq \|(u(0),v(0))\|_{N^{\delta_0,0}}$ as well as

$$C\delta^\kappa\|(u(0),v(0))\|^2_{N^{\delta_0,0}} \leq 1.$$

Suppose now that (3.16) and (3.17) hold for some $n \in \{1,\ldots,m\}$ and we prove that it holds for $n+1$. We estimate

$$\begin{aligned}&\sup_{t\in[nT_0,(n+1)T_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \\ \leq\ & \|(u(nT_0),v(nT_0))\|^2_{N^{\delta,0}} + C\delta^\kappa\|(u(nT_0),v(nT_0))\|^4_{N^{\delta,0}} \\ \leq\ & \|(u(nT_0),v(nT_0))\|^2_{N^{\delta,0}} + C\delta^\kappa 2^2\|(u(0),v(0))\|^4_{N^{\delta_0,0}} \\ \leq\ & \|(u(0),v(0))\|^2_{N^{\delta,0}} + nC\delta^\kappa 2^2\|(u(0),v(0))\|^4_{N^{\delta_0,0}} \\ +\ & C\delta^\kappa 2^2\|(u(0),v(0))\|^4_{N^{\delta_0,0}},\end{aligned}$$

satisfying (3.16) with n replaced by $n+1$. To get (3.17) with n replaced by $n+1$, it is then enough to have

$$(n+1)C\delta^\kappa 2^2\|(u(0),v(0))\|^2_{N^{\delta_0,0}} \leq 1,$$

but this holds by (3.15), since

$$n+1 \leq m+1 \leq \frac{T}{T_0}+1 < \frac{2T}{T_0}.$$

Finally, the condition (3.15) is satisfied for $\delta \in (0,\delta_0)$ such that

$$\frac{2T}{T_0}C\delta^\kappa 2^2 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}} = 1.$$

Thus,

$$\delta = C_1 T^{-\frac{1}{\kappa}},$$

where

$$C_1 = \left(\frac{1}{C2^3 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}} (16C^3 + 32C^3 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}})^{1/\varepsilon}} \right)^{\frac{1}{\kappa}}.$$

The General Case

For all s, by (5.3), we have $u_0, v_0 \in G^{\delta_0,s}(\mathbb{R}) \subset G^{\delta_0/2,0}(\mathbb{R})$.
For case $s=0$, it is proved that there is a $T_2 > 0$, such hat

$$(u,v) \in C\left([0,T_2), G^{\delta_0/2,0}(\mathbb{R})\right) \times C\left([0,T_2), G^{\delta_0/2,0}(\mathbb{R})\right),$$

and

$$(u,v) \in C\left([0,T], G^{2\sigma T^{-\frac{1}{\kappa}a},0}(\mathbb{R})\right) \times C\left([0,T], G^{2\sigma T^{-\frac{1}{\kappa}},0}(\mathbb{R})\right), \;\; for\; T' \geq T_2,$$

where $\sigma > 0$ depends on u_0, v_0, δ_0 and κ.
Applying again the embedding (2.32), we now conclude that

$$(u,v) \in C\left([0,T_2), G^{\delta_0/4,s}(\mathbb{R})\right) \times C\left([0,T_2), G^{\delta_0/4,s}(\mathbb{R})\right),$$

and

$$(u,v) \in C\left([0,T], G^{\sigma T^{-\frac{1}{\kappa}},s}(\mathbb{R})\right) \times C\left([0,T], G^{\sigma T^{-\frac{1}{\kappa}},s}(\mathbb{R})\right), \;\; for\; T \geq T_2,$$

which imply (3.14). The proof of Theorem 3.14 is now completed.

3.7 Regularity of the solution to coupled system (3.1)

In this section we will show that for $x \in \mathbb{R}$, for every $t \in [0,T]$ and $j,l \in \{0,1,2,\dots\}$, there exist $C > 0$ such that,

$$|\partial_t^j \partial_x^l u(x,t)| \leq C^{j+l+1} (j!)^{3\sigma} (l!)^{\sigma},$$

$$|\partial_t^j \partial_x^l v(x,t)| \leq C^{j+l+1} (j!)^{3\sigma} (l!)^{\sigma}.$$

i.e, $(u(\cdot,t), v(\cdot,t)) \in G^{\sigma}(\mathbb{R}) \times G^{\sigma}(\mathbb{R})$ in spacial variable and $(u(x,\cdot), v(x,\cdot)) \in G^3([0,T]) \times G^3([0,T])$ in time variable .

Theorem 3.18. *Let $s > -\frac{1}{2}$, $0 < \beta < 1, \sigma \geq 1$ and $\delta > 0$. If $(u_0,v_0) \in G^{\sigma,\delta,s}(\mathbb{R}) \times G^{\sigma,\delta,s}(\mathbb{R})$, then the solution $(u,v) \in C\left([0,T], G^{\sigma,\delta,s}(\mathbb{R})\right) \times C\left([0,T], G^{\sigma,\delta,s}(\mathbb{R})\right)$ given by Theorem 3.9 belongs to the Gevrey class $G^{3\sigma}([0,T]) \times G^{3\sigma}([0,T])$ in time variable. Furthermore, it is not belong to $G^d([0,T]) \times G^d([0,T])$, $1 \leq d < 3\sigma$ in t.*

Corollary 3.19. *Let $s > -\frac{1}{2}$, $0 < \beta < 1$ and $\delta > 0$. If $(u_0,v_0) \in G^{\delta,s}(\mathbb{R}) \times G^{\delta,s}(\mathbb{R})$, then the solution $(u,v) \in C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right) \times C\left([0,T], G^{\delta(T),s}(\mathbb{R})\right)$ given by Theorem 3.14 belongs to the Gevrey class $G^3([0,T]) \times G^3([0,T])$ in time variable. Furthermore, it is not belong to $G^d([0,T]) \times G^d([0,T])$, $1 \leq d < 3$ in t.*

3.7.1 $G^{3\sigma}$-regularity in the time variable

We will now prove the temporal regularity of solution on the line.

Proposition 3.20. *Let $s \geq -\frac{1}{4}$, $\delta > 0$, $\sigma \geq 1$ and $(u,v) \in C\left([0,T];G^{\sigma,\delta,s}(\mathbb{R})\right) \times C\left([0,T];G^{\sigma,\delta,s}(\mathbb{R})\right)$ be the solution of (3.1). Then $(u,v) \in G^{\sigma}(\mathbb{R}) \times G^{\sigma}(\mathbb{R})$ in $x, \forall t \in [0,T]$, i. e., for some $C > 0$, we have*

$$|\partial_x^l u| \leq C^{l+1}(l!)^{\sigma}, l \in \{0,1,...\}, \quad \forall x \in \mathbb{R}, t \in [0,T], \tag{3.18}$$

and

$$|\partial_x^l v| \leq C^{l+1}(l!)^{\sigma}, l \in \{0,1,...\}, \quad \forall x \in \mathbb{R}, t \in [0,T]. \tag{3.19}$$

Proof. See proof of Proposition 2.24

□

The sequences m_q and M_q are the same as in the first chapter. For some $C > 0$, we define the following constants

$$M_0 = \frac{c}{8} \text{ and } M = max\{\sqrt{2}, \frac{8C}{c}, \frac{4C^2}{c}\}. \tag{3.20}$$

The next Lemma is the main idea for the proof of Theorem 3.18.

Lemma 3.21. *Let (u,v) be the solution of (3.1) satisfied (3.18) and (3.19), then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ we have*

$$|\partial_t^j \partial_x^l u| \leq M^{2j+1} M_{l+3j}, j \in \{0,1,2,...\}, l \in \{0,1,2,...\}, \tag{3.21}$$

and

$$|\partial_t^j \partial_x^l v| \leq M^{2j+1} M_{l+3j}, j \in \{0,1,2,...\}, l \in \{0,1,2,...\}, \tag{3.22}$$

for all $x \in \mathbb{R}$, $t \in [0,T]$.

For this end, we need the next results

Lemma 3.22. *Given $l,k \in \{0,1,2,...\}$ we have*

$$\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q}L_{(n-p)+3(k-q)}L_{p+3q} \leq \sum_{r=1}^{m}\binom{m}{r}L_r L_{m-r},$$

where $L_j, j = 0,1,...,m$ positive real numbers with $m = n+3k$.

Proof. (Of Lemma 3.21)
We will prove (3.21) and (3.22) by induction. Let $j = 0$, for $l = 0$, it follows from (3.18), (3.19) and the definition of M in (3.20) that

$$|u| \leq C \leq MM_0, \quad \forall x \in \mathbb{R}, t \in [0,T],$$

and

$$|v| \leq C \leq MM_0, \quad \forall x \in \mathbb{R}, t \in [0,T].$$

Similarly, for $l = 1$, we have

$$|\partial_x u| \leq C^2 \leq MM_1, \forall x \in \mathbb{R}, t \in [0,T],$$

and

$$|\partial_x v| \leq C^2 \leq MM_1, \forall x \in \mathbb{R}, t \in [0,T].$$

By (3.18), (3.19) and (2.70), for $l \geq 2$ there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, we have

$$|\partial_x^l u| \leq C^{l+1}(l!)^\sigma \leq M_l \leq MM_l, \forall x \in \mathbb{R}, t \in [0,T],$$

and

$$|\partial_x^l v| \leq C^{l+1}(l!)^\sigma \leq M_l \leq MM_l, \forall x \in \mathbb{R}, t \in [0,T].$$

This completes the proof of (3.21) and (3.22) for $j = 0$ and $l \in \{0,1,...\}$.
Next, we will assume that (3.21) and (3.22) is true for $0 \leq q \leq j$ and $l \in \{0,1,...\}$ and we will prove it for $q = j+1$ and $l \in \{0,1,...\}$. We begin by noting that

$$|\partial_t^{j+1}\partial_x^l u| = |\partial_t^j \partial_x^l(\partial_t u)| \quad \leq |\partial_t^j \partial_x^{l+3} u| + |\partial_t^j \partial_x^{l+1}(uv^2)|,$$

and

$$|\partial_t^{j+1}\partial_x^l v| = |\partial_t^j \partial_x^l(\partial_t v)| \quad \leq |\partial_t^j \partial_x^{l+3} v| + |\partial_t^j \partial_x^{l+1}(u^2 v)|.$$

Using the induction hypotheses and the condition $M > \sqrt{2}$, we estimate the second term $\partial_t^j \partial_x^{l+3} u$ and $\partial_t^j \partial_x^{l+3} v$ as follows

$$\begin{aligned} |\partial_t^j \partial_x^{l+3} u| &\leq M^{2j+1} M_{l+3+3j} = M^{-2} M^{2(j+1)+1} M_{l+3(j+1)} \\ &\leq \frac{1}{2} M^{2(j+1)+1} M_{l+3(j+1)}, \end{aligned} \tag{3.23}$$

and

$$|\partial_t^j \partial_x^{l+3} v| \quad \leq \frac{1}{2} M^{2(j+1)+1} M_{l+3(j+1)}. \tag{3.24}$$

