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# Master

**Domaine :** Mathématiques et Informatiques

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# Mémoire

Fixed Point Methods for Fractional differential Equations with Retarded and Advanced Arguments

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# *Dedication*

To my beloved family, both near and far, to my amazing parents, my pillars of strength, to my wonderful siblings, my partners in crime, to my extended family, the bonds that intertwine.

To my loving aunts and their beautiful daughters, Marwa, Chazi, Hanine, Imane, you are my sisters.

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# Abstract

In this work, we establish the existence and uniqueness of solutions for implicit fractional differential equations with retarded and advanced arguments involving the Caputo-Katugampola fractional derivative operator. Using the Banach contraction principle, Schauder′s theorem and Schaefer′s theorem.

**Key words and phrases :** Compactness in metric space, Completely continuous operators, Existence solutions, Functional differential equations, Retarded arguments, Advanced arguments, Fixed point.

**AMS Subject Classification :** 34A08, 34K32, 34K37.

# Résumé

Dans ce travail, nous établissons l'existence de solutions de l′équations différentielles fractionnaires implicites avec condition de retard et anticipation, comprend l'opérateur dérivée fractionnaire de Caputo-Katugampola. En utilisant la contraction de Banach, théorème de Schauder et théorème de Schaefer.

**Mots clés :** Compacité dans les espaces métriques, Opérateurs complètement continus, Existence de solutions, Équations différentielles fractionnaire, Condition de retard, Condition d′anticipation, Point fixe.

**الملخص**

فًي هذا العمل، ندرس وجود و وحدانية الحلول للمعادلات التفاضلية الكسرّية بواسطة شروط متقدمة و متأخرة التًي تتضمن مؤثر المشتق لكابتو كاتيقامبولا.

باستعمال نظرية النقطة الصامدة لبناخ, نظرية النقطة الصامدة لشاودر و نظرية النقطة الصامدة لشيفر.

**الكلمات المفتاحية** : التراص فيً الفضاء المتري، المؤثرات التامة المستمرة، وجود الحلول، المعادلات التفاضلية الكسرية، شروط متقدمة ومتأخرة، النقطة الصامدة

# Notation

| | |
|---|---|
| $\mathbb{R}_+$ | set of all nonnegative real numbers |
| $\mathbb{R}^n$ | set of all $n$-tuples $x=(x_1,x_2,\ldots,x_n)$ |
| $\mathbb{R}^n_+$ | set of all $x\in\mathbb{R}^n$ with $x_i\geq 0$ for all $i$ |
| $x\leq y$ | natural order relation in $\mathbb{R}^n: x_i\leq y_i$ for all $i$ |
| $x<y$ | strict order relation in $\mathbb{R}^n: x_i<y_i$ for all $i$ |
| $x\leq a$ | for $x\in\mathbb{R}^n$ and $a\in\mathbb{R}: x_i\leq a$ for all $i$ |
| $\vert\,.\,\vert$ | norm in $X$, also denoted by $\vert.\vert_X$ |
| $B_r(u;X)$ | open ball $\{v\in X: \vert u-v\vert<r\}$ ( $B_r(u)$ for short) |
| $\bar{B}_r(u;X)$ | closed ball $\{v\in X; \vert u-v\vert\leq r\}$ $\left(\bar{B}_r(u)\right.$ for short $\left.\right)$ |
| $\bar{\Omega}, \operatorname{int}\Omega$ | closure of $\Omega$, interior of $\Omega$ |
| $\partial\Omega$ | boundary of $\Omega: \partial\Omega=\bar{\Omega}\backslash\operatorname{int}\Omega$ |
| $\operatorname{conv} A$ | convex hull of $A$ |
| $\overline{\operatorname{conv}}A$ | closed convex hull of $A$ |
| $C^k\left(\Omega;\mathbb{R}^n\right)$ | set of $k$-times continuously differentiable functions $u:\Omega\to\mathbb{R}^n$ $\left(\Omega\subset\mathbb{R}^N\right.$ open$\left.\right)$ |
| $C^k\left(\bar{\Omega};\mathbb{R}^n\right)$ | space of all functions $u\in C^k\left(\Omega;\mathbb{R}^n\right), u=(u_1,\ldots,u_n)$ such that $D^\alpha u_i$ admits a continuous extension to $\bar{\Omega}$ for all $i$ and $\alpha=(\alpha_1,\ldots,\alpha_N)$ with $\vert\alpha\vert=\sum_i^N\alpha_j\leq k$. Here $D^\alpha=\partial^{\vert\alpha\vert}/\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}$ |
| $\vert\vert u\vert\vert_\infty$ | $\max_{x\in\bar{\Omega}}\vert u(x)\vert$ $\left(\Omega\subset\mathbb{R}^N\right.$ bounded open, $\left.u\in C\left(\bar{\Omega};\mathbb{R}^n\right)\right)$ |
| $C^k(\bar{\Omega})$ | stands for $C^k(\bar{\Omega};\mathbb{R})$ |
| $C^k[a,b]$ | stands for $C^k([a,b])$ |
| $L^p\left(\Omega;\mathbb{R}^n\right)$ | space of all measurable functions $u:\Omega\to\mathbb{R}^n$ with $\int_\Omega\vert u(x)\vert^p dx<\infty$ $\left(\Omega\subset\mathbb{R}^N\right.$ open, $\left.1\leq p<\infty\right)$ |
| $\vert\vert\cdot\vert\vert_p$ | norm in $L^p\left(\Omega;\mathbb{R}^n\right), \vert\vert u\vert\vert_p=\left(\int_\Omega\vert u(x)\vert^p dx\right)^{1/p}$ |
| $L^\infty\left(\Omega;\mathbb{R}^n\right)$ | space of all measurable functions $u:\Omega\to\mathbb{R}^n$ for which there is a constant $c$ with $\vert u(x)\vert\leq c$ for a.e. $x\in\Omega$ |
| $\vert\vert\cdot\vert\vert_\infty$ | norm in $L^\infty\left(\Omega;\mathbb{R}^n\right)$, $\vert\vert u\vert\vert_\infty=\inf\{c:\vert u(x)\vert\leq c$ a.e. on $\Omega\}$ |
| $L^p(\Omega)$ | stands for$L^p(\Omega,\mathbb{R})(1\leq p\leq\infty)$ |

# Contents

# Introduction

A fixed point of a map $f$ defined of a space $E$ in itself is a element $x \in E$ which satisfies $f(x) = x$. The fixed point theorem is one of the mathematical techniques that allows usto solve many problems. Usually find solutions for differential or integral equations amounts to finding fixed points for operators associated with them. In 1922, Stefan Banach proved a very strong theorem that guarantees the existence and uniqueness of a fixed point for any contracting map of a metric space into itself, this theorem is known as the Banach contraction principle and it has many applications, especially in numerical analysis and in the theory of differential equations.
The concept of the fixed point has been developed by several mathematicians so we These include Brouwer, Schauder and Schaefer .

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator $D^{\alpha}_{a^+}$ where $a, \alpha \in \mathbb{R}$. Several approaches to fractional derivatives exist : Riemann-Liouville (RL), Hadamard, katugampola, Grunwald-Letnikov (GL), Weyl and Caputo etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to [15, 35, 37, 39, 46] and the references therein. Throughout the memory, we will use the Caputo-Katugampola derivative.

Fractional differential equations are found to be of great interest in view of their utility in modeling and explaining natural phenomena occurring in biophysics, quantum mechanics, wave theory, polymers, continuum mechanics, etc . For current advances of fractional calculus, we refer the reader to the monographs [3, 4, 5, 33, 37, 40, 43, 48] and the references therein. From the last few years fractional differential equations (FDE) theory has gained significant attraction and importance. For example mechanics, signal processing, memory mechanism, considerable attention has been given to the existence of solutions of boundary value problem and boundary conditions for implicit fractional differential equations and integral equations with Caputo and Caputo-Katugampola derivative. See for example [9, 10, 11, 12, 18, 22, 36, 45] and references therein.

The differential equation with delay is a special type of functional differential equations. Delay differential equations arise in many biological and physical applications and it often forces us to consider variable or state-dependent delays. The functional differential equations with state-dependent delay have many important applications in mathematical models of real phenomena and the study of this type of equations has received great attention in recent years. We refer the reader to the monographs [6, 19, 20, 21, 23, 24, 25, 26, 27, 32].

Recently there have been special situations in decision making, organizational transformation, chaotic equations, wavelet theory and so on, where specific equations with anticipation as well as retardation and anticipation appear in modeling [14, 30, 31]. This lead to the initiation of the study of the general theory of differential equations involving anticipation as well as retardation and anticipation in [38, 44] and continued in [28, 29]. The authors studied the existence and uniqueness of solutions for fractional differential equations and inclusions with retarded and advanced arguments, see [8, 21, 23] and the references therein.

In the following we give an outline of our memory organization consisting of four chapters.

**In Chapter 1**, we gives some notations, definitions, lemmas which are used throughout this memory.

**In Chapter 2**, we establish the compactness on metric space and some characterization of compactness on continuous function space.

**In Chapter 3**, we will study some properties and definitions of completely continuous operators, and the theory of fixed points.

**In Chapter 4**, we establish the existence and uniqueness of solutions for a class of problems for nonlinear implicit of Caputo-Katugampola fractional differential equations with retarded and advanced arguments, given by

$$
\begin{cases}
{}^{\varrho}_{c}D^{\beta}_{a+}(x(t)-h(t,x^t)) = l(t,x^t,{}^{\varrho}_{c}D^{\beta}_{a+}x(t)), \text{ for } t \in J := [a,T],\ 1<\beta\leq 2,\\
x(t)=\varphi(t), \qquad t\in[a-b,a],\ b>0,\\
x(t)=\vartheta(t), \qquad t\in[T,T+\sigma],\ \sigma>0,
\end{cases} \tag{1}
$$

where ${}^{\varrho}_{c}D^{\beta}_{a+}$ is the Caputo-Katugampola fractional derivative,

$$l : J\times C([-b,\sigma],\mathbb{R})\times\mathbb{R}\to\mathbb{R},$$

and

$$h : J\times C([-b,\sigma],\mathbb{R})\to\mathbb{R},$$

with $h(a,x^a)=h(T,x^T)=0$, is a given function, $\varphi\in C([a-b,a],\mathbb{R})$ with $\varphi(a)=0$ and $\vartheta\in C([T,T+\sigma],\mathbb{R})$ with $\vartheta(T)=0$.
We denote by $x^t$ the element of $C([-b,\sigma])$ defined by:

$$x^t(\tau)=x(t+\tau) : \tau\in[-b,\sigma],$$

here $x^t(\cdot)$ represents the history of the state from time $t-b$ up to time $t+\sigma$.

The result based on the Banach contraction principle, Schauder's and Schaefer's fixed point theorems.

# Chapter 1

# Basic tools and fractional calculus

In this chapter, we present definitions and some properties of the Gamma and beta functions, and some properties required to establish our main results.