All this estimates are taken for the linear terms. For the nonlinear terms F_1 and F_2, using Leibniz's rule twice and the induction hypothesis, we have a different cases
For the nonlinear terms $F_1 = \partial_x(uv^2)$

$$\begin{aligned} |\partial_t^j \partial_x^{l+1}(uv^2)| &= \Bigg| \sum_{p_1=0}^{l+1} \sum_{p_2=0}^{p_1} \sum_{q_1=0}^{j} \sum_{q_2=0}^{q_1} \binom{l+1}{p_1} \binom{p_1}{p_2} \binom{j}{p_1} \binom{p_1}{p_2} \\ &\quad .\partial_t^{j-q_1} \partial_x^{l+1-p_1} u \partial_t^{q_1-q_2} \partial_x^{p_1-p_2} v \partial_t^{q_2} \partial_x^{p_2} v \Bigg| \\ &\leq \sum_{p_1=0}^{l+1} \sum_{p_2=0}^{p_1} \sum_{q_1=0}^{j} \sum_{q_2=0}^{q_1} \binom{l+1}{p_1} \binom{p_1}{p_2} \binom{j}{p_1} \binom{p_1}{p_2} \\ &\quad .|\partial_t^{j-q_1} \partial_x^{l+1-p_1} u||\partial_t^{q_1-q_2} \partial_x^{p_1-p_2} v||\partial_t^{q_2} \partial_x^{p_2} v|. \end{aligned}$$

Thus, using the induction hypotheses the last equality gives

$$\begin{aligned}|\partial_t^j\partial_x^{l+1}(uv^2)| \;&\le \sum_{p_1=0}^{l+1}\sum_{p_2=0}^{p_1}\sum_{q_1=0}^{j}\sum_{q_2=0}^{q_1}\binom{l+1}{p_1}\binom{p_1}{p_2}\binom{j}{p_1}\binom{p_1}{p_2}\\ &M^{2(j-q_1)+1}M_{l+1-p_1+3(j-q_1)}M^{2(q_1-q_2)+1}\\ &M_{p_1-p_2+3(q_1-q_2)}M^{2q_2+1}M_{p_2+3q_2}.\end{aligned} \tag{3.25}$$

Next, using Lemma 3.22 with $p=p_2, l=p_1, q=q_2, k=q_1, L_j=M_j, m=p_1+3q_1$, we obtain

$$\begin{aligned}&\sum_{p_2=0}^{p_1}\sum_{q_2=0}^{q_1}\binom{p_1}{p_2}\binom{q_1}{q_2}M_{(p_1-p_2)+3(q_1-q_2)}M_{p_2+3q_2}\\ &\le \sum_{r=1}^{m}\binom{m}{r}M_rM_{m-r}\le (M_0+\varepsilon)M_m\\ &=(M_0+\varepsilon)M_{p_1+3q_1}.\end{aligned} \tag{3.26}$$

Similarly, using Lemma 3.22 with $p=p_1, l=l+1, q=q_1, k=j, L_j=M_j, m=l+3j+1$, we obtain

$$\begin{aligned}&\sum_{p_1=0}^{l+1}\sum_{q_1=0}^{j}\binom{l+1}{p_1}\binom{j}{q_1}M_{(l+1-p_1)+3(j-q_1)}M_{p_1+3q_1}\\ &\le \sum_{r=1}^{m}\binom{m}{r}M_rM_{m-r}\le (M_0+\varepsilon)M_m\\ &=(M_0+\varepsilon)M_{l+3j+1}.\end{aligned} \tag{3.27}$$

Continuing this way we obtain all possible inequalities as (3.25), (3.26) and (3.27). we obtain

$$\begin{aligned}|\partial_t^j\partial_x^{l+1}(uv^2)| \;&\le \sum_{p_1=0}^{l+1}\sum_{p_2=0}^{p_1}\sum_{q_1=0}^{j}\sum_{q_2=0}^{q_1}\binom{l+1}{p_1}\binom{p_1}{p_2}\binom{j}{p_1}\binom{p_1}{p_2}\\ &M^{2(j-q_1)+1}M_{l+1-p_1+3(j-q_1)}M^{2(q_1-q_2)+1}\\ &M_{p_1-p_2+3(q_1-q_2)}M^{2q_2+1}M_{p_2+3q_2}\\ &\le M^{2(j+1)+1}(M_0+\varepsilon)^2M_{l+3j+1}\\ &\le M^{2(j+1)+1}\varepsilon^2(M_0+\varepsilon)^2M_{l+3(j+1)}.\end{aligned}$$

Noting that in the last inequality we have used the fact that $l+3j+1\ge 2$, since we are assuming that either $j\ne 0$ or $l\ne 0$.

Now, choosing $\varepsilon\le\varepsilon_0=\left(\frac{1}{2(M_0+\varepsilon)^2}\right)^{\frac{1}{2}}<1$, to get

$$\varepsilon^2(M_0+\varepsilon)^2\le\varepsilon^2(M_0+1)^2\le(M_0+1)^2\left(\frac{1}{2(M_0+1)^2}\right)=\frac{1}{2}.$$

Hence,

$$|\partial_t^j \partial_x^{l+1}(uv^2)| \le \frac{1}{2} M^{2(j+1)+1} M_{l+3(j+1)}.$$

Next for the nonlinear terms $F_2 = \partial_x(u^2 v)$, we get

$$|\partial_t^j \partial_x^{l+1}(u^2 v)| \le \frac{1}{2} M^{2(j+1)+1} M_{l+3(j+1)}.$$

Which completes the proof.

□

Proof the first part of Theorem 3.18.

By Lemma 3.21, we have

$$|\partial_t^j \partial_x^l u| \le M^{2j+1} M_{l+3j}, \quad j \in \{0,1,2,...\},\ l \in \{0,1,2,...\},$$

and

$$|\partial_t^j \partial_x^l v| \le M^{2j+1} M_{l+3j}, \quad j \in \{0,1,2,...\},\ l \in \{0,1,2,...\}.$$

Applying this inequality for $j \in \{1,2,...\}$ and $l = 0$ gives

$$\begin{aligned} |\partial_t^j u| &\le M^{2j+1} M_{3j} = M M^{2j} \varepsilon^{1-3j} \frac{c((3j)!)^\sigma}{(3j+1)^2} \\ &\le M\varepsilon c \left(\frac{M^2}{\varepsilon^3}\right)^j ((3j)!)^\sigma \\ &\le L_0 L^j ((3j)!)^\sigma \\ &\le L_0 L^j A^{3\sigma j} (j!)^3)^\sigma \\ &\le A_0^{j+1} (j!)^{3\sigma}, \end{aligned} \tag{3.28}$$

and

$$|\partial_t^j v| \le A_0^{j+1} (j!)^{3\sigma}, \tag{3.29}$$

where $L_0 = M\varepsilon c,\ L = \dfrac{M^2}{\varepsilon^3}$ since $(3j)! \le A^{3j}(j!)^3$ for $A > 0$ and $A_0 = max\{L_0, LA^{3\sigma}\}$. We also have from (3.21) and (3.22) for $l = j = 0$, that

$$|u| \le MM_0 = M\frac{c}{8}, \quad \forall x \in \mathbb{R}\ t \in [0,T], \tag{3.30}$$

and

$$|v| \le MM_0 = M\frac{c}{8}, \quad \forall x \in \mathbb{R}\ t \in [0,T]. \tag{3.31}$$

Setting $C = max\{M\frac{c}{8}, A_0\}$, it follows from (3.28) and (3.30) that for $j \in \{0,1,2,...\}$, we have

$$|\partial_t^j u| \le C^{j+1} (j!)^{3\sigma}, \quad \forall x \in \mathbb{R}\ t \in [0,T].$$

And from (3.29) and (3.31) that for $j \in \{0,1,2,...\}$, we have

$$|\partial_t^j v| \le C^{j+1} (j!)^{3\sigma}, \quad \forall x \in \mathbb{R}\ t \in [0,T].$$

Hence, $(u,v) \in G^{3\sigma}(\mathbb{R}) \times G^{3\sigma}(\mathbb{R})$ in t.
Which completes the proof of first part of Theorem 3.18.

3.7.2 Failure of Gevrey-d regularity in time

In this sub-section, we prove the second part of the Theorem 3.18. Replacing t with $-t$ we can write our system as follows

$$\begin{cases} \partial_t u = \partial_x^3 u + \partial_x(uv^2), \\ \partial_t v = \beta\partial_x^3 v + \partial_x(u^2 v), \qquad 0<\beta<1, \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x). \end{cases} \tag{3.32}$$

The following Lemma will be used to estimate the higher-order derivatives of a solution with respect to t.

Lemma 3.23. *([37]) If (u,v) is a solution to (3.32) then for every $j\in\{1,2,...\}$ we have*

$$\partial_t^j u = \partial_x^{3j} u + \sum_{m=1}^{j}\sum_{|\lambda|+2m=3j} C_\lambda^m(\partial_x^{\lambda_1}u)\cdots(\partial_x^{\lambda_m}v), \tag{3.33}$$

and

$$\partial_t^j v = \partial_x^{3j} v + \sum_{m=1}^{j}\sum_{|\lambda|+2m=3j} C_\lambda^m(\partial_x^{\lambda_1}u)\cdots(\partial_x^{\lambda_m}v). \tag{3.34}$$

Definition 3.24. Let $\{\omega_k\}$ be a sequence of positive numbers. We denote by $\mathscr{C}(\omega_k)$ the class of all functions $g(x)$, infinitely differentiable on $[-1,1]$, for each of which there is an $C>0$ such that

$$|g^{(k)}(x)| \le C^{k+1}\omega_k, \quad x\in[-1,1] \text{ and } k=0,1,2,...$$

Lemma 3.25. *([39]) For every $\sigma>1$ and every sequence of complex numbers $\{\varphi_k\}$, satisfying*

$$|\varphi_k| \le C_1^{k+1}k^{k\sigma},$$

for some $C_1>0$, there exists a function $g(x)\in\mathscr{C}(k^{k\sigma})$ for which $g^{(k)}(0)=\varphi_k$.

We will use this result for the sequence of real numbers

$$|g^{(k)}(x)| \le C^{k+1}k^{k\sigma} \le C^{k+1}(k!)^\sigma e^{k\sigma}, \quad k=0,1,2,...$$

where $g(x)\in\mathscr{C}(k^{k\sigma})$ such that $g^{(k)}(0)=\varphi_k=(k!)^\sigma$.
We choose a cut-off function $u_0,v_0\in G_c^\sigma(-2,2)$ such that

$$\begin{cases} \theta(x)=1 \ for \ |x|\le 1, \\ and \\ \theta(x)=0 \ for \ |x|>2, \end{cases}$$

by modifying $g(x)$ to become having a compact support in .
If u_0 and v_0 are extension of θg, then by the algebra property for Gevrey functions we have $u_0,v_0\in G^\sigma(\mathbb{R})$. We have then the relation by $g(x)$

$$u_0^{(k)}(0) = g^{(k)}(0) = (k!)^\sigma \ and \ v_0^{(k)}(0) = g^{(k)}(0) = (k!)^\sigma. \tag{3.35}$$

We will show that (u,v) need not be $G^d(\mathbb{R})\times G^d(\mathbb{R})$, with $1\le d<3\sigma$, in the time variable t.

Theorem 3.26. *Let $s > -\frac{1}{2}$, $0 < \beta < 1, \sigma \geq 1$ and $\delta > 0$. The real-valued solution to (3.32) with real-valued initial data $(u_0, v_0) \in G^{\sigma,\delta,s}(\mathbb{R}) \times G^{\sigma,\delta,s}(\mathbb{R})$ may not be in $G^d(\mathbb{R}) \times G^d(\mathbb{R})$, with $1 \leq d < 3\sigma$, in the time variable t.*

Proof. By using (3.33), (3.34) and (3.35) we get

$$\begin{aligned}
\partial_t^j u(0,0) &= \partial_x^{3j} u(0,0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} v(0,0)) \\
&= u_0^{3j}(0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} v_0(0)) \\
&\geq u_0^{3j}(0) = ((3j)!)^\sigma \geq (j!)^{3\sigma},
\end{aligned}$$

and

$$\begin{aligned}
\partial_t^j v(0,0) &= \partial_x^{3j} v(0,0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} v(0,0)) \\
&= v_0^{3j}(0) + \sum_{m=1}^{j} \sum_{|\lambda|+2m=3j} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} v_0(0)) \\
&\geq v_0^{3j}(0) = ((3j)!)^\sigma \geq (j!)^{3\sigma},
\end{aligned}$$

we have proved that $(u(0,\cdot), v(0,\cdot)) \notin G^d(\mathbb{R}) \times G^d(\mathbb{R})$ for $1 \leq d < 3\sigma$ and for t near 0.

□

Which completes the proof of Theorem 3.18.

Chapter 4

Fifth order Kadomtsev-Petviashvili I equation [1]

4.1 Introduction

The Kadomtsev-Petviashvili equation (or simply the KP equation) originates from a 1970 paper by two Soviet physicists, Boris Kadomtsev (1928-1998) and Vladimir Petviashvili (1936-1993). The two researchers derived the equation that now bears their name as a model to study the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. In the absence of transverse dynamics, this problem is described by the Korteweg-de Vries (KdV) equation. The KP equation was soon widely accepted as a natural extension of the classical KdV equation to two spatial dimensions, and was later derived as a model for surface and internal water waves by Ablowitz and Segur (1979), and in nonlinear optics by Pelinovsky, Stepanyants and Kivshar (1995), as well as in other physical settings.

We consider a class of Cauchy problem for fifth-order Kadomtsev-Petviashvili I equation

$$\begin{cases} \partial_t u+\alpha\partial_x^3 u+\partial_x^5 u+\partial_x^{-1}\partial_y^2 u+u\partial_x u=0, \\ u(x,y,0)=f(x,y), \end{cases} \tag{4.1}$$

where $u=u(x,y,t),\ (x,y,\alpha)\in\mathbb{R}^3, t\in\mathbb{R}^+$.

4.2 Function spaces

Now, we will define the anisotropic Gevrey space that contains the initial data of the considered IVP (4.1). For $s_1,s_2\in\mathbb{R}$ and $\delta>0$, let

$$G^{\delta,s_1,s_2}(\mathbb{R}^2)=\left\{f\in L^2(\mathbb{R}^2);\|f\|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)}<\infty\right\}, \tag{4.2}$$

where

$$\|f\|^2_{G^{\delta,s_1,s_2}(\mathbb{R}^2)}=\int_{\mathbb{R}^2}e^{2\delta(|\xi|+|\mu|)}(1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}|\widehat{f}(\xi,\mu)|^2 d\xi d\mu.$$

[1] A. Boukarou, Kh. Zennir, K. Guerbati and S. G. Georgiev.Wellposedness and regularity of the fifth order KadomtsevPetviashvili I equation in the analytic Bourgain spaces. Ann Univ Ferrara (2020),doi :10.1007/s11565-020-00340-8.

In the particular case where $\delta = 0$, the space $G^{0,s_1,s_2}(\mathbb{R}^2)$ is reduced to the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$, defined by the norm

$$\|f\|^2_{H^{s_1,s_2}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}|\widehat{f}(\xi,\mu)|^2 d\xi d\mu.$$

Then, we define the analytic Bourgain spaces related to the fifth-order Kadomtsev-Petviashvili I equation. The completion of the Schwartz space $S(\mathbb{R}^3)$ is given by $X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)$, for $s,b \in \mathbb{R}$ and $\delta > 0$, defined as

$$X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3); \|u\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} < \infty \right\},$$

where

$$\|u\|^2_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} e^{2\delta(|\xi|+|\mu|)}(1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}(1+|\tau-\phi(\xi,\mu)|)^{2b} \mid \widehat{u}(\xi,\mu,\tau) \mid^2 d\xi d\mu d\tau.$$

with $\phi(\xi,\mu) = -\xi^5 + \alpha\xi^3 - \dfrac{\mu^2}{\xi}$.

Lemma 4.1. *Let $b > \frac{1}{2}$, $s_1,s_2 \in \mathbb{R}$ and $\delta > 0$. Then, for all $T > 0$, we have*

$$X^{s_1,s_2}_{\delta,b} \hookrightarrow C\left([0,T], G^{\delta,s_1,s_2}(\mathbb{R}^2)\right).$$

Proof. First, we observe that the operator Λ^δ, defined by

$$\widehat{\Lambda^\delta u}^{x,y}(\xi,\mu,t) = e^{\delta(|\xi|+|\mu|)}\widehat{u}^{x,y}(\xi,\mu,t), \tag{4.3}$$

satisfies

$$\|u\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} = \|\Lambda^\delta u\|_{X_{s_1,s_2,b}(\mathbb{R}^3)} \quad \text{and} \quad \|u\|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)} = \|\Lambda^\delta u\|_{H^{s_1,s_2}(\mathbb{R}^2)}, \tag{4.4}$$

where $X_{s_1,s_2,b}(\mathbb{R}^3)$ is introduced in [58]. We observe that Au belongs to $C\left([0,T], H^{s_1,s_2}(\mathbb{R}^2)\right)$ and for some $C_0 > 0$, we have

$$\| \Lambda^\delta u \|_{C\left([0,T],H^{s_1,s_2}(\mathbb{R}^2)\right)} \leq C_0 \| \Lambda^\delta u \|_{X_{s_1,s_2,b}(\mathbb{R}^3)}. \tag{4.5}$$

Thus, it follows that $u \in C\left([0,T], G^{\delta,s_1,s_2}(\mathbb{R}^2)\right)$ and

$$\| u \|_{C\left([0,T],G^{\delta,s_1,s_2}(\mathbb{R}^2)\right)} \leq C_0 \| u \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)}. \tag{4.6}$$

This completes the proof.

□

4.3 Linear Estimates

Lemma 4.2. *Let* $s_1, s_2 \geq 0$, $\frac{1}{2} < b < 1$ *and* $\delta > 0$. *For some constant* $C > 0$, *we have*

$$\|\psi(t)S(t)f\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq C\|f\|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)}, \tag{4.7}$$

for all $f \in G^{\delta,s_1,s_2}(\mathbb{R}^2)$.

Proof. We have

$$\begin{aligned}\psi(t)S(t)f &= C\psi(t)\int_{\mathbb{R}^2} e^{i(x\xi+y\mu+t\phi(\xi,\mu))}\widehat{f}(\xi,\mu)d\xi d\mu \\ &= C\int_{\mathbb{R}^3} e^{i(x\xi+y\mu+t\tau)}\widehat{\psi}(\tau-\phi(\xi,\mu))\widehat{f}(\xi,\mu)d\xi d\mu d\tau.\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\| \psi(t)S(t)f \|^2_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \\ &= C\int_{\mathbb{R}^3} e^{2\delta(|\xi|+|\mu|)}(1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}(1+|\tau-\phi(\xi,\mu)|)^{2b} |\widehat{\psi}(\tau+\phi(\xi,\mu))|^2|\widehat{f}(\xi,\mu)|^2 d\xi d\mu d\tau \\ &= C\int_{\mathbb{R}^2} e^{2\delta(|\xi|+|\mu|)}(1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}|\widehat{f}(\xi,\mu)|^2 \\ &\times \left(\int_{\mathbb{R}} |\widehat{\psi}(\tau-\phi(\xi,\mu))|^2 (1+|\tau-\phi(\xi,\mu)|)^{2b} d\tau\right) d\xi d\mu.\end{aligned}$$

By the fact that $b > 1/2$, we get

$$\begin{aligned}&\int_{\mathbb{R}} |\widehat{\psi}(\tau-\phi(\xi,\mu))|^2 (1+|\tau-\phi(\xi,\mu)|)^{2b} d\tau \\ &\leqslant \int_{\mathbb{R}} |\widehat{\psi}(\tau-\phi(\xi,\mu))|^2 d\tau + C\int_{\mathbb{R}} |\widehat{\psi}(\tau-\phi(\xi,\mu))|^2|\tau-\phi(\xi,\mu)|^{2b} d\tau \\ &\leq C.\end{aligned}$$

This completes the proof.

□

Proposition 4.3. *([58]) Let* $s_1, s_2 \geq 0$, $-\frac{1}{2} < b' \leq 0 \leq b \leq b'+1$ *and* $0 < T < 1$. *Then for some constant* $C > 0$, *we have*

$$\left\|\psi_T(t)\int_0^t S(t-t')F(x,y,t')\mathrm{d}t'\right\|_{X_{s_1,s_2,b}(\mathbb{R}^3)} \leq CT^{1-b+b'}\|F\|_{X_{s_1,s_2,b'}(\mathbb{R}^3)}.$$

Lemma 4.4. *Let* $s_1, s_2 \geq 0$, $-\frac{1}{2} < b' \leq 0 \leq b \leq b'+1$, $0 < T < 1$ *and* $\delta > 0$. *Then for some constant* $C > 0$, *we have*

$$\left\|\psi_T(t)\int_0^t S(t-t')F(x,y,t')\mathrm{d}t'\right\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq CT^{1-b+b'}\|F\|_{X^{s_1,s_2}_{\delta,b'}(\mathbb{R}^3)}.$$

Proof. Define $U = \psi_T(t)\int_0^t S(t-t')F(x,y,t')\mathrm{d}t'$. Let us consider the operator Λ^δ given in (4.3). Then we have

$$\widehat{\Lambda^\delta U}^{x,y}(\xi,t) = \psi_T(t)\int_0^t \left(e^{-i(t-t')\phi(\xi,\mu)}\right)e^{\delta(|\xi|+|\mu|)}\widehat{F}^{x,y}(\xi,\mu,t')\mathrm{d}t'$$

$$= \psi_T(t)\int_0^t \widehat{[S(t-t')(\Lambda^\delta F)]}^{x,y}(\xi,\mu,t')\mathrm{d}t'.$$

Thus,

$$\| U \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} = \| \Lambda^\delta U \|_{X_{s_1,x_2,b}(\mathbb{R}^3)} = \left\| \psi_T(t)\int_0^t S(t-t')\Lambda^\delta F(x,y,t')dt' \right\|_{X_{s_1,x_2,b}(\mathbb{R}^3)}.$$

Using Proposition 4.3, we get

$$\left\| \psi_T(t)\int_0^t S(t-t')\Lambda^\delta F(x,y,t')dt' \right\|_{X_{s_1,x_2,b}(\mathbb{R}^3)} \leq CT^{1-b+b'}\|\Lambda^\delta F\|_{X_{s_1,x_2,b'}(\mathbb{R}^3)}.$$

□

4.4 Bilinear estimate

Theorem 4.5. *([58]) If $s_1,s_2 \geq 0$, $b' = -\frac{1}{2}+2\varepsilon$, $b = \frac{1}{2}+\varepsilon$, then*

$$\| \partial_x(u_1u_2) \|_{X_{s_1,s_2,b'}(\mathbb{R}^3)} \leq C \| u_1 \|_{X_{s_1,s_2,b'}(\mathbb{R}^3)} \| u_2 \|_{X_{s_1,s_2,b'}(\mathbb{R}^3)}.$$

Lemma 4.6. *If $s_1,s_2 \geq 0$, let $\delta > 0$, $b' = -\frac{1}{2}+2\varepsilon$, $b = \frac{1}{2}+\varepsilon$, then*

$$\| \partial_x(u_1u_2) \|_{X^{s_1,s_2}_{\delta,b'}(\mathbb{R}^3)} \leq C \| u_1 \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \| u_2 \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)}. \tag{4.8}$$