## 1.1 Notations and Definitions

*Let $C([-b,\sigma],E)$ be the Banach space of all continuous functions from $[-b,\sigma]$ into the Banach space $E$ equipped with the norm:*

$$\|x\|_{[-b,\sigma]} = \sup\{\|x(t)\| : -b \leq t \leq \sigma\}, \quad b,\sigma > 0,$$

*and $C([a,T],E)$ is the Banach space endowed with the norm:*

$$\|x\|_{[a,T]} = \sup\{\|x(t)\| : a \leq t \leq T\}.$$

*Also, let $E_1 = C([a-b,a],E)$ , $E_2 = C([T,T+\sigma],E)$, $a,T \in \mathbb{R}_+$ and let the space:*

$$AC^1(J) := \{v : J \longrightarrow E : v' \in AC(J)\},$$

*where*

$$v'(t) = t\frac{d}{dt}v(t), \; t \in J = [a,T],$$

*$AC(J,E)$ is the space of absolutely continuous functions on $J$,*

$$\mathcal{C} = \{x : [a-b,T+\sigma] \longmapsto E : x\mid_{[a-b,a]} \in C([a-b,a]), x\mid_{[a,T]} \in AC^1([a,T]),$$

$$\text{and } x\mid_{[T,T+\sigma]} \in C([T,T+\sigma])\},$$

be the spaces endowed, respectively, with the norms:

$$\|x\|_{[a-b,a]} = \sup\{\|x(t)\| : a-b \leq t \leq a\},$$

*and*

$$\|x\|_{[T,T+\sigma]} = \sup\{\|x(t)\| : T \leq t \leq T+\sigma\},$$
$$\|x\|_{\mathcal{C}} = \sup\{\|x(t)\| : a-b \leq t \leq T+\sigma\}.$$

*Consider the space $X_o^p(a,b)$, $(o \in \mathbb{R}, 1 \leq p \leq \infty)$ of those complex-valued Bochner measurable functions $h$ on $[a,b]$ for which $\|h\|_{X_o^p} < \infty$, where the norm is defined by :*

$$\|h\|_{X_o^p} = \left(\int_a^b |t^o h(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, o \in \mathbb{R}).$$

*In particular, where $o = \frac{1}{p}$ the space $X_o^\alpha(a,b)$ coincides $L^p(a,b)$ space, i.e., $X_{\frac{1}{p}}^p(a,b) = L^p(a,b)$.*

*Denote by, $L^\infty(J,\mathbb{R})$ the Banach space of essentially bounded measurable functions, $h : J \longrightarrow \mathbb{R}$ equipped with the norm:*

$$\|h\|_{L^\infty} = \inf\{o \geq 0; \;\; |h(t)| \leq o \text{ a.e. on } \;\; J\}.$$

## 1.2 Functional analyses

### Topological spaces

**Definition 1.1.** *A topology on a set $E$ is collection $\tau$ of subsets of $E$ having the following propreties:*

1. $\emptyset$ *and $E$ are in $\tau$ .*
2. *The union of the elements of any subcollection of $\tau$ is in $\tau$ .*
3. *The intersection of the elements of any finite subcollection of $\tau$ is in $\tau$ .*

### Metric spaces

**Definition 1.2.** *A metric $d$ on a set $E$ is an application $d : E \times E \longrightarrow \mathbb{R}^+$ such that for all $u, v$ and $w$ in $E$ :*

1. $d(u, v) = 0 \Leftrightarrow u = v$ *(positivity).*
2. $d(u, v) = d(v, u)$ *(symmetry).*
3. $d(u, w) \leq d(u, v) + d(v, w)$ *(triangle inequality).*

*A metric space (E,d) is a set $E$ with a metric $d$ defined on $E$ .*

**Definition 1.3.** *A normed vector space (E,$\|.\|$) is a vector space $E$ together, with an application $\|\cdot\|$ : $E \longrightarrow \mathbb{R}^+$ , called a norm on $E$ , such that for all $u, v \in E$ and $\lambda \in \mathbb{R}$ :*

1. $\|u\| = 0 \Leftrightarrow u = 0.$
2. $\|\lambda u\| = |\lambda|\|u\|.$
3. $\|u + v\| \leq \|u\| + \|v\|.$

**Remark 1.4.** *If (E,$\|\cdot\|$) is a normed vector space $E$ , then $d : E \times E \longrightarrow \mathbb{R}^+$ defined by :*

$$d(u, v) = \|u - v\|,$$

*is a metric on $E$ .*

**Definition 1.5.** *A sequence $\{u_n\}$ in a metric space $(E, d)$, is a Cauchy sequence if $\forall\delta > 0$, there is an $M \in \mathbb{N}$, such that:*

$$d(u_n, u_m) < \delta,$$

*whenever $n, m \geq M$.*

**Definition 1.6.** *A metric space $E$ is called complete, if all Cauchy sequences in $E$ converges to a point in $E$.*

### Banach space

**Definition 1.7.** *A normed vector space $E$ is called a Banach space, if every Cauchy sequence in $E$ converges.*

### Compact spaces

**Definition 1.8.** *A collection $B$ of subsets of a space $E$ is said to cover $E$, or to be a covering of $E$, if the union of the elements of $B$ is equal to $E$.*
*It is called an open covering of $E$, if its elements are open subsets of $E$.*

**Definition 1.9.** *A space $E$ is said to be compact, if every open covering $B$ of $E$ contains a finite subcollection, that also covers $E$.*

**Definition 1.10.** *Let $E$ be a compact metric space, and $F$ be a complete space.*
*We denote by $C(E, F)$ the vector space of continuous functions, defined on $E$ to value in $F$.*
*A subset $\mathfrak{D} \subset C(E, F)$ is said to be equicontinuous in $y_0$ with $y_0 \in E$, if $\forall\delta > 0$, $\exists\eta > 0$, such that:*

$$\forall h \in \mathfrak{D}, \quad \forall y \in E, \quad d_E(y, y_0) < \eta \Longrightarrow d_F(h(y), h(y_0)) < \delta.$$

*$\mathfrak{D}$ is said to be equicontinuous, if it is equicontinuous at any point of $E$.*
*It is said uniformly equicontinuous, if $\eta$ depends only on $\delta$.*

# 1.3 Special Function

## 1.3.1 Gamma Function

The Gamma function is a complex function, it extends the factorial to the real numbers and even complex numbers.

**Definition 1.11.** *the Gamma function is defined by:*

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \qquad \forall z \in \mathbb{C}, \quad Re(z) > 0.$$

**Properties 1.12.**

1. *The Gamma function checks the following recurrence relation:*

$$\Gamma(z+1) = z\Gamma(z), \quad Re(z) > 0.$$

2. *Euler's Gamma function generalizes the factorial:*

$$\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}^*.$$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

4. $\Gamma(n + \dfrac{1}{2}) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}$.

*Proof.* 1. We have

$$\Gamma(z+1) = \int_0^{+\infty} x^{(z+1)-1} e^{-x} dx.$$

Integration by part, we get

$$\Gamma(z+1) = \int_0^{+\infty} x^z e^{-x} dx = \left[-x^z e^{-x}\right]_0^{\infty} + z\int_0^{+\infty} x^{z-1} e^{-x} dx = z\Gamma(z).$$

2. Euler's Gamma function generalizes the factorial:

$$\Gamma(n+1) = n!, \quad \forall n \in \mathbb{N}.$$

We have

$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x} dx = 1,$$

and using the property (1) we get

$$\begin{aligned} \Gamma(2) =& 1.\Gamma(1) = 1! \\ \Gamma(3) =& 2.\Gamma(2) = 2.1! = 2! \\ \Gamma(4) =& 3.\Gamma(3) = 3.2! = 3! \\ &\cdots \\ &\cdots \\ \Gamma(n+1) =& n \cdot \Gamma(n) = n \cdot (n-1)! = n!. \end{aligned}$$

3. We now show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
From the Definition (1.11) we have:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-\frac{1}{2}} e^{-x} dx.$$

If we set $x = u^2$, then $dx = 2udu$, we get :

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-u^2} du. \tag{1.1}$$

Equivalently, we can write:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-v^2}dv. \tag{1.2}$$

If we multiply (1.1) and (1.2) we get

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-\left(u^2+v^2\right)}dudv.$$

It is a double integral, which can be evaluated in polar coordinates to obtain:

$$\begin{cases} u = r\cos\theta \\ v = r\sin\theta, \qquad r \in [0,+\infty), \theta \in [0,\frac{\pi}{2}], \end{cases}$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}rdrd\theta = \pi.$$

Hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

4. We will demonstrate the formula

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}, \tag{1.3}$$

by recurrence on $n \in \mathbb{N}$.

(a) for $n = 0$, we have $\Gamma\left(0+\frac{1}{2}\right) = \frac{(0)!\sqrt{\pi}}{4^0(0!)} = \sqrt{\pi}$.

(b) Suppose the formula (1.3) is verified for $(n-1)$ and consider $n$.
That is, suppose that :

$$\Gamma\left((n-1)+\frac{1}{2}\right) = \frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!},$$

is verified.
Then

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!}$$
$$= \frac{(2n)!\sqrt{\pi}}{4^n n!}.$$

Therefore, the formula is checked for $n$.

□

**Remark 1.13.** *Determination of the Gamma function for negative values by the formula :*

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

*The Gamma function does not exist for integer negative values.*

**Example**

1. $\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4\sqrt{\pi}}{3}$.
2. $\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = \frac{\Gamma\left(\frac{-3}{2}\right)}{-\frac{5}{2}} = \frac{\frac{4\sqrt{\pi}}{3}}{\frac{-5}{2}} = -\frac{8\sqrt{\pi}}{15}$.

### 1.3.2 Beta Function

The Beta function is a special type of function, which is also known as the first kind Euler′ integral.

**Definition 1.14.** *We define the Beta function by:*

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad \forall p,q \in \mathbb{C}, Re(p) > 0, Re(q) > 0.$$

$\forall p,q \in \mathbb{C}, Re(p) > 0, Re(q) > 0$.
*We have*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

**Example 1.15.**

$$B(\frac{1}{2},\frac{1}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})}=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}=\pi.$$

**Properties 1.16.**

1. *The Beta function is symmetric:*

$$B(p,q)=B(q,p),\quad Re(p)>0, Re(q)>0.$$

2. *The Beta function can be given by:*

$$B(p,q)=B(p+1,q)+B(p,q+1),\quad Re(p)>0, Re(q)>0.$$

*Proof.* 1. We have

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx.$$

The change of variable $x=1-s$

$$B(p,q)=-\int_1^0 (1-s)^{p-1}x^{q-1}dx=B(q,p).$$

2. Indeed

$$\begin{aligned} B(p,q)&=\int_0^1 [x+(1-x)]x^{p-1}(1-x)^{q-1}dx\\ &=\int_0^1 x^p(1-x)^{p-1}dx+\int_0^1 x^{p-1}(1-x)^q dx\\ &=B(p+1,q)+B(p,q+1). \end{aligned}$$

□

### 1.3.3 Mittag–Leffer Function

The Mittag–Leffer function represents a generalization of the exponential function, and play an important role in the theory of differential equations linear fractions with constant coefficients.