Proof. We observe, by considering the operator A^δ in (4.3), that

$$e^{\delta(|\xi|+|\mu|)}\widehat{u_1u_2} = (\sqrt{2\pi})^{-3}e^{\delta(|\xi|+|\mu|)}\widehat{u_1} * \widehat{u_2}$$

$$\leq (\sqrt{2\pi})^{-3}\int_{\mathbb{R}^3} e^{\delta(|\xi-\xi_1|+|\mu-\mu_1|)}\widehat{u_1}(\xi-\xi_1,\mu-\mu_1,\tau-\tau_1)$$

$$\times e^{\delta(|\xi_1|+|\mu_1|)}\widehat{u_2}(\xi_1,\mu_1,\tau_1)d\xi_1 d\mu_1 d\tau_1$$

$$= \widehat{\Lambda^\delta u_1 \Lambda^\delta u_2}.$$

Then

$$\| \partial_x(u_1u_2) \|_{X^{s_1,s_2}_{\delta,b'}(\mathbb{R}^3)}$$

$$=\| e^{\delta(|\xi|+|\mu|)}(1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}(1+|\tau-\phi(\xi,\mu)|)^{2b}\widehat{\partial_x(u_1u_2)}(\xi,\mu,\tau) \|_{L^2_{\xi,\mu,\tau}(\mathbb{R}^3)}$$

$$=\| (1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}(1+|\tau-\phi(\xi,\mu)|)^{2b}e^{\delta(|\xi|+|\mu|)}\xi\widehat{u_1u_2}(\xi,\mu,\tau) \|_{L^2_{\xi,\mu,\tau}(\mathbb{R}^3)}$$

$$\leq\| (1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}(1+|\tau-\phi(\xi,\mu)|)^{2b}(\widehat{\partial_x\Lambda^\delta u_1\Lambda^\delta u_2})(\xi,\mu\tau) \|_{L^2_{\xi,\mu,\tau}(\mathbb{R}^3)}$$

$$=\| \partial_x(\Lambda^\delta u_1\Lambda^\delta u_2) \|_{X_{s_1,s_2,b}(\mathbb{R}^3)} .$$

Now, by using Theorem 4.5, there exists $C>0$ such that

$$\begin{aligned}\| \partial_x(\Lambda^\delta u_1\Lambda^\delta u_2) \|_{X_{s_1,s_2,b'}(\mathbb{R}^3)} &\leq C \| \Lambda^\delta u_1 \|_{X_{s_1,s_2,b}(\mathbb{R}^3)}\| \Lambda^\delta u_2 \|_{X_{s_1,s_2,b}(\mathbb{R}^3)} \\ &= C \| u_1 \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)}\| u_2 \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} .\end{aligned}$$

□

4.5 Local well-posedness

Junfeng Li and Jie Xiao establish the local and global well-posedness of the real valued fifth order Kadomtsev-Petviashvili I equation in the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ with nonnegative indices. In particular, they are improves Saut-Tzvetkov's one and our global well-posedness gives an affirmative answer to Saut-Tzvetkov's $L^2(\mathbb{R}^2)$-data conjecture. We improved this result, states local well-posedness in $G^{\delta,s_1,s_2}, s_1,s_2(\mathbb{R}^2)\geqslant 0$, and $\delta>0$.

Theorem 4.7. *Let $s_1,s_2\geq 0,\delta>0$ and $b=\frac{1}{2}+\varepsilon$. Then, for any $f\in G^{\delta,s_1,s_2}(\mathbb{R}^2)$ and $|\xi|^{-1}\widehat{f}(\xi,\mu)\in L^2(\mathbb{R}^2)$, the problem (4.1) with the initial condition f has a solution u, satisfying*

$$u\in C\left([0,T],G^{\delta,s_1,s_2}(\mathbb{R}^2)\right).$$

Moreover the solution depends continuously on the data f.

4.5.1 Existence of solution

By Duhamel's formula, (4.1) can be reduced to the integral representation below:

$$u(t)=S(t)f-\frac{1}{2}\int_0^t S(t-t')\left(\partial_x u^2(t')\right)dt',$$

where the unit operator related to the corresponding linear equation is given by

$$u(t)(x,y)=(S(t)f)(x,y)=\int_{\mathbb{R}^2} e^{i(x\xi+y\mu+t\phi(\xi,\mu))}\widehat{f}(\xi,\mu)d\xi d\mu.$$

We localize it in t by using a cut-off function satisfying $\psi \in C_0^\infty(\mathbb{R})$, supp$\psi \subset [-2,2]$ $\psi = 1$ in $[-1,1]$ and $\psi_T(t) = \psi(\frac{t}{T})$.

$$\Phi(u) = \psi(t)S(t)f - \frac{\psi_T(t)}{2}\int_0^t S(t-t')\left(\partial_x u^2(t')\right)dt'. \tag{4.9}$$

We are now ready to estimate all the terms in (4.9) by using the bilinear estimates in the above Lemmas.

Lemma 4.8. *Let $s_1, s_2 \geq 0$, $\delta > 0$ and $b > \frac{1}{2}$. Then, for all $f \in G^{\delta,s_1,s_2}(\mathbb{R}^2)$ and $0 < T < 1$, with some constant $C > 0$, we have*

$$\| \Phi(u) \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq C\left(\| f \|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)} + T^{1-b+b'}(\mathbb{R}^2) \| u \|^2_{X^{s_1,s_2}_{\delta,b}}(\mathbb{R}^3)\right), for\ all\ u \in X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3), \tag{4.10}$$

and

$$\| \Phi(u) - \Phi(v) \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq CT^{1-b+b'} \| u - v \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \| u + v \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)}, for\ all\ u, v \in X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3). \tag{4.11}$$

Proof. To prove estimate (4.10), we have

$$\begin{aligned}
\| \Phi(u) \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} &\leq \|\psi(t)S(t)f\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} + \left\|\psi_T(t)\int_0^t S(t-t')\left(\partial_x u^2(t')\right)dt'\right\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \\
&\leq C \| f \|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)} + CT^{1-b+b'} \| \partial_x u^2 \|_{X^{s_1,s_2}_{\delta,b'}(\mathbb{R}^3)} \\
&\leq C \| f \|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)} + CT^{1-b+b'} \| u \|^2_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)}.
\end{aligned}$$

For the estimate (4.11), we observe that

$$\Phi(u) - \Phi(v) = \psi_T(t)\int_0^t S(t-t')\left(\partial_x u^2 - \partial_x v^2\right)(x,y,t')dt',$$

where $\omega = \partial_x u^2 - \partial_x v^2$ is now given by

$$\omega = \partial_x(u^2 - v^2) = \partial_x[(u+v)(u-v)],$$

Thus, from the previous results, we obtain (4.11).

□

We will show that the map Φ is a contraction on the ball $\mathbb{B}(0,r)$ to $\mathbb{B}(0,r)$.

Lemma 4.9. *Let $s_1, s_2 \geq 0$ and $\delta > 0$, $b > \frac{1}{2}$. Then, for all $f \in G^{\delta,s_1,s_2}(\mathbb{R}^2)$, the map $\Phi : \mathbb{B}(0,r) \to \mathbb{B}(0,r)$ is a contraction, where $\mathbb{B}(0,r)$ is given by*

$$\mathbb{B}(0,r) = \left\{u \in X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3); \|u\|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq r, with\quad r = 4C\|f\|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)}\right\}.$$

Proof. From Lemma 4.8, for all $u \in \mathbb{B}(0,r)$, we have

$$\| \Phi(u) \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq C\left(\| f \|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)} + T^{1-b+b'} \| u \|^2_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \right) \leq \frac{r}{4} + CT^{1-b+b'} r^2.$$

We choose T sufficiently small such that $T^{1-b+b'} \leq \frac{1}{4Cr}$. Hence,

$$\| \Phi(u) \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \leq r, \quad \forall u \in \mathbb{B}(0,r).$$

Thus, Φ maps $\mathbb{B}(0,r)$ into $\mathbb{B}(0,r)$, which is a contraction, since

$$\begin{aligned} \| \Phi(u) - \Phi(v) \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} &\leq CT^{1-b+b'} \| u - v \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \| u + v \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \\ &\leq CT^{1-b+b'} 2r \| u - v \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)} \\ &\leq \frac{1}{2} \| u - v \|_{X^{s_1,s_2}_{\delta,b}(\mathbb{R}^3)}, \quad \forall u, v \in \mathbb{B}(0,r). \end{aligned}$$

□

The rest of the proof follows the standard argument.

4.6 Time regularity

Theorem 4.10. *Let $s_1, s_2 \geq 0, \delta > 0$ and $\alpha = 1$. If $f \in G^{\delta,s_1,s_2}(\mathbb{R}^2)$ then the solution $u \in C\left([0,T], G^{\delta,s_1,s_2}(\mathbb{R}^2)\right)$, given by Theorem 4.7, belongs to the Gevrey class $G^5([0,T])$ in time variable.*

This section is devoted to the proof of Theorem 4.10, we study the Gevrey's regularity in t of the unique solution of the problem (4.1). The proof of time regularity on the circle and on the line is analogous.

Proposition 4.11. *Let $s_1, s_2 \geq 0$, $\delta > 0$ and $u \in C\left([0,T], G^{\delta,s_1,s_2}(\mathbb{R}^2)\right)$ be the solution of (4.1). Then u is analytic in x, y for all t near the zero. More precisely,*

$$|\partial_x^l \partial_y^n u(x,y,t)| \leq C^{l+n+1}(l!)(n!), \tag{4.12}$$

for all $(x,y) \in \mathbb{R}^2$, $t \in [0,T]$, $\quad (l,n) \in \{0,1,...\}^2$, and $C > 0$.

Proof. For any $t \in [0,T]$, we have

$$\begin{aligned} \|\partial_x^l \partial_y^n u(\cdot,\cdot,t)\|^2_{H^{s_1,s_2}(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\xi|^{2l} |\mu|^{2n} (1+|\xi|)^{2s_1} (1+|\mu|)^{2s_2} |\widehat{u}(\xi,\mu,t)|^2 d\xi d\mu \\ &= \int_{\mathbb{R}^2} |\xi|^{2l} |\mu|^{2n} e^{-2\delta(|\xi|+|\mu|)} (1+|\xi|)^{2s_1} (1+|\mu|)^{2s_2} e^{2\delta(|\xi|+|\mu|)} |\widehat{u}(\xi,\mu,t)|^2 d\xi d\mu. \end{aligned}$$

We observe that

$$e^{2\delta|\xi|} = \sum_{j=0}^{\infty} \frac{1}{j!} (2\delta|\xi|)^j \geq \frac{1}{(2l)!} (\delta)^{2l} |\xi|^{2l}, \quad \forall l \in \{0,1,...\}, \xi \in \mathbb{R}.$$

And

$$e^{2\delta|\mu|} = \sum_{j=0}^{\infty} \frac{1}{j!}(2\delta|\mu|)^j \geq \frac{1}{(2n)!}(\delta)^{2n}|\mu|^{2n}, \quad \forall n \in \{0,1,...\}, \mu \in \mathbb{R}.$$

This implies that

$$|\xi|^{2l}e^{-2\delta|\xi|} \leq C_\delta^{2l}(2l)!,$$
$$|\mu|^{2n}e^{-2\delta|\mu|} \leq C_\delta^{2n}(2n)!.$$

Thus,

$$\begin{aligned}\|\partial_x^l\partial_y^n u(\cdot,\cdot,t)\|^2_{H^{s_1,s_2}(\mathbb{R}^2)} &\leq C_\delta^{2l+2n}(2l)!(2n)!\int_{\mathbb{R}^2} e^{2\delta(|\xi|+|\mu|)}(1+|\xi|)^{2s_1}(1+|\mu|)^{2s_2}|\widehat{u}(\xi,\mu,t)|^2 d\xi d\mu \\ &= C_\delta^{2l+2n}(2l)!(2n)!\|u(\cdot,\cdot,t)\|^2_{G^{\delta,s_1,s_2}(\mathbb{R}^2)}.\end{aligned}$$

Since $(2l)! \leq A_1^{2l}(l!)^2$ and $(2n)! \leq A_2^{2n}(n!)^2$, for some $A_1, A_2 > 0$, by Sobolev Lemma ($\|.\|_{L^2(\mathbb{R}^2)} \leq \|.\|_{H^{s_1,s_2}(\mathbb{R}^2)}$), we have for all $(l,n) \in \{0,1,2,\dots\}^2$

$$|\partial_x^l\partial_y^n u(x,y,t)| \leq \|\partial_x^l\partial_y^n u(\cdot,\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq \|\partial_x^l\partial_y^n u(\cdot,\cdot,t)\|_{H^{s_1,s_2}(\mathbb{R}^2)} \leq C_0 C_1^{l+n}(l)!(n)!,$$

where $C_0 = \|u(\cdot,\cdot,t)\|_{G^{\delta,s_1,s_2}(\mathbb{R}^2)}$ and $C_1 = A_0^2 C_\delta$ and $A_0 = \max(A_1,A_2)$. This implies that u is analytic in x,y for all t near zero and $s_1,s_2 \geq 0$. This completes the proof.

□

Lemma 4.12. *For $(j,l,n) \in \{0,1,2,...\}^3$ the following inequality*

$$|\partial_t^j\partial_x^l\partial_y^n u| \leq C^{j+l+n+1}(l+n+5j)!L^j, \tag{4.13}$$

holds, where $L = C^4 + \frac{C^2}{20} + \frac{1}{120} + \frac{C}{4}$, for all $(x,y) \in \mathbb{R}^2$, $t \in [0,T]$.

Proof. We will prove it by using induction on j.
Firstly, for $j = 0$ and $(l,n) \in \{0,1,2,...\}^2$, by (4.12), we have

$$|\partial_x^l\partial_y^n u(x,y,t)| \leq C^{l+n+1}(l!)(n!) \leq C^{l+n+1}(l+n)!. \tag{4.14}$$

Secondly, for $j = 1$ and $(l,n) \in \{0,1,2,...\}^2$, we get

$$|\partial_t\partial_x^l\partial_y^k u| \leq |\partial_x^{l+5}\partial_y^n u| + |\partial_x^{l+3}\partial_y^n u| + |\partial_x^{l-1}\partial_y^{n+2}u| + |\partial_x^l\partial_y^n(u\partial_x u)|.$$

We estimate these terms as

$$\begin{aligned}|\partial_x^{l+5}\partial_y^n u| &\leq C^{l+5+n+1}(l+5+n)! \\ &\leq C^{l+n+1+1}(l+n+5\cdot 1)!C^4,\end{aligned} \tag{4.15}$$

$$\begin{aligned}|\partial_x^{l+3}\partial_y^n u| &\leq C^{l+3+n+1}(l+3+n)! \\ &\leq C^{l+n+1+1}(l+n+5\cdot 1)!C^2\frac{1}{(l+n+4)(l+n+5)} \\ &\leq C^{l+n+1+1}(l+n+5\cdot 1)!\frac{C^2}{20}.\end{aligned} \tag{4.16}$$

Where

$$\frac{1}{(l+n+4)(l+n+5)} \le \frac{1}{20},$$

and

$$\begin{aligned}
|\partial_x^{l-1}\partial_y^{n+2}u| &\le C^{l-1+n+2+1}(l-1+n+2)! \\
&\le C^{l+n+1+1}(l+n+5\cdot 1)!\frac{1}{(l+n+2)(l+n+3)(l+n+4)(l+n+5)} \\
&\le C^{l+n+1+1}(l+n+5\cdot 1)!\frac{1}{120},
\end{aligned} \tag{4.17}$$

where

$$\frac{1}{(l+n+2)(l+n+3)(l+n+4)(l+n+5)} \le \frac{1}{120},$$

For the nonlinear terms

$$|\partial_x^l\partial_y^n(u\partial_x u)| = \Big|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\Big|.$$

We shall recall that for $p \le l$ and $k \le n$, we have the following inequality

$$\binom{l}{p}\binom{n}{k} \le \binom{l+n}{p+k}. \tag{4.18}$$

Now, using (4.18), to get

$$\begin{aligned}
\left|\partial_x^l\partial_y^n(u\partial_x u)\right| &\le \Big|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l+n}{p+k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\Big| \\
&\le \sum_{p=0}^{l}\sum_{k=0}^{n}\frac{(l+n)!}{(p+k)!(l+n-p-k)!}C^{l-p+n-k+1}(l+n-p-k)! \\
&\quad C^{p+1+k+1}(p+1+k)! \\
&= C^{l+n+3}(l+n)!\sum_{p=0}^{l}\sum_{k=0}^{n}(p+1+k).
\end{aligned}$$

Now we use the fact that

$$\sum_{p=0}^{l}\sum_{k=0}^{n}(p+1+k) = \frac{(l+1)(n+1)(l+n+2)}{2}. \tag{4.19}$$

Next,

$$\begin{aligned}|\partial_x^l\partial_y^n(u\partial_x u)| &\le C^{l+n+3}(l+n)!\frac{(l+1)(n+1)(l+n+2)}{2}\\ &\le C^{l+n+1+1}(l+n)!(l+n+3)(l+n+4)(l+n+5)\frac{C}{2}\\ &= C^{l+n+1+1}(l+n+5)!\frac{1}{(l+n+1)(l+n+2)}\frac{C}{2}\\ &\le C^{l+n+1+1}(l+n+5)!\frac{C}{4}.\end{aligned} \tag{4.20}$$

From (4.15), (4.16), (4.17) and (4.20), it follows that

$$|\partial_t\partial_x^l\partial_y^k u| \le C^{l+n+1+1}(l+n+5\cdot 1)!L^1,$$

for all $(x,y)\in\mathbb{R}^2, t\in[0,T]$.
Now, we will assume that (2.73) is correct for $1\le m\le j$ where $(l,n)\in\{0,1,2,...\}^2$ and we will prove it for $m=j+1$ and $(l,n)\in\{0,1,2,...\}^2$. We have

$$|\partial_t^{j+1}\partial_x^l\partial_y^k u| \le |\partial_t^j\partial_x^{l+5}\partial_y^n u| + |\partial_t^j\partial_x^{l+3}\partial_y^n u| + |\partial_t^j\partial_x^{l-1}\partial_y^{n+2} u| + |\partial_t^j\partial_x^l\partial_y^n(u\partial_x u)|.$$

We estimate these terms as

$$\begin{aligned}|\partial_t^j\partial_x^{l+5}\partial_y^n u| &\le C^{j+l+5+n+1}(l+5+n+5j)!L^j\\ &\le C^{(j+1)+l+n+1}(l+n+5(j+1))!C^4L^j,\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}|\partial_t^j\partial_x^{l+3}\partial_y^n u| &\le C^{j+l+3+n+1}(l+3+n+5j)!L^j\\ &\le C^{j+l+n+1+1}(l+n+5(j+1))!C^2L^j\frac{1}{(l+n+4)(l+n+5)}\\ &\le C^{(j+1)+l+n+1}(l+n+5(j+1))!\frac{C^2}{20}L^j,\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}|\partial_t^j\partial_x^{l-1}\partial_y^{n+2} u| &\le C^{j+l-1+n+2+1}(l-1+n+2+5j)!L^j\\ &\le C^{j+l+n+1+1}(l+n+5(j+1))!\frac{1}{(l+n+2)(l+n+3)(l+n+4)(l+n+5)}L^j\\ &\le C^{(j+1)+l+n+1}(l+n+5(j+1))!\frac{L^j}{(120)}.\end{aligned} \tag{4.23}$$

For the nonlinear terms, we have

$$\begin{aligned}\partial_t^j\partial_x^l\partial_y^n(u\partial_x u) &= \sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_t^j\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\\ &+\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^j\partial_x^{p+1}\partial_y^k u\right)\\ &+\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j}{q}\binom{l}{p}\binom{n}{k}\left(\partial_t^{j-q}\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^q\partial_x^{p+1}\partial_y^k u\right).\end{aligned} \tag{4.24}$$

Using (4.18), we estimate the first term of (4.24)

$$\left|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_t^j\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\right|$$

$$\leq \sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l+n}{p+k}C^{j+l-p+n-k+1}(l+n-p-k+5j)!L^jC^{p+1+k+1}(p+1+k)!$$

$$= C^{j+l+n+3}L^j\sum_{p=0}^{l}\sum_{k=0}^{n}\frac{(l+n)!}{(p+k)!(l+n-p-k)!}(p+1+k)!$$

$$(l+n-p-k+5j)!$$

$$= C^{j+l+n+3}L^j(l+n)!\sum_{p=0}^{l}\sum_{k=0}^{n}(p+1+k)(l+n-p-k+1)$$

$$(l+n-p-k+2)\cdots(l+n-p-k+5j).$$

Since $p\leq l$, $k\leq n$, for any $m\in\mathbb{N}$, we have

$$p+k+m\leq l+n+m,$$

and

$$l+n-p-k+m\leq l+n+m.$$

From these inequalities and from (4.19), we obtain

$$
\begin{aligned}
&\left|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_t^j\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^k u\right)\right| \\
&\leq C^{j+l+n+3}L^j(l+n)!(l+n+1)(l+n+2)\cdots(l+n+5j) \\
&\cdot\frac{(l+1)(n+1)(l+n+2)}{2} \\
&\leq C^{j+l+n+3}L^j\frac{1}{2}(l+n+5j)!(l+n+5j+1)(l+n+5j+2)(l+n+5j+3) \\
&= C^{(j+1)+l+n+1}(l+n+5j+5)!\frac{1}{(l+n+5j+4)(l+n+5j+5)}\frac{C}{2}L^j \\
&\leq \frac{1}{3}C^{(j+1)+l+n+1}(l+n+5(j+1))!\frac{C}{4}L^j.
\end{aligned}
\tag{4.25}
$$

The $(4.24)_2$ is estimated as

$$
\begin{aligned}
&\left|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^j\partial_x^{p+1}\partial_y^k u\right)\right| \\
&\leq \sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l+n}{p+k}C^{l-p+n-k+1}(l+n-p-k)!C^{j+p+1+k+1}(p+1+k+5j)!L^j \\
&= \sum_{p=0}^{l}\sum_{k=0}^{n}\frac{(l+n)!}{(p+k)!(l+n-p-k)!}C^{j+l+n+3}L^j(l+n-p-k)!(p+1+k+5j)! \\
&= C^{j+l+n+3}L^j(l+n)!\sum_{p=0}^{l}\sum_{k=0}^{n}(p+k+1)(p+k+2)\cdots(p+1+k+5j).
\end{aligned}
$$