**Definition 1.17.** *The Mittag-Leffter function is defined by:*

$$E_\alpha(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha>0.$$

*and the generalized Mittag-Leffter function is defined by:*

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (\alpha,\beta)>0.$$

**Example 1.18.**

1.

$$E_1(z)=E_{1,1}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(n+1)}=\sum_{n=0}^{+\infty}\frac{z^n}{n!}=e^z.$$

2.

$$E_2(z)=E_{2,1}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(2n+1)}=\sum_{n=0}^{+\infty}\frac{z^n}{(2n)!}=\cosh\sqrt{z}.$$

3.

$$E_{1,2}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(n+2)}=\sum_{n=0}^{+\infty}\frac{z^n}{(n+1)!}=\frac{1}{z}\sum_{n=0}^{+\infty}\frac{z^{n+1}}{(n+1)!}=\frac{1}{z}\left(e^z-1\right).$$

## 1.4 Fractional integral

### 1.4.1 Riemann–Liouville fractional integral

*Let $g$ be a continuous function on the interval $[a,T]$, we consider the integral*

$$I^{(1)}g(x) = \int_a^x g(t)dt.$$

$$I^{(2)}g(x) = \int_a^x dx_1 \int_a^{x_1} g(t)dt.$$

*According to the Fubini theorem we find*

$$I^{(2)}g(x) = \frac{1}{1!}\int_a^x (x-t)^{2-1}g(t)dt.$$

*By repeating the same operation $n$ times we get*

$$\begin{aligned} I^{(\mathrm{n})}g(x) &= \int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_2} \cdots \int_a^{x_{n-1}} (x-t)^{n-1}g(t)dt \\ &= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}g(t)dt. \end{aligned}$$

*For all $n \in \mathbb{N}$. This formula is called Cauchy's formula and we have*

$$(n-1)! = \Gamma(n).$$

*Hence*

$$I^{(\mathrm{n})}g(x) = \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}g(t)dt.$$

**Definition 1.19.** *Let $g : [a,T] \longrightarrow \mathbb{R}$. Riemann–Liouville fractional integral of $g$, is defined by the following formula:*

$$I_a^\alpha g(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}g(t)dt.$$

*Where $\alpha$ is a positive real number.*

**Theorem 1.20.** *[37] Let $g \in L_1[a,T]$ and $\alpha > 0$, then the integral $I_a^\alpha g(x)$ exists for all $x \in [a,T]$ and the function $I_a^\alpha g$ is an element of $L_1[a,T]$.*

**Theorem 1.21.** *[37] Let $\alpha, \beta > 0$ and $g \in L_1[a,T]$.*
*Then:*

$$I_a^\alpha I_a^\beta g(x) = I_a^{\alpha+\beta}g(x),$$

*for all $x \in [a,T]$.*

### 1.4.2 Hadamard fractional integral

**Definition 1.22.** *The Hadamard fractional integral of order $\alpha$, for a continuous function*

$$g : [a,b] \longrightarrow \mathbb{R}, \quad a > 0,$$

*is defined by :*

$$I_a^\alpha g(x) = \int_a^x log^{\alpha-1}(\frac{x}{t})g(t)\frac{dt}{t}, \qquad \alpha > 0.$$

**Theorem 1.23.** *[37], [42] Let $g : [a,T] \longrightarrow \mathbb{R}$ a continuous function. So for all $\alpha, \beta > 0$.*
*We have*

$$I_a^\alpha[I_a^\beta g(x)] = I_a^{\alpha+\beta}[g(x)].$$

*And*

$$I_a^\alpha[I_a^\beta g(x)] = I_a^\beta[I_a^\alpha g(x)].$$

### 1.4.3 Katugampola fractional integral

Katugampola gave a new fractional integration which generalizes both the Riemann-Liouville and Hadamard fractional integrals into a single form.

**Definition 1.24.** *Let $\beta \in \mathbb{R}, o \in \mathbb{R}$ and $g \in X_o^p(a,T)$, Katugampola fractional integral of order $\beta$ is defined by :*

$$({}^{\varrho}I_{a+}^{\beta} g)(x) = \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \int_a^x (x^{\varrho} - t^{\varrho})^{\beta-1} t^{\varrho-1} g(t)dt, \qquad x > a, \quad \varrho > 0.$$

**Theorem 1.25.** *Let $\beta > 0, 1 \leq p \leq \infty$, and $\varrho \geq o$. for $g(x) \in X_o^p(a,T)$.*
*We have*

$${}^{\varrho}\mathcal{I}_a^{\beta}\, {}^{\varrho}\mathcal{I}_a^{\gamma} g = {}^{\varrho}\mathcal{I}_a^{\beta+\gamma} g.$$

## 1.5 Fractional derivative

### 1.5.1 Riemann–Liouville fractional derivative

**Definition 1.26.** *Let $g$ an integrable function on $[a,T]$. Then The Riemann–Liouville fractional derivative of order $p$ $(n-1 \leq p < n)$ is defined by:*

$$\begin{aligned} {}_a^R D_T^p g(x) &= \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-p-1} g(t)dt \\ &= \frac{d^n}{dx^n} \left( I_a^{n-p} g(x) \right). \end{aligned}$$

**Example 1.27.** *In this example, we will calculate the Riemann–Liouville fractional derivative of the function:*

$$g(x) = (x-a)^{\alpha}.$$

*Let $n-1 \leq p < n$ and $\alpha > -1$.*
*Then we have*

$${}_a^R D_t^p (x-a)^{\alpha} = \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-p-1} (x-a)^{\alpha} dt.$$

*By changing variable $t = a + \tau(x-a)$, we get*

$$\begin{aligned} {}_a^R D_t^p (x-a)^{\alpha} &= \frac{1}{\Gamma(n-p)} \frac{d^n}{dx^n} (x-a)^{n+\alpha-p} \int_0^1 (1-\tau)^{n-p-1} \tau^{\alpha} d\tau \\ &= \frac{\Gamma(n+\alpha-p+1) B(n-p, \alpha+1)}{\Gamma(n-p)\Gamma(\alpha-p+1)} (x-a)^{\alpha-p} \\ &= \frac{\Gamma(n+\alpha-p+1)\Gamma(n-p)\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-p+1)\Gamma(n+\alpha-p+1)} (x-a)^{\alpha-p} \\ &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)} (x-a)^{\alpha-p}. \end{aligned}$$

**Remark 1.28.** *In general, The Riemann–Liouville fractional derivative of a constant is not zero or constant, but it is defined by:*

$$\begin{aligned} {}_a^R D_t^p C &= \frac{d}{dx} \left( I_a^{1-p}[c] \right) \\ &= \frac{d}{dx} \left( \frac{C}{\Gamma(1-p)} \int_a^x (x-t)^{-p} dt \right) \\ &= \frac{d}{dx} \left( \frac{C}{(1-p)\Gamma(1-p)} (x-a)^{1-p} \right) \\ &= \frac{C}{\Gamma(1-p)} (x-a)^{-p}, \qquad 0 < p < 1. \end{aligned}$$

*If $C = 1$, then:*

$${}_a^R D_t^p [1] = \frac{1}{\Gamma(1-p)} (x-a)^{-p}, \qquad 0 < p < 1.$$

### 1.5.2 Hadamard fractional derivative

**Definition 1.29.** *Let $g \in C([a,T])$. The Hadamard fractional derivative of order $\alpha$ ( $n-1 \leq \alpha < n$), is defined by :*

$$ {}_H D_a^\alpha [g(x)] = \left(x\frac{d}{dx}\right)^n \int_a^x \left(log\left(\frac{x}{t}\right)\right)^{n-\alpha-1} f(t)\frac{dt}{t}, \qquad n \in \mathbb{N}^*. $$

**Remark 1.30.** *The Hadamard fractional derivative of a constant is not zero. We have*

$$ \begin{aligned} {}_H D_a^\alpha [C] &= t\frac{d}{dt}\left(I_a^{1-\alpha}[C]\right) = t\frac{d}{dt}\left(\frac{C}{\Gamma(1-\alpha)}\int_a^t \log^{-\alpha}\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}\right) \\ &= t\frac{d}{dt}\left(\frac{C}{(1-\alpha)\Gamma(1-\alpha)}\log^{1-\alpha}\left(\frac{t}{a}\right)\right) = \frac{C}{\Gamma(1-\alpha)}\log^{-\alpha}\left(\frac{t}{a}\right). \end{aligned} $$

*If $C=1$, we get*

$$ {}_H D_a^\alpha [1] = \frac{1}{\Gamma(1-\alpha)}\log^{-\alpha}\left(\frac{t}{a}\right). $$

### 1.5.3 Katugampola fractional derivative

**Definition 1.31.** *The Katugampola fractional derivative ${}^\varrho\mathcal{D}_a^\beta g$ of order $\beta > 0$ is defined by:*

$$ \begin{aligned} \left({}^\varrho D_a^\beta g\right)(x) &= \left(x^{1-\varrho}\frac{d}{dx}\right)^n \circ \left({}^\varrho\mathcal{I}_\beta^{n-\beta} g\right)(x) \\ &= \frac{\varrho^{\beta-n+1}}{\Gamma(n-\beta)}\left(x^{1-\varrho}\frac{d}{dx}\right)^n \int_a^x \frac{t^{\varrho-1}g(t)}{(x^\varrho - t^\varrho)^{\beta-n+1}}dt, \quad x > a. \end{aligned} $$

**Lemma 1.32.** *Let $\beta > 0$ , $0 \leq p < 1$ and $g \in X_o^p(a,T)$, then*

$$ ({}^\varrho D_a^\beta\, {}^\varrho I_a^\beta g)(x) = g(x). $$

### 1.5.4 Caputo fractional derivative

**Definition 1.33.** *Let $\rho > 0$, $n \in \mathbb{N}^*$ with $n-1 \leq \rho < n$ and $\frac{d^n}{dx^n}g \in [a,T]$. The Caputo fractional derivative ${}^C D_a^\rho g$ of order $\rho$ is defined by:*

$$ \begin{aligned} \left({}^C D_a^\rho g\right)(x) &= \frac{1}{\Gamma(n-\rho)}\int_a^x (x-\tau)^{n-\rho-1} g^{(n)}(\tau)d\tau \\ &= I^{n-\rho}\left(\frac{d^n}{dx^n}g(x)\right). \end{aligned} $$

### 1.5.5 Caputo-Katugampola fractional derivative

**Definition 1.34.** *([10],[13]) The Caputo-Katugampola fractional derivative ${}_c^\varrho D_{a^+}^\beta$ is defined by :*

$$ \left({}_c^\varrho D_{a^+}^\beta g\right)(x) = \left({}^\varrho D_{a^+}^\beta\left[g(x) - \sum_{m=0}^{n-1}\frac{g^m(a)}{m!}(\tau-a)^m\right]\right). $$

**Lemma 1.35.** *([10]) Let $\beta, \varrho \in \mathbb{R}^+$, then*

$$ \left({}^\varrho I_{a^+}^\beta\, {}_c^\varrho D_{a^+}^\beta g\right)(x) = g(x) - \sum_{m=0}^{n-1} c_m \left(\frac{x^\varrho - a^\varrho}{\varrho}\right)^m, $$

*for some $c_m \in \mathbb{R}$, $n = [\beta]+1$.*

# Chapter 2

# Compactness in Metric spaces

In this chapter, we state and prove Hausdorff's theorem of characterization of the relatively compact subsets of a complete metric space, in terms of finite or relatively compact $\delta$-nets. And prove the Ascoli-Arzela compactness criterion on continuous function space see; [41].

## 2.1 Hausdorff's Theorem

**Proposition 2.1.** *Let $(E, d)$ be a metric space. The following statements are equivalent:*

1. *Every sequence of elements of $E$ has a convergent subsequence in $E$ .*
2. *The space $E$ is complet and for each $\delta > 0$, it admits a finite covering by open balls of radius $\delta$ .*

*Proof.* (1) $\Longrightarrow$ (2)
Let′s show that: $E$ is a complete space:

a) Let $(y_n)_{n\geq 1}$ Cauchy sequence in $E$, according to (1) we can extract a convergent subsequence, based on the result (any Cauchy sequence that admits a convergent subsequence, then this sequence is convergent) from where $(y_n)_{n\geq 1}$ is converged which shows the completeness of $E$.

b) Let′s show that: for all $\delta > 0$, $E$ admits a finite covering of open balls of radius $\delta$, we proceed by the absurd:
Suppose that, for all $\delta > 0$ the space $E$ does not admit a finite covering balls of radius $\delta$.
Then:
for each element $v_1 \in E$ fixed, there exists $v_2 \in E$ such that:

$$d\left(v_1, v_2\right) \geq \delta,$$

there exists $v_3 \in E$ such that:

$$d\left(v_1, v_3\right) \geq \delta,$$

and

$$d\left(v_2, v_3\right) \geq \delta.$$

By recurrence, we define a sequence of elements $(v_i)_{i\in\mathbb{N}}$ such that:

$$d((v_i, v_\zeta) \geq \delta,$$

for $i \neq \zeta$, this shows that $(v_i)$ has no convergent subsequences, which contradicts (1) .