Since $p \leq l$, $k \leq n$, for any $m \in \mathbb{N}$ we have $p+k+m \leq l+n+m$. From these inequalities and from (4.19), we get

$$\begin{aligned}
&\left|\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{l}{p}\binom{n}{k}\left(\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_x^{p+1}\partial_y^{k}u\right)\right| \\
&\leq C^{j+l+n+3}L^j(l+n)!\frac{(l+1)(n+1)(l+n+2)}{2}(l+n+2)(l+n+3) \\
&\cdots(l+1+n+5j) \\
&\leq C^{j+l+n+3}L^j\frac{1}{2}(l+n+5j+1)!(l+n+5j+2)(l+n+5j+3) \qquad (4.26)\\
&= C^{(j+1)+l+n+1}(l+n+5j+5)!\frac{1}{(l+n+5j+4)(l+n+5j+5)}\frac{C}{2}L^j \\
&\leq C^{(j+1)+l+n+1}(l+n+5(j+1))!\frac{C}{40}L^j \\
&\leq \frac{1}{3}C^{(j+1)+l+n+1}(l+n+5(j+1))!\frac{C}{4}L^j.
\end{aligned}$$

To estimate $(4.24)_3$, we shall recall that for $q \leq j$, $p \leq l$ and $k \leq n$, we have the following inequality

$$\binom{j}{q}\binom{l}{p}\binom{n}{k} \leq \binom{j+l+n}{q+p+k}.$$

Then

$$\begin{aligned}
&\left|\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j}{q}\binom{l}{p}\binom{n}{k}\left(\partial_t^{j-q}\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^{q}\partial_x^{p+1}\partial_y^{k}u\right)\right| \\
&\leq \sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j+l+n}{q+p+k}C^{j-q+l-p+n-k+1}(l-p+n-k+5(j-q))!L^{j-q} \\
&C^{q+p+1+k+1}(p+1+k+5q)!L^q \\
&\leq C^{j+l+n+3}L^j\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\frac{(j+l+n)!}{(q+p+k)!(j-q+l+n-p-k)!}(l+n-p-k+5(j-q))! \\
&(p+1+k+5q)! \\
&= C^{j+l+n+3}L^j(j+l+n)!\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}(q+p+k+1)(q+p+k+2)\cdots(q+p+k+1+4q) \\
&((j-q)+l+n-p-k+1)((j-q)+l+n-p-k+2)\cdots((j-q)+l+n-p-k+4(j-q)).
\end{aligned}$$

Since $q \le j-1$, $p \le l$, $k \le n$, for any $m \in \mathbb{N}$, we have

$$q+p+k+m \le j+l+n+m-1, \; j+l+n-q-p-k+m \le j+l+n+m-1,$$

$$\left|\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j}{q}\binom{l}{p}\binom{n}{k}\left(\partial_t^{j-q}\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^{q}\partial_x^{p+1}\partial_y^{k}u\right)\right|$$

$$\le C^{j+l+n+3}L^j(j+l+n)!\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}(q+p+k+1)(j+l+n+1)(j+l+n+2)$$

$$\cdots\cdot(j+l+n+4q)(j+l+n)(j+l+n+1)\cdots(j+l+n+4(j-q)-1).$$

Since $j+l+n+m \le j+l+n+m+4q+1$, for any $m \in \mathbb{N}$, we have

$$\left|\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}\binom{j}{q}\binom{l}{p}\binom{n}{k}\left(\partial_t^{j-q}\partial_x^{l-p}\partial_y^{n-k}u\right)\left(\partial_t^{q}\partial_x^{p+1}\partial_y^{k}u\right)\right|$$

$$\le C^{j+l+n+3}L^j(j+l+n)!\sum_{q=1}^{j-1}\sum_{p=0}^{l}\sum_{k=0}^{n}(q+p+k+1)(j+l+n+1)(j+l+n+2)$$

$$\cdots\cdot(j+l+n+4q)(j+l+n+4q+1)(j+l+n+4q+2)\cdots(j+l+n+4j)$$

$$= C^{j+l+n+3}L^j(l+n+5j)!\frac{(j-1)(l+1)(n+1)(j+l+n+2)}{2} \tag{4.27}$$

$$\le C^{j+l+n+3}L^j(l+n+5j)!(l+n+5j+1)(l+n+5j+2)(l+n+5j+3)\frac{(j-1)}{2}$$

$$= C^{j+l+n+3}L^j(l+n+5(j+1))!\frac{(j-1)}{(l+n+5j+4)(l+n+5j+5)}\frac{1}{2}$$

$$\le \frac{1}{3}C^{(j+1)+l+n+1}(l+n+5(j+1))!\frac{C}{4}L^j.$$

Finally by using (4.21), (4.22), (4.23), (4.25), (4.26) and (4.27) we arrive at

$$|\partial_t^{j+1}\partial_x^l\partial_y^k u| \le C^{(j+1)+l+n+1}(l+n+5(j+1))!L^{j+1},$$

for all $(x,y) \in \mathbb{R}^2, t \in [0,T]$.
This completes the proof.

□

Chapter 5

Fifth order Kadomtsev-Petviashvili II equation[1]

5.1 Introduction

The fifth-order Kadomtsev-Petviashvili II equation is the partial differential equation

$$\begin{cases} \partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u(x,y,0) = f(x,y), \end{cases} \tag{5.1}$$

where $u = u(x,y,t)$ and $(x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

In the present work, we will consider the Cauchy problem for equation (5.1) with initial data in an anisotropic Gevrey space $G^{\delta_1,\delta_2}(\mathbb{R}^2)$, which we define as the completion of the Schwartz functions with respect to the norm

$$\|f\|^2_{G^{\delta_1,\delta_2}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} e^{2\delta_1|\xi|} e^{2\delta_2|\eta|} |\hat{f}(\xi,\eta)|^2 d\xi d\eta.$$

The primary reason for considering initial data in these spaces is because of the following theorem:

Proposition 5.1 (Paley-Wiener Theorem)**.** *Let $\delta > 0$, and suppose $f \in L^2(\mathbb{R})$. Then the following are equivalent:*

1. *The function f is the restriction to the real line of a function F which is holomorphic in the strip*
$$S_\delta = \{x + iy \in \mathbb{C} : \ |y| < \delta\},$$
and satisfies
$$\sup_{|y|<\delta} \|F(x+iy)\|_{L^2(\mathbb{R})} < \infty.$$
2. $e^{\delta|\xi|}\hat{f}(\xi) \in L^2_\xi(\mathbb{R})$.

In addition to the holomorphic extension property, Gevrey spaces satisfy the embeddings $G^{\delta_1,\delta_2}(\mathbb{R}^2) \hookrightarrow G^{\delta_1',\delta_2'}(\mathbb{R}^2)$ for $\delta_i' < \delta_i$, which follow from the corresponding estimates

$$\|f\|_{G^{\delta_1',\delta_2'}(\mathbb{R}^2)} \leq C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)}. \tag{5.2}$$

[1] Boukarou, A.; Oliveira da Silva, D; Guerbati, Kh.; Zennir, Global well-posedness for the fifth-order Kadomtsev-Petviashvili II equation in anisotropic Gevrey Spaces, http://arxiv.org/abs/2006.12859v1.

5.2 Local and global well-posedness

Saut and Tzvetkov [74] proved that problem (5.1) is locally well-posed for initial data in $H^{0,0}(\mathbb{R}^2) = L^2(\mathbb{R}^2)$. This result was improved in [42] by Isaza, López and Mejía, who reduced the minimal regularity for initial data to $s_1 > -5/4$ and $s_2 \geq 0$. They also showed that the problem is globally well-posed in $H^{s_1,0}(\mathbb{R}^2)$ with $s_1 > -4/7$.

The first result relates to the short-term persistence of analyticity of solutions.

Theorem 5.2. *Let $\delta_1 \geq 0$ and $\delta_2 \geq 0$. Then for all initial data $f \in G^{\delta_1,\delta_2}(\mathbb{R}^2)$, there exists $T = T(\| f \|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)}) > 0$ and a unique solution u of (5.1) on the time interval $T > 0$ such that*

$$u \in C\left([0,T], G^{\delta_1,\delta_2}(\mathbb{R}^2)\right).$$

Moreover the solution depends continuously on the data f. In particular, the time of existence can be chosen to satisfy

$$T = \frac{c_0}{(1 + \|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)})^\gamma},$$

for some constants $c_0 > 0$ and $\gamma > 1$. Moreover, the solution u satisfies

$$\sup_{t\in[0,T]} \|u(t)\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)} \leq 2C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)}.$$

Our second main result concerns the evolution of the radius of analyticity for the x-direction.

Theorem 5.3. *Let $\delta_1 > 0$ and $\delta_2 \geq 0$, and assume $f \in G^{\delta_1,\delta_2}(\mathbb{R}^2)$. Then the solution u given by Theorem 5.2 extends globally in time, and for any $T > 0$, we have*

$$u \in C\left([0,T], G^{\delta(T),0}(\mathbb{R}^2)\right) \quad \text{with} \quad \delta(T) = \min\left\{\delta_1, CT^{-1}\right\},$$

where $C > 0$ is a constant which does not depend on T.

5.3 Function spaces and bilinear estimate

To simplify the notation, we introduce some operators which will be useful later. We first introduce the operator $\Lambda^{\delta_1,\delta_2}$, which we define as

$$\widehat{\Lambda^{\delta_1,\delta_2} f}^{x,y} = e^{\delta_1|\xi|}e^{\delta_2|\eta|}\hat{f}^{x,y}. \tag{5.3}$$

With this, we may also define another useful operator by

$$N(f) = \partial_x\left[(\Lambda^{\delta_1,\delta_2} f)^2 - \Lambda^{\delta_1,\delta_2}(f^2)\right]. \tag{5.4}$$

Since our proofs will rely heavily on the theory developed by Isaza, López and Mejía, let us state the function spaces they used explicitly, so that we can state their useful properties which we will exploit in our modifications of their spaces.

$$\|u\|^2_{X^{s_1,s_2,b,\varepsilon}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \lambda^2(s_1,s_2,b,\varepsilon)|\hat{u}(\xi,\eta,\tau)|^2 d\xi d\eta d\tau.$$

Where

$$\lambda(s_1,s_2,b,\varepsilon) = \langle\xi\rangle^{s_1}\langle\eta\rangle^{s_2}\langle\tau - m(\xi,\eta)\rangle^{b}\left\langle\frac{\tau - m(\xi,\eta)}{1+|\xi|^5}\right\rangle^{\varepsilon}.$$

with $m(\xi,\eta) = \xi^5 - \frac{\eta^2}{\xi}$ and $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

We remark that we will not need to make use of the case $\varepsilon \neq 0$. Thus, we will employ the simplified notation $X^{s_1,s_2,b}(\mathbb{R}^3) = X^{s_1,s_2,b,0}(\mathbb{R}^3)$.

Since our interest is in constructing analytic solutions, we will require a version of the Bourgain spaces which are adapted to the Gevrey spaces. For this, we define the spaces $Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)$ by the norm

$$\|u\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} = \|\Lambda^{\delta_1,\delta_2}u\|_{X^{0,0,b}(\mathbb{R}^3)}.$$

As is well-known, the spaces $X^{s_1,s_2,b}(\mathbb{R}^3)$ satisfy the embedding $X^{s_1,s_2,b} \hookrightarrow C\left(\mathbb{R};H^{s_1,s_2}(\mathbb{R}^2)\right)$ for $b > \frac{1}{2}$. An immediate consequence of this is that $Y^{\delta_1,\delta_2,b}(\mathbb{R}^3) \hookrightarrow C(\mathbb{R};G^{\delta_1,\delta_2}(\mathbb{R}^2))$ when $b > \frac{1}{2}$. Thus, solutions constructed in $Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)$ belong to the natural solution space.

The final preliminary fact we must state, is the following bilinear estimate, which is Lemma 1.1 of [42]:

Lemma 5.4. *Let $s_1 > -5/4$, $s_2 \geq 0$, $b > 1/2$, and define $s = \max\{0, -s_1\}$. If ε and β satisfy the inequalities*

$$0 \leq \varepsilon \leq \min\left\{\frac{2}{5}\left(\frac{5}{4} - s\right), \frac{3}{20}\right\},$$

and

$$\max\left\{\frac{9}{20}, \frac{1}{2} - \frac{1}{2}\left(\frac{5}{4} - s\right) + \varepsilon\right\} \leq \beta < \frac{1}{2},$$

then

$$\|\partial_x(uv)\|_{X^{s_1,s_2,-\beta,\varepsilon}(\mathbb{R}^3)} \leq C\|u\|_{X^{s_1,s_2,b,\varepsilon}(\mathbb{R}^3)}\|v\|_{X^{s_1,s_2,b,\varepsilon}(\mathbb{R}^3)}.$$

This estimate yields the following result as a lemma

Lemma 5.5. *For $\delta_1 \geq 0$, $\delta_2 \geq 0$, and $b > 1/2$, we have*

$$\|\partial_x(uv)\|_{Y^{\delta_1,\delta_2,-9/20}(\mathbb{R}^3)} \leq C\|u\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)}\|v\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)}.$$

5.4 Proof of Theorem 5.2

We may now begin the proof of Theorem 5.2 by Picard iteration, where the iteration space is $Y^{\delta_1,\delta_2,b}(I)$ and $I = [0,T]$. Since this is a modification of the proof of Isaza, López, and Mejía in [42], we will merely outline the essential steps. To begin, consider the linear problem

$$\begin{aligned}&\partial_t u - \partial_x^5 u + \partial_x^{-1}\partial_y^2 u = F,\\ &u(0) = f.\end{aligned}$$

By Duhamel's principle the solution can be written as

$$u(t) = S(t)f - \frac{1}{2}\int_0^t S(t-t')F(t')dt'. \tag{5.5}$$

Where

$$\widehat{S(t)f}(\xi,\eta) = e^{itm(\xi,\eta)}\widehat{f}(\xi,\eta).$$

Rather than work directly with the integral formulation in equation (5.5), we instead use the following modified form: let $\psi \in C_0^\infty(\mathbb{R})$ be supported in the interval $[-2,2]$ such that $0 \le \psi(t) \le 1$ and $\psi = 1$ on $[-1,1]$, and let $\phi_T \in C_0^\infty(\mathbb{R})$ be supported on $(-2T^{-1/2}, 2T^{-1/2})$ such that $0 \le \phi_T(t) \le 1$ and $\phi_T = 1$ on $[-T^{-1/2}, T^{-1/2}]$. Then, we observe that we may decompose the integral operator on the right-hand side of equation (5.5) in the form

$$-\frac{1}{2}\int_0^t S(t-t')f(t')dt' = I_{\phi_T}(f) + II_{\phi_T}(f) + III_{\phi_T}(f), \tag{5.6}$$

where

$$I_{\phi_T}(f) = C\int e^{i(x\xi+y\eta)}e^{itm(\xi,\eta)}\frac{e^{itp(\xi,\eta,\tau)}-1}{ip(\xi,\eta,\tau)}\phi_T(p(\xi,\eta,\tau))\hat{f}(\xi,\eta,\tau)\, d\xi d\eta d\tau,$$

$$II_{\phi_T}(f) = C\int e^{i(x\xi+y\eta)}\frac{e^{it\tau}}{ip(\xi,\eta,\tau)}(1-\phi_T(p(\xi,\eta,\tau)))\hat{f}(\xi,\eta,\tau)\, d\xi d\eta d\tau,$$

$$III_{\phi_T}(f) = -C\int e^{i(x\xi+y\eta)}e^{itm(\xi,\eta)}\frac{(1-\phi_T(p(\xi,\eta,\tau))}{ip(\xi,\eta,\tau)}\hat{f}(\xi,\eta,\tau)\, d\xi d\eta d\tau.$$

Here, $p(\xi,\eta,\tau) = \tau - m(\xi,\eta)$. Then, we define a modification $G_T(f)$ of the integral operator in (5.6),

$$G_T(f) = \psi(t/T)I_{\phi_T}(f) + II_{\phi_T}(f) + \psi(t/T)III_{\pi_T}(f).$$

It can be shown that if $0 < T < 1$ and $t \in [-T,T]$, then

$$G_T(F)(t) = -\frac{1}{2}\int_0^t S(t-t')F(t')dt'.$$

The next step is to define a sequence $\{u_n\}_{n=0}^\infty$ of functions which are solutions to the equations

$$\partial_t u_0 - \partial_x^5 u_0 + \partial_x^{-1}\partial_y^2 u_0 = 0, \qquad \partial_t u_n - \partial_x^5 u_n + \partial_x^{-1}\partial_y^2 u_n = -u_{n-1}\partial_x u_{n-1},$$
$$u_0(x,y,0) = f(x,y). \qquad u_n(x,y,0) = f(x,y).$$

By the discussion above, for $t \in (0,T)$, we have the identity $u_n(x,y,t) = \Phi(u_{n-1}(x,y,t))$, where

$$\Phi(u) = \psi(t)S(t)f + G_T\left(\partial_x(u^2)\right).$$

Using this decomposition, we obtain the following estimate, which will be crucial to our iteration argument:

Lemma 5.6. *Let* $\delta_1 \ge 0$, $\delta_2 \ge 0$, $\frac{1}{2} < b < 1$, $\beta \in (0, 1-b)$, *and* $0 < T \le 1$. *Then*

$$\|u_n\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} \le C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)} + CT^\gamma\|\partial_x(u_{n-1}^2)\|_{Y^{\delta_1,\delta_2,-\beta}(\mathbb{R}^3)},$$

for some $0 < \gamma < 1$.

Proof. Since $u_n = \Phi(u_{n-1})$ for $t \in (0,T]$, it suffices to show that

$$\|\Phi(v)\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} \leq C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)} + CT^{\gamma}\|\partial_x(v^2)\|_{Y^{\delta_1,\delta_2,-\beta}(\mathbb{R}^3)},$$

for any function v. To show that this estimate holds, we recall equations (1.2) and (1.4) of [42], which state that

$$\|\psi(t)S(t)f\|_{X^{s_1,s_2,b,\varepsilon}(\mathbb{R}^3)} \leq C\|f\|_{H^{s_1,s_2}(\mathbb{R}^2)},$$

for $s_1, s_2, b \in \mathbb{R}$ and $\varepsilon \geq 0$, and

$$\|G_T(F)\|_{X^{s_1,s_2,b,\varepsilon}(\mathbb{R}^3)} \leq CT^{\gamma}\|F\|_{X^{s_1,s_2,-\beta,\varepsilon}(\mathbb{R}^3)},$$

for $s_1, s_2 \in \mathbb{R}$, $b \in (\frac{1}{2}, 1)$, $\varepsilon \geq 0$, and $\beta \in (0, 1-b)$. From these estimates, it follows that

$$\begin{aligned}
\|\Phi(v)\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} &= \|\Lambda^{\delta_1,\delta_2}\Phi(v)\|_{X^{0,0,b,0}(\mathbb{R}^3)} \\
&\leq C\|\Lambda^{\delta_1,\delta_2} f\|_{H^{0,0}(\mathbb{R}^2)} + CT^{\gamma}\|\partial_x(u_{n-1}^2)\|_{X^{0,0,-\beta,0}(\mathbb{R}^3)} \\
&\leq C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)} + CT^{\gamma}\|\partial_x(u_{n-1}^2)\|_{Y^{\delta_1,\delta_2,-\beta}(\mathbb{R}^3)},
\end{aligned}$$

as desired.

□

If we now apply Lemma 5.5, it is a simple matter to show that

$$\|u_n\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} \leq C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)} + CT^{\gamma}\|u_{n-1}\|^2_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)}.$$

Using a simple proof by induction, one may show that

$$\|u_n\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} \leq 2C\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)},$$

for all $n \in \mathbb{N} \cup \{0\}$, if we choose T such that

$$T < \frac{1}{(4C^2\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)})^{1/\gamma}}. \tag{5.7}$$

The final step is to show that the sequence converges. Applying Lemmas 5.6 and 5.5 once again, a similar computation will show that

$$\begin{aligned}
\|u_n - u_{n-1}\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} &= \|\Phi(u_n) - \Phi(u_{n-1})\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} \\
&\leq CT^{\gamma}M_{n-1}\|u_{n-1} - u_{n-2}\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} \\
&\leq 4C^2T^{\gamma}\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)}\|u_{n-1} - u_{n-2}\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)},
\end{aligned}$$

where

$$M_{n-1} = \|u_{n-1}\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)} + \|u_{n-2}\|_{Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)}.$$

Thus, the sequence will converge if T satisfies

$$T < \frac{1}{(8C^2\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)})^{1/\gamma}}. \tag{5.8}$$

Sufficiently small. Thus, the sequence converges in $Y^{\delta_1,\delta_2,b}(\mathbb{R}^3)$ to a solution

$$u \in Y^{\delta_1,\delta_2,b}(\mathbb{R}^3) \subset C([0,T],G^{\delta_1,\delta_2}(\mathbb{R}^2)).$$

As a concluding remark, we observe that for later convenience, we may choose the time of existence to be

$$T = \frac{c_0}{(1+\|f\|_{G^{\delta_1,\delta_2}(\mathbb{R}^2)})^{1/\gamma}}. \tag{5.9}$$

For appropriate choice of c_0, this will satisfy inequalities (5.7) and (5.8).