Show (2) $\Longrightarrow$ (1)
Let $(v_i)_{i\in\mathbb{N}}$ be an arbitrary sequence of elements of $E$, and $(\delta_i)_{i\in\mathbb{N}}$ be any decreasing sequence of positive numbers converging to 0 .
According to (2) there exists $w_1 \in E$ such that, the open ball $B_{\delta_1}(w_1)$ (of center $(w_1)$ and radius $\delta_1$ ) contains infinitely many terms of the sequence $(v_i)_{i\in\mathbb{N}}$.
Moreover, there exists $w_2 \in E$ such that $B_{\delta_2}(w_2)$ contains infinitely many terms from those contained by $B_{\delta_1}(w_1)$.
By recurrence we define a subsequence $\left(v_{i_\zeta}\right)$ de $(v_i)_{i\in\mathbb{N}}$ such that:

$$v_{i1} \in B_{\delta 1}\left(w_1\right), v_{i2} \in B_{\delta 1}\left(w_1\right) \cap B_{\delta 2}\left(w_2\right),$$

generally:

$$v_{i_\zeta} \in \bigcap_{\sigma=1}^{\zeta} B_{\delta_\sigma}(w_\sigma),$$

for $\zeta = 1, 2, \cdots$
since $v_{i_\sigma}, v_{i_\zeta} \in B_{\delta_\sigma}(w_\sigma)$ for $\sigma \leq \zeta$,
which gives:

$$\begin{aligned} d\left(v_{i_\sigma}, v_{i_\zeta}\right) &\leq d\left(v_{i_\sigma}, v_\zeta\right) + d\left(w_\zeta, v_{i_\zeta}\right) \\ &\leq 2\delta_\sigma. \end{aligned}$$

As $\delta_\sigma \to 0$ when $\sigma \longrightarrow \infty$.
Hence $\left(v_{i_\zeta}\right)$ is a Cauchy sequence. Since $E$ is completey this subsequence is convergent. □

**Definition 2.2.** *A metric space $E$ is said to be compact, if it satisfies condition (1) (equivalently (2)) in Proposition (2.1).*

*A subset $F$ of a metric space $E$ is said to be relatively compact, if its closure $\overline{F}$ is compact (as a metric subspace of $E$ ).*

**Definition 2.3.** *Let $(E, d)$ be a metric space, $F$ a subset of $E$ and $\delta > 0$. A subset $K \subset E$ is said to be an $\delta$-net for $F$, if for every $x \in F$ there exists $y \in K$, such that:*

$$d(x, y) < \delta.$$

*Notice that a metric space is compact if and only if it is complete, and for every $\delta > 0$ it admits a finite $\delta$-net. Consequently every compact metric space is complete and bounded (in the sense that its metric takes values in a bounded real interval).*

**Theorem 2.4.** *Let $(E, d)$ be a complete metric space and $F \subset E$ be a subset. The following statements are equivalent:*

1. *$F$ is relatively compact.*
2. *For every $\delta > 0$, there exists in $E$ a finite $\delta$-net.*
3. *For every $\delta > 0$, there exists in $E$ a relatively compact $\delta$-net for $F$ .*

*Proof.* Let′s show that:
(1) $\Longrightarrow$ (2)
It is assumed that $F$ is relatively compact. Then $\overline{F}$ is compact. So for all $\delta > 0$, there exists $\mathcal{K}$ a finite $\delta$-net for $\overline{F}$.

Therefore, Clearly $\mathcal{K}$ is also finite $\delta-$net in $E$ for $F$, (i.e. $\forall x \in F (F \subset \overline{F}), \exists w \in \mathcal{K} / d(x, w) < \delta$).
(2)$\Longrightarrow$(1)
We assume (2) and show that $\overline{F}$ is compact (i.e; $\forall \delta > 0$ there exists a finite $\delta-$net of $\overline{F}$ and $\overline{F}$ is complete ).
Since $\overline{F}$ is close and $\overline{F} \subset E$ ($E$ complete ) deduces that $\overline{F}$ is complete, according to (2), for $\delta > 0$ there exists $\mathcal{K}$ a finite $\frac{\delta}{3}$-net for $F$,
for all $x \in \mathcal{K}$ we choose an element $W_x \in F$, such that:

$$d(x, W_x) < \frac{\delta}{3},$$

then we get a set:

$$\mathcal{K}' = \{W_x, x \in \mathcal{K}\}.$$

$\mathcal{K}'$ is a finite $\frac{2\delta}{3}$-net for $F$ is a finite $\delta-$net for $\overline{F}$, because if

$$y \in \overline{F} \Longrightarrow \forall \frac{\delta}{3} > 0 \quad B\left(y, \frac{\delta}{3}\right) \cap F \neq \varnothing,$$

(i.e; there exists $j \in F / d(j, y) < \frac{\delta}{3}$) .
Since $j \in F$, there exists $W_x \in \mathcal{K}'$ such that:

$$d(W_x, j) < \frac{2\delta}{3},$$

and we have:

$$d(y, W_x) \leq d(y, j) + d(j, W_x) < \frac{\delta}{3} + \frac{2\delta}{3} = \delta.$$

Then $\overline{F}$ is compact. Hence $F$ is relatively compact.
$(2) \Longrightarrow (3)$
According to (2) we have the set is finite then is relatively compact hence(3).
$(3) \Longrightarrow (2)$
It is assumed (3) that is for all $\delta > 0$, there exists $\mathcal{K}$ a relatively compact $\frac{\delta}{2}$-net for $F$.
Then $\overline{\mathcal{K}}$ is compact (i.e; there exists a finite $\frac{\delta}{2}$-net $\mathcal{K}'$ ). It is clear that $\mathcal{K}'$ is a finite $\delta-$net in $E$ for $F$. □

**Proposition 2.5.** *Let $(E,d)$ be a complete metric space and $\{F_\gamma : \gamma > 0\}$, be a family of relatively compact subsets of $E$. Assume that $F \subset E$ is the uniform limit of $F_\gamma$ as $\gamma \to 0$.*
*That is for each $\delta > 0$ there is $\gamma_\delta > 0$, such that for every $x \in F$ and $\gamma \in (0, \gamma_\delta)$, there exists $x_\gamma \in F_\gamma$ with*

$$d(x, x_\gamma) < \delta.$$

*Then $F$ is relatively compact.*

## 2.2 The Ascoli-Arzèla Theorem

Let $E$ be a compact metric space and $F$ be a complete space. We denote by $C(E,F)$ the vector space of continuous functions defined on $E$ to value in $F$. We denote by:

$$d_\infty(h,l) = \sup_{y \in E} d(h(y), l(y)).$$

uniform distance. For this distance, the space $C(E,F)$ is a complete space, the Ascoli–Arzèla theorem characterizes compact subsets of $C(E,F)$.

**Definition 2.6.** *(Ascoli-Arzèla Theorem)*
*Let $E$ be a compact metric space, $F$ be a complete metric space, and $\mathfrak{D}$ a subset of $C(E,F)$.*
*$\mathfrak{D}$ is compact in $C(E,F)$ if and only if $\mathfrak{D}$ is closed, equicontinuous and for all $y \in E$, Sets*

$$\mathfrak{D}(y) = \{h(y), \quad h \in \mathfrak{D}\}.$$

*are relatively compact in $F$.*

*Proof.* $(\Longrightarrow)$
Suppose that $\mathfrak{D}$ is compact in $C(E,F)$, then $\mathfrak{D}$ is closed on the other hand the application :

$$\begin{aligned} \mathfrak{D} &\longrightarrow F \\ h &\longmapsto h(y), \end{aligned}$$

for a fixed $y$ is continuous because:

$$d_F(h(y), l(y)) \leq d_\infty(h,l).$$

So if

$$h \longrightarrow l \quad \text{therefore} \quad d_F(h(x), l(x)) \longrightarrow 0.$$

Then $\mathfrak{D}(y)$ is the image of a compact $\mathfrak{D}$ by a continuous function which is a compact in $F$.
Let us show the equicontinuity of $\mathfrak{D}$.
Since $\mathfrak{D}$ is compact, for all $\delta > 0$, there exists a finite number of functions ( $\frac{\delta}{3}$-net)

$$h_1, h_2, \cdots, h_N \in C(E,F),$$

such that for all $h \in \mathfrak{D}$ is at a distance of less than $\frac{\delta}{3}$ .
On the other hand of one of these functions. In other words

$$\forall h \in \mathfrak{D}, \exists h_\zeta \quad \zeta \in [1,N] : d_\infty(h, h_\zeta) < \frac{\delta}{3}.$$

Since $h_\zeta$ are continuous, for all $y_0 \in E$, $\exists \eta_\zeta(\delta, y_0)$ such that:

$$\forall \mu \in E, d_E(y_0, \mu) < \eta_\zeta \Longrightarrow d_F(h_\zeta(y_0), h_\zeta(\mu)) < \frac{\delta}{3}.$$

If we take $\delta(\delta, y_0) = \min_\zeta \delta_\zeta(\delta, y_0)$, we get:

$$\forall \zeta = \overline{1,N} \quad \forall \mu \in E, d_E(y_0, \mu) < \eta \Longrightarrow d_F(h_\zeta(y_0), h_\zeta(\mu)) < \frac{\delta}{3}.$$

Let $h \in \mathfrak{D}$, then there exists $\zeta \in [1, N]$ such that:

$$d_\infty (h, h_\zeta) < \frac{\delta}{3}.$$

Then

$$\forall \mu \in E, d(y_0, \mu) < \eta.$$

Then

$$\begin{aligned} d(h(y_0), h(\mu)). &\leq d(h(y_0), h_\zeta(y_0)) + d(h_\zeta(y_0), h_\zeta(\mu)) + d(h_\zeta(\mu), h(\mu)) \\ &\leq 2d_\infty (h, h_\zeta) + d(h_\zeta(y), h_\zeta(\mu)) \\ &< \delta. \end{aligned}$$

Which shows that $\mathfrak{D}$ is equicontinuous.
$(\Longleftarrow)$
Suppose that $\mathfrak{D}$ is closed, equicontinuous and that for all $y \in E$, $\mathfrak{D}(y)$ is relatively compact in $F$.
Let's show that $\mathfrak{D}$ is compact. It is shown to be complete and admits a finite $\delta-$net, for all $\delta > 0$.
Let fixed $\delta > 0$, the equicontinuity of $\mathfrak{D}$ shows that:

$$\forall y \in E, \exists \eta(\delta, y) : d_E(y, \mu) < \eta \Longrightarrow d_F(h(y), h(\mu)) < \frac{\delta}{3}, \forall h \in \mathfrak{D}.$$

Since $E$ is compact, it can be covered with a finite number of open balls,

$$B(y_\zeta, \eta), 1 \leq \zeta \leq \varrho.$$

$$E \subseteq \bigcup_{\zeta=1,\varrho} B(y_\zeta, \eta).$$

In addition, since $\mathfrak{D}(y)$ is relatively compact in $F$, for all $y \in E$.
Then the sets

$$\mathfrak{D}(y_1), \mathfrak{D}(y_2), \cdots, \mathfrak{D}(y_\varrho),$$

are relatively compact in $F$, so it can be covered by a finite number of open balls of radius less than $\frac{\delta}{3}$.
That is there exists

$$h_1, h_2, \cdots, h_i \in \mathfrak{D},$$

such that:

$$\mathfrak{D}(y) \subset \bigcup_{\sigma=1,i} B\left(h_\sigma(y), \frac{\delta}{3}\right).$$

Let $f \in \mathfrak{D}$, and $\sigma \in [1, i]$, such that $h$ either in the open ball $B(y_\zeta, \eta)$.
As a result:

$$\begin{aligned} d(h(y), h_\sigma(y)) &\leq d(h(y), h(y_\sigma)) + d(h(y_\zeta), h_\sigma(y_\zeta)) + d(h_\sigma(y_\zeta), h_\sigma(y)) \\ &< \delta. \end{aligned}$$

This means that:

$$d_\infty (h, h_\zeta) < \delta.$$

Since $\mathfrak{D}$ is closed, therefore complete and therefore $\mathfrak{D}$ is compact. □

**Corollary 2.7.** *Let $\varphi$ be a bounded open subset of $\mathbb{R}^n$. Every bounded subset of the space $\left(C^1(\overline{\varphi}; \mathbb{R}^n), |\cdot|_{1,\infty}\right)$ is relatively compact in $(C(\overline{\varphi}; \mathbb{R}^n), |\cdot|_\infty)$.*

*Proof.* Let $F$ be a bounded subset of $\left(C^1(\overline{\varphi}; \mathbb{R}^n), |, |_{1,\infty}\right)$. Then $F$ is also bounded in $(C(\overline{\varphi}; \mathbb{R}^n), |\cdot|_\infty)$. In addition, all functions in $F$ are Lipschitz with the same Lipschitz constant, and therefore $F$ is equicontinuous. Now the Ascoli-Arzèla theorem guarantees that $F$ is relatively compact in $(C(\overline{\varphi}; \mathbb{R}^n), |\cdot|_\infty)$. □

**Remark 2.8.** *Let $\varphi$ be a bounded open subset of $R^n$ and $l \in N\backslash\{0\}$. Every bounded subset of the space $\left(C^l(\overline{\varphi}; \mathbb{R}^n), |\cdot|_{l,\infty}\right)$ is relatively compact in $\left(C^{l-1}(\overline{\varphi}; \mathbb{R}^n), |\cdot|_{l-1,\infty}\right)$.*

# Chapter 3

# Completely Continuous Operators on Banach Spaces

Is devoted to the concept of a completely continuous operator. We prove the theorem of representation of the completely continuous operators as uniform limits of sequences of continuous operators of finite rank. Also we present the proof of one of the fundamental results of nonlinear analysis, namely Schauder's fixed point theorem see [41].

## 3.1 Completely Continuous Operators

**Definition 3.1.** *Let $E, F$ be Banach spaces and $N : S \subset E \to F$.*

1. *The operator $N$ is said to be bounded, if it maps any bounded subset of $S$ into a bounded subset of $F$.*
2. *The operator $N$ is said to be completely continuous, if it is continuous and maps any bounded subset of $S$ into a relatively compact subset of $F$.*
3. *The operator $N$ is said to be of finite rank if $N(s)$ lies in a finite dimensional subspace of $F$.*

*It is clear that a continuous operator $N : S \subset E \to F$ is completely continuous, if and only if for every bounded sequence $(x_i)$ with $x_i \in S$, the sequence $(N(x_i))$ has a convergent subsequence.*
*Notice that any completely continuous operator is a bounded operator.*

**Theorem 3.2.**

1. *If the operators $N_1, N_2 : S \subset E \to F$ are bounded (completely continuous) then for every $\gamma, \lambda \in \mathbb{R}$ the operator $\gamma N_1 + \lambda N_2$ is bounded (respectively, completely continuous).*
2. *Let $E, F, G$ be Banach spaces and $N_1, N_2$ be two operators which are defined as follows:*

$$S_1 \xrightarrow{N_1} N_1(S_1) \subset S_2 \xrightarrow{N_2} G, \quad S_1 \subset E, \quad S_2 \subset F.$$

*If both operators $N_1, N_2$ are bounded then the composite operator $N_2 N_1$ is also bounded. If one of the operators $N_1, N_2$ is bounded continuous and the other one is completely continuous, then $N_2 N_1$ is completely continuous.*

**Lemma 3.3.** *Let $\Upsilon$ be a compact subset of $\mathbb{R}^n$ and $x_1, x_2, x_3, \cdots, x_m$ open such that:*

$$\Upsilon \subset \bigcup_{j}^{m} x_j,$$

*then there exist function*

$$\phi_1, \phi_2, \cdots, \phi_m \in C^{\infty}(\mathbb{R}^n),$$

*such that:*

1. $0 \leq \phi_j \leq, \forall j = 1, 2, \cdots, m$ *and* $\sum_{j=1}^{m} \phi_j = 1$ *in* $\mathbb{R}^n$.

*2.*
$$\begin{cases} supp(\phi_j) \text{ is compact and } \quad supp(\phi_j) \subset x_j, \forall j = 1, \cdots, m. \\ supp(\phi_0) \subset \mathbb{R}^n \setminus \Upsilon. \end{cases}$$

*When $\Omega$ is a open bounded set and $\Upsilon = \partial\Omega$ then $\phi_0 \in C_c^\infty(\Omega)$ sur $\Omega$.*

**Theorem 3.4.** *1. If the operators $N_i : S \to F, S \subset E, i = 1, 2, \cdots$ are completely continuous and $N : S \to F$ is such that:*

$$N(x) = \lim_{i \to \infty} N_i(x). \tag{3.1}$$

*uniformly on any bounded subset of $S$, then $N$ is completely continuous too.*

*2. Let $S \subset E$ be a bounded closed set and $N : S \to F$ a completely continuous operator. Then there exists a sequence of continuous operators $N_i : S \to F$ of finite rank such that (3.1) holds, uniformly on $S$, and*

$$N_i(S) \subset \text{ conv } (N(S)),$$

*for every $i$.*

*Proof.* Let $N_i$ be a completely continuous sequence of operators.

1. $N_i : S \longrightarrow F, (S \subset E)$, (then $i = 1, 2, \cdots$) completely continuous operator,
$N : S \longrightarrow F$, such that $N_i$ converges to $N$ uniformly on each bounded $Z$ for $E$,
$\big(i.e, \forall\delta > 0, \exists G \in \mathbb{N}^*/\forall i > G \Longrightarrow \sup_{x\in Z} \|N(x) - N_i(x)\|_F < \frac{\delta}{3}\big)$.
Let's first show that $N$ is continuous.
By hypothesis $N_i, i \in \mathbb{N}$ is continuous $\forall i \in \mathbb{N}$, $\big(i.e; \forall\delta > 0, \exists\eta_i > 0, / \|x - x_0\|_E < \eta_i \Longrightarrow \|N_i(x) - N_i(x_0)\|_F < \frac{\delta}{3}\big)$.
In addition $\forall x_0 \in E$, $(\eta_i)_{i\geq 1} \subset \mathbb{R}$ minore $\Longrightarrow \inf (\eta_i)_{i\geq 1}$ exists and that in note $\eta$.
Let $\delta > 0, x_0 \in S, \eta = \inf_i(\eta_i)$, if $\|x - x_0\|_E < \eta$, then

$$\begin{aligned}\|N(x) - N(x_0)\|_F &\leq \|N(x) - N_i(x)\|_F + \|N_i(x) - N_i(x_0)\|_F + \|N_i(x_0) - N(x_0)\|_F \\ &\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.\end{aligned}$$

Hence the continuity of the operator $N$ (1.1)
Show that the operator $N$ is compact, (i.e; $\forall Z \subset S$ bounded, $N(Z)$ is relatively compact in $F$ ).
Let $Z \subset S$ a bounded part in $E$, for $N(Z)$ is relatively compact, we demonstrate $N(Z) \subset F$ is the uniform limit of $N_{\frac{1}{\gamma}}(Z)$ when $\gamma \longrightarrow 0$.
Since $N_i$ Completely continuous, Then $N_i(Z)$ is relatively compact in $F$, $(andi = 1, 2, \cdots)$.
So the family $\Big\{N_{\frac{1}{\gamma}}(Z), i = \frac{1}{\gamma} > 0\Big\}$ is relatively compact.
We have:

$$\lim_{i\to+\infty} N_i(x) = N(x), \quad x \in Z,$$

$$\forall\delta > 0, \quad \exists G_\delta \in \mathbb{N}, \quad \forall i \geq G_\delta \Longrightarrow \sup_{v\in Z}\|N_i(x) - N(x)\|_F < \delta.$$

And Let

$$\delta > 0, \exists\gamma_\delta = \frac{1}{G_\delta} > 0,$$

such that:

$$\forall b \in N(Z), (\exists x \in Z/b = N(x)),$$

and

$$\forall\frac{1}{i} \leq \frac{1}{G_\delta}, (\gamma \leq \gamma_\delta),$$

$\forall\gamma \in (0, \gamma_\delta], \exists a_\gamma = N_{\frac{1}{\gamma}}(x)$ such that:

$$\|a - a_\gamma\|_E < \delta \Longrightarrow \sup_{x\in Z}\Big\|N(x) - N_{\frac{1}{\gamma}}\Big\|_F < \delta.$$

According to the proposal (2.5) $N(Z)$ is relatively compact in $F$ (1.2)
According to (1.1) and (1.2) $N$ is completely continuous.

2. $N$ is completely continuous and $S$ a bounded set, then $N(S)$ is relatively compact by consequence, for all $\delta > 0$ there exists a finite element $w_\zeta \in N(S)$, $\zeta = 1, 2, \cdots, n_\delta$, such that:

$$\overline{N(S)} \subset \bigcup_{\zeta=1}^{n_\delta} B_\delta\left(w_\zeta\right).$$

$\overline{N(S)}$ is compact, then according to the lemma (3.3), there is $(\psi_\zeta)$ a partition of unity relative to $\overline{N(S)}$. Therefore,

$$\psi_\zeta \in C(\overline{N(S)}, [0.1]), \quad supp(\psi)_\zeta \subset \overline{B_\delta}\left(w_\zeta\right), \quad \sum_{\zeta=1}^{m_\delta} \psi_\zeta(w) = 1,$$

for all $w \in \overline{N(S)}$, we define the operator $N_\delta : S \to F$ by :

$$N_\delta(x) = \sum_{\zeta=1}^{m_\delta} \psi_\zeta(N(x)) w_\zeta, \quad (x \in S),$$

by its construction $N_\delta$ a continuous operator, and it is of finite rank because $Ran(N(S))$ is a vector subspace generated by $w_\zeta$ such that $\zeta = 1, 2, \cdots, m_\delta$.
And $N_\delta(S) \subset conv(N(S))$, because:

$$N_\delta(S) \subset [w_\zeta]_{\zeta=1}^{m_\delta} \subset conv\,[w_\zeta]_{\zeta=1}^{m_\delta},$$

and $conv\,[w_\zeta]_{\zeta=1}^{m_\delta} \subset (N(S))$.
Than

$$N_\delta(S) \subset conv(N(S)).$$

In addition to $i \in S$, we have :

$$\begin{aligned} \|N(x) - N_\delta(x)\|_F &= \left\| \sum_{\zeta=1}^{m_\delta} \psi_\zeta(N(x))\left(w_\zeta - N(x)\right) \right\|_F \\ &\leq \sum_{\zeta=1}^{m_\delta} \psi_\zeta(N(x)) \left\|w_\zeta - N(x)\right\|_F \\ &\leq \delta \sum_{\zeta=1}^{m_\delta} \psi_\zeta(N(x)) = \delta. \end{aligned}$$

On more $N(x) \in B_\delta\left(w_\zeta\right)$ and for all $\psi_\zeta(N(x)) > 0$, then $N(x) = \lim_{\delta\to 0} N_\delta(x)$, that converges unifornately for each bounded $S$. Take $N_i := N_\delta$ such that $\delta = \frac{1}{i}$

□

**Definition 3.5.** *Let $(E, |\cdot|_E)$ and $(F, |\cdot|_F)$ be Banach spaces such that $F \subset E$. We say that the embedding $F \subset F$ is continuous (respectively, completely continuous) if the injection map $\zeta : F \to E, \zeta(x) = x$ $(x \in F)$ is continuous, respectively, completely continuous.*

## 3.2 Banach′s Fixed Point Theorem

Banach′*s* fixed point theorem, also known as the contraction theorem, concerns certain contracting application of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point.
This theorem proved in 1922 by Stefan Banach is based essentially on the concepts of contracting application [47],[17],[16].