5.5 Proof of Theorem 5.3

In this section, we begin the proof of Theorem 5.3. The first step is to obtain estimates on the growth of the norm of the solutions. For this, we will need the following approximate conservation law

Theorem 5.7. *Let $\delta_1 \geq 0$. Then there is a $b \in (1/2,1)$ and a $C > 0$, such that $u \in Y^{\delta_1,0,b}(I)$ is a solution to the Cauchy problem (5.1) on the time interval $[0,T)$, we have the estimate*

$$\sup_{t\in[0,T)} \|u(t)\|^2_{G^{\delta_1,0}} \leq \|f\|^2_{G^{\delta_1,0}} + C\delta_1 \|u\|^3_{Y^{\delta_1,0,b}(I)}. \tag{5.10}$$

Before we may state the proof, let us first state some preliminary lemma. This will be used to prove the following key estimate

Lemma 5.8. *Let $N(u)$ be as in equation (5.4) for $\delta_1 \geq 0$ and $\delta_2 = 0$. Then for b and β as in Lemma 5.4, we have*

$$\|N(u)\|_{Y^{0,0,-\beta}} \leq C\delta_1 \|u\|^2_{Y^{\delta_1,0,b}}.$$

Proof. We first observe that the inequality in Lemma 5.4, for the case $\varepsilon = 0$, is equivalent to

$$\begin{aligned}&\left\| \xi\lambda(s_1,s_2,-\beta,0) \int \frac{\hat{f}(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)}{\langle \xi-\xi_1\rangle^{s_1}\langle \eta-\eta_1\rangle^{s_2}\langle \phi(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)\rangle^b} \times \right.\\ &\left. \times \frac{\hat{g}(\xi_1,\eta_1,\tau_1)}{\langle \xi_1\rangle^{s_1}\langle \eta_1\rangle^{s_2}\langle \phi(\xi_1,\eta_1,\tau_1)\rangle^b}\, d\xi_1 d\eta_1 d\tau_1 \right\|_{L^2_{\xi,\eta}} \leq C\|f\|_{L^2_{x,y}}\|g\|_{L^2_{x,y}},\end{aligned}$$

where we denote $\phi(\tau,\xi,\eta) = \langle \tau - m(\xi,\eta)\rangle$. With this in mind, we observe that the left side of the inequality in Lemma 5.8 can be estimated by Lemma 2.21 as

$$\begin{aligned}\|N(u)\|_{Y^{0,0,-\beta}} \leq C\delta_1 &\left\| \frac{\xi\langle\xi\rangle^{-1}}{\langle\phi(\tau,\xi,\eta)\rangle^\beta} \int \frac{e^{\delta_1|\xi-\xi_1|}\hat{u}(\xi-\xi_1,\eta-\eta_1,\tau-\tau_1)}{\langle \xi-\xi_1\rangle^{-1}} \times \right.\\ &\left. \times \frac{e^{\delta_1|\xi_1|}\hat{u}(\xi_1,\eta_1,\tau_1)}{\langle\xi_1\rangle^{-1}}\, d\xi_1 d\eta_1 \right\|_{L^2_{\xi,\eta}}.\end{aligned}$$

If we apply Lemma 5.4 with $s_1 = -1$, $s_2 = 0$, it will follow that

$$\|N(u)\|_{Y^{0,0,-\beta}} \leq C\delta_1 \|u\|^2_{Y^{\delta_1,0,b}}.$$

□

5.5.1 Proof of Theorem 5.7

Begin by applying the operator $\Lambda^{\delta_1,0}$ to equation (5.1). If we let $U = \Lambda^{\delta_1,0}u$, then equation (5.1) becomes

$$U_t - U_{xxxxx} + \partial_x^{-1}U_{yy} + UU_x = N(u),$$

where $N(u)$ is as defined in Lemma 5.8. Multiplying this by U and integrating with respect to the spatial variables, we obtain

$$\int UU_t - UU_{xxxxx} + U\partial_x^{-1}U_{yy} + U^2U_x \; dxdy = \int UN(u) \; dxdy.$$

If we apply integration by parts, we may rewrite the left-hand side as

$$\frac{d}{dt}\int \frac{1}{2}U^2 \; dxdy + \int U_{xx}U_{xxx}dxdy - \int U_y\partial_x^{-1}U_y \; dxdy + \int U^2U_x \; dxdy,$$

which can then be rewritten as

$$\frac{1}{2}\frac{d}{dt}\int U^2 \; dxdy + \frac{1}{2}\int \partial_x(U_{xx}^2)dxdy - \frac{1}{2}\int \partial_x[(\partial_x^{-1}U_y)^2] \; dxdy$$
$$+\frac{1}{3}\int \partial_x(U^3) \; dxdy.$$

To proceed, we observe that U and its derivatives vanish at infinity. We thus obtain the formal identity

$$\frac{d}{dt}\int U^2(x,y,t) \; dxdy = 2\int U(x,y,t)N(u)(x,y,t) \; dxdy.$$

Integrating with respect to time yields

$$\int U^2(x,y,t) \; dxdy = \int U^2(x,y,0) \; dxdy$$
$$+2\int_0^t \int U(x,y,t')N(u)(x,y,t') \; dxdydt'.$$

Applying Cauchy-Schwarz and the definition of U, we obtain

$$\|u(t)\|^2_{G^{\delta_1,0}} \leq \|f\|^2_{G^{\delta_1,0}} + \|u\|_{Y^{\delta_1,0,b}}\|N(u)\|_{Y^{0,0,-\beta}(I)},$$

where β is as defined previously. If we now apply Proposition 5.7 and the fact that $\beta < 1/2 < b$, we can further estimate this by

$$\|u(t)\|^2_{G^{\delta_1,0}} \leq \|f\|^2_{G^{\delta_1,0}} + C\delta_1\|u\|^3_{Y^{\delta_1,0,b}}, \tag{5.11}$$

as desired.

5.5.2 Proof of Theorem 5.3

With the tools established in the previous section, we may begin the proof of Theorem 5.3. By the embedding in equation (5.2), it suffices to consider the case $\delta_2 = 0$. To begin, let us first suppose that T^* is the supremum of the set of times T for which

$$u \in C([0,T]; G^{\delta_1,0}).$$

If $T^* = \infty$, there is nothing to prove, so let us assume that $T^* < \infty$. In this case, it suffices to prove that

$$u \in C\left([0,T], G^{\delta(T),0}\right), \tag{5.12}$$

for some $\delta(T) > 0$ and all $T > T^*$. To show that this is the case, we will use Theorem 5.2 and Proposition 5.7 to construct a solution which exists over subintervals of width T_0, using the parameter δ_1 to control the growth of the norm of the solution. Thus, the desired result will follow from the following proposition

Proposition 5.9. *Let $T > 0$ and $T_0 > 0$ be numbers such that $nT_0 \leq T < (n+1)T_0$. Then the solution u to the Cauchy problem (5.1) satisfies*

$$\sup_{t\in[0,nT_0]} \|u(t)\|^2_{G^{\delta(T),0}} \leq \|f\|^2_{G^{\delta(T),0}} + 2^3 C\delta(T) n \|f\|^3_{G^{\delta_1,0}}, \tag{5.13}$$

and

$$\sup_{t\in[0,nT_0]} \|u(t)\|^2_{G^{\delta(T),0}} \leq 4\|u(t)\|^2_{G^{\delta_1,0}}, \tag{5.14}$$

if

$$\delta(T) \leq \delta_1 \quad \textit{and} \quad \delta(T) \leq \frac{C}{T},$$

for some constant $C > 0$.

Proof. By induction on n. The base case $n = 1$ follows from equation (5.11), Theorem 5.2, and the embedding $G^{\delta_1,\delta_2} \hookrightarrow G^{\delta_1',\delta_2'}$ when $\delta_1' \leq \delta_1$ and $\delta_2' \leq \delta_2$. Suppose, then, that the result holds for $n \leq k$. The inductive hypothesis then tells us that

$$\sup_{t\in[0,kT_0]} \|u(t)\|_{G^{\delta(T),0}} \leq \|f\|^2_{G^{\delta(T),0}} + C\delta(T)\|f\|^3_{G^{\delta_1,0}},$$

and

$$\sup_{t\in[0,kT_0]} \|u(t)\|^2_{G^{\delta(T),0}} \leq 4\|f\|^2_{G^{\delta_1,0}}.$$

If we apply the inductive hypothesis on the interval $[kT_0,(k+1)T_0]$, then

$$\sup_{t\in[kT_0,(k+1)T_0]} \|u(t)\|_{G^{\delta(T),0}} \leq \|u(kT_0)\|^2_{G^{\delta(T),0}} + C\delta(T)\|u(kT_0)\|^3_{G^{\delta_1,0}},$$

and

$$\sup_{t\in[kT_0,(k+1)T_0]} \|u(t)\|^2_{G^{\delta(T),0}} \leq 4\|u(kT_0)\|^2_{G^{\delta_1,0}}.$$

If we apply equations (5.13) and (5.14) to these, we get

$$\|u(kT_0)\|_{G^{\delta(T),0}} \leq \|f\|^2_{G^{\delta(T),0}} + 2^3 C k \delta(T) \|f\|^3_{G^{\delta_1,0}},$$

and

$$\|u(kT_0)\|^2_{G^{\delta_1,0}} \leq 4\|f\|^2_{G^{\delta_1,0}}.$$

Combining these together, we obtain

$$\begin{aligned}\sup_{t\in[kT_0,(k+1)T_0]} \|u(t)\|_{G^{\delta(T),0}} &\leq \left(\|f\|^2_{G^{\delta(T),0}} + 2^3 C k \delta(T) \|f\|^3_{G^{\delta_1,0}}\right) + 2^3 C \delta(T) \|f\|^3_{G^{\delta_1,0}} \\ &= \|f\|^2_{G^{\delta(T),0}} + 2^3 C (k+1) \delta(T) \|f\|^3_{G^{\delta_1,0}}.\end{aligned}$$

It follows that

$$\sup_{t\in[0,(k+1)T_0]} \|u(t)\|_{G^{\delta(T),0}} \leq \|f\|^2_{G^{\delta(T),0}} + 2^3 C (k+1) \delta(T) \|f\|^3_{G^{\delta_1,0}}.$$

To complete the proof, we need to show that

$$\sup_{t\in[0,(k+1)T_0]} \|u(t)\|^2_{G^{\delta(t),0}} \leq 4\|f\|^2_{G^{\delta_1,0}}. \tag{5.15}$$

Next, we observe that the assumption $nT_0 \leq T < (n+1)T_0$ implies that

$$n \leq \frac{T}{T_0} < n+1 \leq \frac{T}{T_0} + 1 \leq \frac{2T}{T_0}.$$

Since $k+1 \leq n+1$, we have

$$2^3 C (k+1) \delta(T) \|f\|_{G^{\delta_1,0}} \leq 2^3 C \frac{2T}{T_0} \delta(T) \|f\|_{G^{\delta_1,0}}.$$

Recalling the definition of T_0 in equation (5.9), we can rewrite this as

$$\begin{aligned}2^3 C \frac{2T}{T_0} \delta(T) \|f\|_{G^{\delta_1,0}} &= 2^3 C \|f\|_{G^{\delta_1,0}} (1 + \|f\|_{G^{\delta_1,0}})^{1/\gamma} T \delta(T) \\ &\leq C (1 + \|f\|_{G^{\delta_1,0}})^{1/\gamma+1} T \delta(T).\end{aligned}$$

Thus, for equation (5.15) to hold, it suffices to have

$$\delta(T) \leq \frac{C}{T},$$

where C is a constant that depends on the norm $\|f\|_{G^{\delta_1,0}}$.

Since we have shown that the result holds for $n = 1$, and we have shown that the result for $n = k$ implies it for $n = k+1$, then the result holds for all n. This completes the proof of Proposition 5.9, from which Theorem 5.3 follows as an immediate corollary.

□

Conclusions.

We have discussed the local well-posedness for Kadomtsev-Petviashvili I, II equation in an anisotropic Gevrey space and the local well-posedness for the Kawahara equation and the m-Korteweg-de Vries system with the initial data in analytical Gevrey spaces. We proved the existence of solutions using the Banach contraction mapping principle. This was done by using the bilinear and trilinear estimates in Gevrey-Bourgain. We used this local result and a Gevrey approximate conservation law to prove that global solutions exist. These solutions are Gevrey class of order m in the time variable with $m = 3\sigma, 5\sigma, 5$.

Open Question[2]

Open Question : Does the KdV radius of analyticity persist for all time?
The best result in the references shows that the analytic radius has a polynomial decay lower bound, which means that analytic radius may shrink to zero as time goes to infinity. In this note, Ming Wang[3] proved that, for the KdV equation with some damping, the analytic radius has a fixed positive lower bound uniformly for all time.

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[3]Ming Wang, Nondecreasing analytic radius for the KdV equation with a weakly damping February 2022 Nonlinear Analysis 215(1):112653

Numerical analysis applications

As an extension of the theoretical works, we will have to prove the exponential convergence rate for a spectral projection of the initial value problem for some partial differential equations[4][5][6].

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