**Theorem 3.6.** *( Banach contraction principle): Let $(E, d)$ be a complete metric space and let $h : E \longrightarrow E$ be a contracting application on $E$. Then $h$ has a unique fixed point $u \in E$ such that $h(u) = u$.*
*In addition, for any initial point $u_0 \in E$, iterated sequence $\{u_n\}_{n\in\mathbb{N}}$*
*given by:*

$$\begin{cases} u_0 \in E \\ u_{n+1} = h(u_n), \quad n \in \mathbb{N}, \end{cases}$$

*converges to $u$.*
*Then*

$$d\left(u, u_n\right) \leq \frac{k^n}{1-k} d\left(u_0, h\left(u_0\right)\right).$$

*Proof.* 1. Existence : Let $u_0$ any initial point and $\{u_n\}_{n\in\mathbb{N}}$ sequence defined by :

$$\begin{cases} u_0 \in E \\ u_{n+1} = h\left(u_n\right), n \in \mathbb{N}. \end{cases}$$

We will establish that$\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. For all $n \in \mathbb{N}$, Since $h$ is contracting, we have:

$$d\left(u_n, u_{n+1}\right) = d\left(h\left(u_{n-1}\right), h\left(u_n\right)\right) \leq kd\left(u_{n-1}, u_n\right) \leq \cdots \leq k^n d\left(u_0, u_1\right).$$

Thus, for $i > n$, where $n \geq 0$, we have:

$$\begin{aligned} d\left(u_n, u_i\right) &\leq d\left(u_n, u_{n+1}\right) + d\left(u_{n+1}, u_{n+2}\right) + \cdots + d\left(u_{i-1}, u_i\right) \\ &\leq k^n d\left(u_1, u_0\right) + k^{n+1} d\left(u_1, u_0\right) + \cdots + k^{i-1} d\left(u_0, u_1\right) \\ &\leq k^n \left[1 + k + k^2 + \cdots + k^{i-n-1}\right] d\left(u_0, u_1\right) \\ &\leq k^n \left(\frac{1-k^{i-n}}{1-k}\right) d\left(u_0, u_1\right) \text{ since } 1 - k^{i-n} < 1. \end{aligned}$$

We obtain:

$$\leq \frac{k^n}{1-k} d\left(u_0, h\left(u_0\right)\right) \longrightarrow 0 \text{ when } n \longrightarrow \infty \text{ because } k \in [0,1].$$

This shows that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and as $E$ is complete space, then there exist $u \in E$ such that $u_n \longrightarrow u$.
Moreover, since $h$ is continuous, we have:

$$u = \lim_{n\longrightarrow\infty} u_{n+1} = \lim_{n\longrightarrow\infty} h\left(u_n\right) = h\left(\lim_{n\longrightarrow\infty} u_n\right) = h(u).$$

So $u$ is a fixed point of $h$.

2. Uniqueness: Suppose there exist points $u, v \in E$ such that $u \neq v, u = h(u)$ et $v = h(v)$.

Then, we have:

$$d(h(u), h(v)) = d(u, v),$$

therefore

$$\frac{d(h(u), h(v))}{d(u, v)} = 1.$$

On the other hand, $h$ is contracting so

$$\frac{d(h(u), h(v))}{d(u, v)} \leq k < 1.$$

This is contradictory, hence the uniqueness.

□

## 3.3 Brouwer′s Fixed Point Theorem

Brouwer′s fixed-point theorem is a result of algebraic topology. It belongs to the large family of fixed point theorems. This theorem gives the existence of a fixed point for a continuous function on a closed ball in a finite-dimensional space.

**Theorem 3.7.** *(Brouwer)[41] Let $S \subset \mathbb{R}^n$ be a nonempty convex compact set and let $N : S \longrightarrow S$ be a continuous mapping.*
*Then there exists at least one $x \in S$ with $N(x) = x$.*

*Proof.* See [41] □

## 3.4 Schauder′s Fixed Point Theorem

This theorem extends the result of Brouwer's theorem to show the existence of a fixed point for a continuous function on a compact convex in a Banach space. Schauder's fixed point theorem is more topological and states that a map continuous on a compact convex admits a fixed point, which is not necessarily unique. Schauder generalized Brouwer's result to infinite dimension.

**Theorem 3.8.** *(Shauder) [41] Let $E$ be a Banach space , $L \subset E$ a nonempty convex compact set and let $N : L \longrightarrow L$ be a continuous operateur. Then $N$ has at least one fixed point.*

*Proof.* 1. We have the operator $N : L \longrightarrow L$ is continuous.

2. We will show that $N$ is a relatively compact operator, ( i.e, $\forall A \subset L$ bounded, then $\overline{N(A)}$ is compact)
   Let $A \subset L$ a bounded set, then $\overline{N(A)}$ A closed set on $L$ and since $L$ is compact, therefore:
   $N(A)$ is compact.
   According to (1) et (2) The operator $N$ is completely continuous.
   Then according to theorem (3.4) there is an operator sequence $N_\zeta : L \longrightarrow L$ of finite rank is continuous that converges to $N$ for all bounded $L$ (because $L$ is compact then $L$ is bounded)
   $$N(x) = \lim_{\zeta \to +\infty} N_\zeta(x),$$
   ( i.e, $\forall \delta > 0, \exists M \in \mathbb{N} \sup_{x \in L} \|N(x) - N_i(x)\| < \delta$).
   Poson $n = n(\zeta)$ the subset dimension $E_n$ where $E_n$ is the smallest subspace generated by $N_\zeta(L)$
   We have : $N_\zeta : L \cap E_n \to L \cap E_n$ and $L \cap E_n$ is compact because ($E_n$ a finite-dimensional subspace is then closed, then $L \cap E_n$ is closed in a compact then $L \cap E_n$ is compact), according to theorem (**??**) (Brouwer) exists $x_\zeta \in L \cap E_n$ such that:
   $$N_\zeta (x_\zeta) = x_\zeta,$$
   and since $L \cap E_n$ is compact, then there exists a sub-sequence $\left(x_{i_\zeta}\right)$ of $(x_\zeta)$ converges to $x \in L$.
   Then
   $$\|N(x) - x\| \leq \left(\left\|N(x) - N\left(x_{i_\zeta}\right)\right\| + \left\|N\left(x_{i_\zeta}\right) - N_{i_\zeta}\left(x_{i_\zeta}\right)\right\| + \left\|x_{i_\zeta} - x\right\|\right) \longrightarrow 0 \text{ when} \zeta \to +\infty.$$
   Hence $N(x) = x$.

□

**Lemma 3.9.** *(Mazur) Let $E$ be a Banach space and $F \subset E$ be a relatively compact subset, then $conv(F)$ is relatively compact.*

*Proof.* Let $F$ be a relatively compact subset such that $F \subset E$, then for all $\delta > 0$ we find a finite number of elements of $E$ say:
$$x_1, x_2, x_3, \cdots, x_n,$$
such that:
$$F \subset \bigcup_{\sigma=1}^{m} B_\delta (x_\sigma) \tag{3.2}$$
Let $\mathcal{K} = conv\{x_1, x_2, x_3, \cdots, x_n\}$, we will show that $\mathcal{K}$ is relatively compact finite $\delta-$net for $conv(F)$.
Let
$$x \in conv(F), \quad x = \sum_{\zeta=1}^{n} \lambda_\zeta v_\zeta, \quad \lambda_j > 0, \quad \sum_{\zeta=1}^{n} \gamma_\zeta = 1, \quad w_\zeta \in F.$$
And according to (3.2) $w_\zeta \in B_\delta (x_{\sigma\zeta})$
Then:
$$\left| x - \sum_{\zeta=1}^{n} \gamma_\zeta x_{\sigma\zeta} \right| = \left| \sum_{\zeta=1}^{n} \gamma_\zeta (w_\zeta - x_{\sigma\zeta}) \right|$$
$$\leq \sum_{\zeta=1}^{n} \gamma_\zeta |w_\zeta - x_{\sigma\zeta}|,$$
and
$$\sum_{\zeta=1}^{n} \gamma_\zeta x_{\sigma\zeta} \in \mathcal{K}.$$
Hence $\mathcal{K}$ is an finite $\delta-$net for $conv(F)$, then according to Hausdorff's theorem(2.4),
$conv(F)$ is relatively compact .

□

**Theorem 3.10.** *(Schauder)[41] Let $E$ be a Banach space and $S \subset E$ a nonempty convex bounded closed set and let $N : S \longrightarrow S$ be a completely continuous operator, Then $N$ has at least one fixed point.*

*Proof.* We have $N$ is completely continuous operatoret and $S$ is bounded set , alors $N(S)$ is relatively compact, According to Mazur's lemma(3.9), then $\overline{conv}(N(S))$ is relatively compact ,
then $K = \overline{conv}(N(S))$ is compact, (and obviously convex ) from $N(S) \subset S$ and $S$ closed convex by following $K \subset S$ by theorem (**??**) ,
the operator $N : K \longrightarrow K$ admits a fixed point . □

**Corollary 3.11.** *If $N : E \longrightarrow E$ is continuous and compact and*

$$N : \overline{B}(0, R) \longrightarrow \overline{B}(0, R).$$

*Then $N$ has a fixed point in $\overline{B}(0, R)$ .*

**Theorem 3.12.** *([7])(Schaefer's fixed point theorem) Let $E$ be a Banach space, and $N : E \longrightarrow E$ be a completely continuous operator.*
*If the set*

$$\chi = \{x \in E : x = \gamma N x, \textit{for some} \quad \gamma \in (0, 1)\},$$

*is bounded, then $N$ has a fixed point.*

*Proof.* Choose $M > 0$ such that

$$\| x \| < M \quad \text{if} \quad x = \gamma N x \quad \text{for some} \quad 0 \leq \gamma \leq 1, \tag{3.3}$$

and define

$$\widetilde{N}(x) = \begin{cases} N(x) & \text{if} \quad \| N(x) \| \leq M \\ \frac{MN(x)}{\|N(x)\|} & if \quad \| N(x) \| \geq M. \end{cases}$$

Then $\widetilde{N}$ is continuous and compact
and

$$\widetilde{N} : \overline{B}(0, R) \longrightarrow \overline{B}(0, R).$$

Thus it follows from the corollary that $\widetilde{N}$ has a fixed point $\widetilde{N}(x) = x$,
it remains to prove that $N(x) = x$ .
Suppose to contrary that $N(x) \neq x$ so $N(x) \neq \widetilde{N}(x)$. Then $\| N(x) \| > M$ and

$$x = \widetilde{N}(x) = \frac{MN(x)}{\| N(x) \|} = \gamma N(x), \quad \gamma = \frac{M}{\| N(x) \|} < 1.$$

Since $\| x \| = \| \widetilde{N}(x) \| = M$ we arrive to a contradiction with 3.3. □

# Chapter 4

# Implicit Neutral Functional Fractional differential equation with Retarded and Advanced Arguments

## 4.1 Introduction

In this chapter, we establish, the existence and uniqueness of solutions for neutral implicit Caputo-Katugampola fractional differential equations with retarded and advanced arguments see [24].

## 4.2 Implicit Neutral FDE with Retarded and Advanced Arguments

[1]

In this section, we study the existence and uniqueness of solutions for a class of problem for nonlinear neutral implicit fractional differential equations involving both retarded and advanced arguments.

$$ {}^{\varrho}_{c}D^{\beta}_{a^+}(x(t)-h(t,x^t)) = l(t,x^t,{}^{\varrho}_{c}D^{\beta}_{a^+}x(t)), \text{ for } t\in J:=[a,T],\ 1<\beta\leq 2, \tag{4.1} $$

$$ x(t)=\varphi(x), \qquad t\in[a-b,a],\ b>0, \tag{4.2} $$

$$ x(t)=\vartheta(t), \qquad t\in[T,T+\sigma],\ \sigma>0, \tag{4.3} $$

where ${}^{\varrho}_{c}D^{\beta}_{a^+}$ is the Caputo-Katugampola fractional derivative,

$$ l: J\times C([-b,\sigma],\mathbb{R})\times\mathbb{R}\to\mathbb{R}, $$

and

$$ h: J\times C([-b,\sigma],\mathbb{R})\to\mathbb{R}, $$

with $h(a,x^a)=h(T,x^T)=0$,
is a given function, $\varphi\in C([a-b,a],\mathbb{R})$, with $\varphi(a)=0$ and $\vartheta\in C([T,T+\sigma],\mathbb{R})$, with $\vartheta(T)=0$.
We denote by $x^t$ the element of $C([-b,\sigma])$ defined by:

$$ x^t(\tau)=x(t+\tau):\tau\in[-b,\sigma], $$

here $x^t(\cdot)$ represents the history of the state from time $t-b$ up to time $t+\sigma$.

**Lemma 4.1.** *Let $1<\beta\leq 2$, $\varphi\in C([a-b,a],\mathbb{R})$ with $\varphi(a)=0$, $\vartheta\in C([T,T+\sigma],\mathbb{R})$ with $\vartheta(T)=0$ and $k:J\to\mathbb{R}$ be a continuous function. Then the linear problem*

$$ {}^{\varrho}_{c}D^{\beta}_{a^+}(x(t)-h(t,x^t)) = k(t), \text{ for a.e. } t\in J:=[a,T],\ 1<\beta\leq 2, \tag{4.4} $$

[1] M. Boumaaza, M. Benchohra, F. Berhoun, Nonlinear implicit Caputo type modification of the Erdélyi-Kober fractional differential equations with retarded and advanced Arguments, *Panam. Math. J*, **30** (2020), 21 - 36.

$$x(t) = \varphi(t), \qquad t \in [a-b, a], \; b > 0, \tag{4.5}$$

$$x(t) = \vartheta(t), \qquad t \in [T, T+\sigma], \; \sigma > 0, \tag{4.6}$$

*has a unique solution, which is given by*

$$x(t) = \begin{cases} \varphi(t), & \textit{if} \quad t \in [a-b, a], \\ h(t, x^t) - \int_a^T G(t,\tau)k(\tau)d\tau, & \textit{if} \quad t \in J, \\ \vartheta(t), & \textit{if} \quad t \in [T, T+\sigma], \end{cases} \tag{4.7}$$

*where*

$$G(t,\tau) = \frac{\varrho^{1-\beta}}{\Gamma(\beta)} \begin{cases} \frac{(t^\varrho - a^\varrho)(T^\varrho - \tau^\varrho)^{\beta-1}\tau^{\varrho-1}}{(T^\varrho - a^\varrho)} - \tau^{\varrho-1}(t^\varrho - \tau^\varrho)^{\beta-1}, & a \le \tau \le t \le T, \\ \frac{(t^\varrho - a^\varrho)(T^\varrho - \tau^\varrho)^{\beta-1}\tau^{\varrho-1}}{(T^\varrho - a^\varrho)}, & a \le t \le \tau \le T. \end{cases} \tag{4.8}$$

*Here $G(t,\tau)$ is called the Green function of the boundary value problem (4.4)-(4.6).*

*Proof.* From (1.35), we have

$$x(t) = h(t, x^t) + a_0 + a_1 \left( \frac{t^\varrho - a^\varrho}{\varrho} \right) +^\varrho I_{a^+}^\beta k(\tau), \quad a_0, a_1 \in \mathbb{R}, \tag{4.9}$$

therefore

$$x(a) = a_0 = 0,$$

$$x(T) = a_1 \left( \frac{T^\varrho - a^\varrho}{\varrho} \right) + \frac{\varrho^{1-\alpha}}{\Gamma(\alpha)} \int_a^T (T^\varrho - \tau^\varrho)^{\beta-1}\tau^{\varrho-1}k(\tau)d\tau,$$

and

$$a_1 = -\frac{\varrho^{2-\beta}}{(T^\varrho - a^\varrho)\Gamma(\beta)} \int_a^T (T^\varrho - \tau^\varrho)^{\beta-1}\tau^{\varrho-1}k(\tau)d\tau.$$

Substitute the value of $a_0$ and $a_1$ into equation (4.9), we get equation (4.7).

$$x(t) = \begin{cases} \varphi(t), & \text{if} \quad t \in [a-b, a], \\ h(t, x^t) - \int_a^T G(t,\tau)k(\tau)d\tau, & \text{if} \quad t \in J, \\ \vartheta(t), & \text{if} \quad t \in [T, T+\sigma], \end{cases}$$

where $G$ is defined by equation (4.8), the proof is complete. □

**Lemma 4.2.** *Let $l : J \times C[-b, \sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. A function $x \in \mathcal{C}$ is solution of problem* $(4.1) - (4.3)$ *if and only if $y$ satisfies the following integral equation*

$$x(t) = \begin{cases} \varphi(t), & \textit{if} \quad t \in [a-b, a], \\ h(t, x^t) - \int_a^T G(t,\tau)k(\tau)d\tau, & \textit{if} \quad t \in J \\ \vartheta(t), & \textit{if} \quad t \in [T, T+\sigma], \end{cases}$$

*where $k \in C(J)$ satisfies the functional equation*

$$k(t) = l(t, x^t, k(t)).$$

The following hypotheses will be used in the sequel:

$(H_1)$ The function $l : J \times C[-b,\sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : J \times C[-b,\sigma] \longrightarrow \mathbb{R}$ is continuous.

$(H_2)$ There exist $R, S > 0$ , $0 < \overline{R} < 1$ such that:

$$|l(t,z,m) - l(t,\bar{z},\bar{m})| \leq R\|z-\bar{z}\|_{[-b,\sigma]} + \overline{R}|m-\bar{m}|,$$

and

$$|h(t,n) - h(t,\bar{n})| \leq S\|n-\bar{n}\|_{[-b,\sigma]},$$

for any $z,\bar{z},n,\bar{n} \in C([-b,\sigma])$ and $m,\bar{m} \in \mathbb{R}$.

$(H_3)$ There exists $\delta, \bar{\delta} \in L^\infty([a,T],\mathbb{R}_+)$ such that:

$$|l(t,z,m)| \leq \delta(t) \text{ for a.e. } t \in J, \text{ and each } z \in C([-b,\sigma]) \text{ and } m \in \mathbb{R}.$$

$$|h(t,z)| \leq \bar{\delta}(t) \text{ for a.e. } t \in J, \text{ and each } z \in C([-b,\sigma]).$$

Set

$$\delta^* = ess \sup_{t\in J} \delta(t),$$

and

$$\bar{\delta}^* = ess \sup_{t\in J} \bar{\delta}(t),$$

$$\widetilde{G} = \sup\left\{\int_a^T |G(t,\tau)|d\tau, t \in J\right\}.$$

$(H_4)$ For each bounded set $D_N$ in $\mathcal{C}$, the set $\{t \to h(t,x^t) : x \in D_N\}$ is equicontinuous in $C(J,\mathbb{R})$.

Now, we state and prove our existence result for (4.1)-(4.3) based on the Banach contraction principle.

**Theorem 4.3.** *Assume* $(H_1)$ *and* $(H_2)$ *hold. If*

$$\left(S + \frac{R\widetilde{G}}{(1-\overline{R})}\right) < 1, \tag{4.10}$$

*then the problem (4.1)-(4.3) has a unique solution.*

*Proof.* Let the operator $I : \mathcal{C} \longmapsto \mathcal{C}$ defined by

$$(Ix)(t) = \begin{cases} \varphi(t), & \text{if } \ t \in [a-b,a], \\ \\ h(t,x^t) - \int_a^T G(t,\tau)k_x(\tau)d\tau, & \text{if } \ t \in J \\ \\ \vartheta(t), & \text{if } \ t \in [T,T+\sigma]. \end{cases} \tag{4.11}$$

By Lemma 4.2 it is clear that the fixed points of $I$ are solutions (4.1)-(4.3) .
Let $x_1, x_2 \in \mathcal{C}$. If $t \in [a-b,a]$ or $t \in [T,T+\sigma]$ then

$$|(Ix_1)(t) - (Ix_2)(t)| = 0.$$

For $t \in J$, we have:

$$|(Ix_1)(t) - (Ix_2)(t)| \leq |h(x_1^t,t) - h(x_2^t,t)| + \int_a^T |G(t,\tau)||k_{x_1}(\tau) - k_{x_2}(\tau)|d\tau, \tag{4.12}$$

and by $(H_2)$ we have:

$$\begin{aligned} |k_{x_1}(t) - k_{x_2}(t)| &= |l(t, x_1^t, {}^{\varrho}_{c}D^{\beta}_{a^+}x_1(t)) - l(t, x_2^t, {}^{\varrho}_{c}D^{\beta}_{a^+}x_2(t))| \\ &\leq R\|x_1 - x_2\|_{[-b,\sigma]} + \overline{R}|k_{x_1}(t) - k_{x_2}(t)|. \end{aligned}$$

Then

$$|k_{x_1}(t) - k_{x_2}(t)| \leq \frac{R}{(1-\overline{R})}\|x_1 - x_2\|_{[-b,\sigma]}. \tag{4.13}$$

By replacing (4.13) in (4.12) we obtain,

$$\begin{aligned}|(Ix_1)(t)-(Ix_2)(t)| &\leq S\|x_1-x_2\|_{[-b,\sigma]}+\frac{R}{(1-\overline{R})}\int_a^T|G(t,\tau)|\|x_1-x_2\|_{[-b,\sigma]}d\tau\\ &\leq \left(S+\frac{R\widetilde{G}}{(1-\overline{R})}\right)\|x_1-x_2\|_{[-b,\sigma]}.\end{aligned}$$

Therefore, for each $t\in J$, we have

$$|(Ix_1)(t)-(Ix_2)(t)|\leq\left(S+\frac{R\widetilde{G}}{(1-\overline{R})}\right)\|x_1-x_2\|_{\mathcal{C}}.$$

Thus

$$\|Ix_1-Ix_2\|_{\mathcal{C}}\leq\left(S+\frac{R\widetilde{G}}{(1-\overline{R})}\right)\|x_1-x_2\|_{\mathcal{C}}.$$

Hence, by the Banach contraction principle, $I$ has a unique fixed point which is a unique solution of the problem (4.1)-(4.3). □

We now prove an existence result for (4.1)-(4.3) by using the Schauder's fixed point theorem.

**Theorem 4.4.** *Assume that the hypotheses $(H_1)$ and $(H_3)$ hold. Then problem (4.1)-(4.3) has at least one solution.*

**Step 1.** $I$ is continuous. Let $\{x_i\}$ be a sequence such that $x_i\longrightarrow x$ in $\mathcal{C}$. If $t\in[a-b,a]$ or $t\in[T,T+\sigma]$ then

$$|(Ix_i)(t)-(Ix)(t)|=0.$$

For $t\in J$, we have

$$|(Ix_i)(t)-(Ix)(t)|\leq|h(x_i^t,t)-h(x^t,t)|+\int_a^T|G(t,\tau)||k_i(\tau)-k(\tau)|d\tau, \tag{4.14}$$

where

$$k_i(t)=l(t,x_i^t,k_i(t)),$$

and

$$k(t)=l(t,x^t,k(t)).$$

Since $x_i\longrightarrow x$, and by $(H_1)$ we get $k_i(t)\longrightarrow k(t)$ and $h(x_i^t,t)\longrightarrow h(x^t,t)$ as $i\longrightarrow\infty$ for each $t\in J$.
By $(H_3)$ we have for each $t\in J$,

$$|k_i(t)|\leq\delta^*. \tag{4.15}$$

Then,

$$\begin{aligned}|G(t,\tau)||k_i(t)-k(t)| &\leq |G(t,\tau)|\left[|k_i(t)|+|k(t)|\right]\\ &\leq 2\delta^*|G(t,\tau)|.\end{aligned}$$

For each $t\in J$ the functions $\tau\longmapsto 2\delta^*|G(t,\tau)|$ are integrable on $[a,t]$, then by Lebesgue dominated convergence theorem, equation (4.14 ) implies

$$|(Ix_i)(t)-(Ix)(t)|\longrightarrow 0 \ as \ i\longrightarrow\infty,$$

and hence

$$\|I(x_i)-I(x)\|_{\mathcal{C}}\longrightarrow 0 \ as \ i\longrightarrow\infty.$$

Consequently, $I$ is continuous.
Let the constant $N$ be such that:

$$N\geq\max\left\{\bar{\delta^*}+\delta^*\widetilde{G},\|\varphi\|_{[a-b,a]},\|\vartheta\|_{[T,T+\sigma]}\right\}, \tag{4.16}$$

and define

$$D_N = \{x \in \mathcal{C} : \|x\|_{\mathcal{C}} \leq N\}.$$

It is clear that $D_N$ is a bounded, closed and convex subset of $\mathcal{C}$.

**Step 2.** $I(D_N) \subset D_N$.

Let $x \in D_N$ we show that $Ix \in D_N$.
If $t \in [a-b, a]$, then

$$|I(x)(t)| \leq \|\varphi\|_{[a-b,a]} \leq N,$$

and if $t \in [T, T+\sigma]$, then

$$|I(x)(t)| \leq \|\vartheta\|_{[T,T+\sigma]} \leq N.$$

For each $t \in J$, we have:

$$|(Ix)(t)| \leq |h(x^t, t)| + \int_a^T |G(t,\tau)||k(\tau)|d\tau.$$

By $(H_3)$, we have:

$$\begin{aligned} |(Ix)(t)| &\leq \bar{\delta^*} + \delta^* \int_a^T |G(t,\tau)|d\tau \\ &\leq \bar{\delta^*} + \delta^* \widetilde{G} \\ &\leq N, \end{aligned}$$

from which it follows that for each $t \in [a-b, T+\sigma]$, we have $|Ix(t)| \leq N$, which implies that $\|Ix\|_{\mathcal{C}} \leq N$. Consequently,

$$I(D_N) \subset D_N.$$

**Step 3:** $I(D_N)$ is bounded and equicontinuous.
By Step 2 we have $I(D_N)$ is bounded.
Let $t_1, t_2 \in J = [a, T], t_1 < t_2,$ *and* $x \in D_N$ then

$$\begin{aligned} |(Ix)(t_2) - (Ix)(t_1)| &\leq |h(t_2, x^{t_2}) - h(t_1, x^{t_1})| + \int_a^T |G(t_2,\tau) - G(t_1,\tau)||k(\tau)|d\tau \\ &\leq |h(t_2, x^{t_2}) - h(t_1, x^{t_1})| + \delta^* \int_a^T |G(t_2,\tau) - G(t_1,\tau)|d\tau. \end{aligned}$$

According to $(H_4)$ we have $t \longrightarrow h(t, x^t)$ is equicontinuous. As $t_1 \longrightarrow t_2$ the right hand side of the above inequality tends to zero. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that $I$ is continuous and completely continuous. From Schauder's theorem, we conclude that $I$ has a fixed point with is a solution of the problem (4.1)-(4.3).

We prove an existence result for the (4.1)-(4.3) problem by using the Schaefer's fixed point theorem.

**Theorem 4.5.** *Assume that* $(H_1)$ *and*
$(H_5)$ *There exist* $j, u, v \in C(J, \mathbb{R})$ *with* $v^* = \sup_{t \in J} v(t) < 1$ *such that*

$$|l(t, z, m)| \leq j(t) + u(t)\|z\|_{[-b,\sigma]} + v(t)|m|,$$

*and there exist two constants* $c_1 > 0, c_2 > 0$ *such that:*

$$|h(t, z)| \leq c_1\|z\|_{[-b,\sigma]} + c_2,$$

*where* $t \in J$, $z \in C([-b, \sigma], \mathbb{R})$ *and* $m \in \mathbb{R}$.
*If*

$$\frac{u^*\widetilde{G}}{(1-v^*)} < 1, \tag{4.17}$$

*then problem (4.1)-(4.3) has at least one solution.*

**Proof.** Consider the operator $I$ defined in (4.11). We shall show that $I$ satisfies the assumption of Schaefer's fixed point theorem. As shown in Theorem 4.4, we see that the operator $I$ is continuous, and completely continuous.
Now it remains to show that the set

$$\eta = \{x \in \mathcal{C} : x = \gamma I x, \quad \text{for some} \quad \gamma \in (0,1)\} \quad \text{is bounded.}$$

Let $x \in \eta$, then $x = \gamma I x$ for some $0 < \gamma < 1$. Thus for each $t \in J$ we have

$$x(t) = -\gamma \left( h(t,x) + \int_a^T G(t,\tau) k_x(\tau) d\tau \right), \tag{4.18}$$

where

$$k_x(t) = l(t, x^t, k_x(t)).$$

By $(H_5)$, we have for each $t \in J$

$$\begin{aligned} |k_x(t)| &\le j(t) + u(t)\|x\|_{[-\beta,\sigma]} + v(t)|k_x(t)| \\ &\le j^* + u^*\|x\|_{[-b,\sigma]} + v^*|k_x(t)|. \end{aligned}$$

Thus

$$|k_x(t)| \le \frac{1}{1-v^*} \left( j^* + u^* \|x\|_{[-\beta,\sigma]} \right).$$

This implies, by (4.18) that for each $t \in J$ we have

$$\begin{aligned} |x(t)| &\le |h(t,x^t)| + \int_a^T |G(t,\tau)| \frac{1}{1-v^*} \left( j^* + u^*\|x\|_{[-\beta,\sigma]} \right) d\tau \\ &\le d_1^* \|z\|_{[-b,\sigma]} + j_2^* + \frac{\left( j^* + u^*\|x\|_{[-b,\sigma]} \right) \widetilde{G}}{(1-v^*)}. \end{aligned}$$

Then

$$\|g\|_{[-b,\sigma]} \le j_1^* \|z\|_{[-b,\sigma]} + j_2^* + \frac{j^* \widetilde{G}}{(1-v^*)} + \frac{u^* \widetilde{G} \|x\|_{[-b,\sigma]}}{(1-v^*)}.$$

Thus

$$\left[ 1 - j_1^* - \frac{u^* \widetilde{G}}{(1-v^*)} \right] \|x\|_{[-b,\sigma]} \le j_2^* + \frac{j^* \widetilde{G}}{(1-v^*)}.$$

Finally, by (4.17) we have

$$\|x\|_{[-b,\sigma]} \le \frac{j_2^* + \frac{j^* \widetilde{G}}{(1-v^*)}}{\left[ 1 - j_1^* - \frac{u^* \widetilde{G}}{(1-v^*)} \right]} = d_0.$$

If $t \in [a-b, a]$, then

$$|x(t)| \le \|\varphi\|_{[a-b,a]} \le d_1,$$

and if $t \in [T, T+\sigma]$, then

$$|x(t)| \le \|\vartheta\|_{[T,T+\sigma]} \le d_2.$$

From which it follows that for each $t \in [a-b, T+\sigma]$, we have $|x(t)| \le \max\{d_2, d_1, d_0\}$, which implies that $\|x\|_{\mathcal{C}} \le \max\{d_2, d_1, d_0\}$, this implies that $\eta$ is bounded As a consequence of Schaefer's fixed point theorem, $I$ admits a fixed point which is a solution of the problem (4.1)-(4.3).

## 4.3 Examples

**Example 1:** Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} x(t) = e^{t-2} - 1, & t \in [1,2], \\ {}_c^{\frac{1}{2}} D_{2^+}^{\frac{3}{2}} x(t) = \dfrac{1}{10e^{t+2}\left(1 + |x^t| + \left| {}_c^{\frac{1}{2}} D_{2^+}^{\frac{3}{2}} x(t) \right| \right)} + \dfrac{\sin(t)}{\ln(t^2+1)}, & t \in J = [2,4] \\ x(t) = t - 4, & t \in [4,6]. \end{cases} \tag{4.19}$$

Set
$$l(t,z,m)=\frac{1}{10e^{t+2}(1+|z|+|m|)}+\frac{\sin(t)}{\ln(t^2+1)},\quad t\in[2,4], z\in C([-b,\sigma])$$
and
$$h(t,z)=0,$$
and
$m\in\mathbb{R}, \beta=\frac{3}{2}, \varrho=\frac{1}{2}, b=1, \sigma=2$. For each $z,\bar{z}\in C([-b,\sigma])$, $m,\bar{m}\in\mathbb{R}$ and $t\in[2,4]$, we have
$$\begin{aligned}|h(t,z,m)-h(t,\bar{z},\bar{m})| &\le \left|\frac{1}{10e^{t+2}(1+|z|+|m|)}-\frac{1}{10e^{t+2}(1+|\bar{z}|+|\bar{m}|)}\right|\\ &\le \frac{1}{10e^{t+2}}\left(|z-\bar{z}|+|m-\bar{m}|\right)\\ &\le \frac{1}{10e^{t+2}}\left(\|z-\bar{z}\|_{[-b,\sigma]}+|m-\bar{m}|\right).\end{aligned}$$
Therefore, $(H_2)$ is verified with $R=\overline{R}=\frac{1}{10e^4}$.
For each $t\in J$ we have
$$\int_a^T|G(t,\tau)|d\tau\le\frac{1}{\Gamma(\beta)}\left(\frac{t^\varrho-a^\varrho}{T^\varrho-a^\varrho}\right)\int_a^T\left|\left(\frac{T^\varrho-\tau^\varrho}{\varrho}\right)^{\beta-1}\tau^{\varrho-1}\right|d\tau$$
$$+\frac{1}{\Gamma(\beta)}\int_a^t\left|\left(\frac{t^\varrho-\tau^\varrho}{\varrho}\right)^{\beta-1}\tau^{\varrho-1}\right|d\tau.$$
Then
$$\int_a^T|G(t,\tau)|d\tau\le\frac{2}{\Gamma(\beta+1)}\left(\frac{T^\varrho-a^\varrho}{\varrho}\right)^{\beta}.$$
Therefore
$$\widetilde{G}\le\frac{2}{\Gamma(\beta+1)}\left(\frac{T^\varrho-a^\varrho}{\varrho}\right)^{\beta}.$$
The condition
$$\begin{aligned}\frac{R\widetilde{G}}{(1-\overline{R})} &\le 2\frac{\frac{1}{10e^4}}{(1-\frac{1}{10e^4})\Gamma(\frac{5}{2})}\left(\frac{2-2^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}}\approx 0.0035008\\ &< 1,\end{aligned}$$
is satisfied with $T=4, a=2$ and $\beta=\frac{3}{2}$. Hence all conditions of Theorem 4.3 are satisfied, it follows that the problem (4.19) admit a unique solution defined on $J$.

**Example 2:** Consider the boundary value problem of implicit Caputo type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases} x(t)=e^t-1, & t\in[-1,0],\\ {}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+}x(t)=\dfrac{\sin(2t)\left(2+|x^t|+\left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+}x(t)\right|\right)}{20e^{t+4}\left(1+|x^t|+\left|{}^{\frac{1}{2}}_{c}D^{\frac{3}{2}}_{0^+}x(t)\right|\right)}, & t\in J=[0,e]\\ x(t)=\ln(t)-1, & t\in[e,4],\end{cases}\tag{4.20}$$

with
$$l(t,z,m)=\frac{\sin(2t)\,(2+|z|+|m|)}{10e^{t+2}(1+|z|+|m|)},\quad t\in J=[0,e], z\in C([-b,\sigma])\ \text{ and } m\in\mathbb{R}$$
$$h(x,z)=0$$
$$\beta=\frac{3}{2}, \varrho=\frac{1}{2}, b=1, \sigma=4-e.$$

Condition $(H_4)$ is satisfied for each $z, \in C([-b,\sigma])$ , $m \in \mathbb{R}$ and $t \in [0,e]$:

$$\begin{aligned} |l(t,z,m)| &\leq \frac{2+|z|+|m|}{20e^{t+4}} \\ &\leq \frac{1}{20e^{t+4}}\left(2+|m|+\|z\|_{[-b,\sigma]}\right). \end{aligned}$$

Therefore, $(H_4)$ is verified with

$$j(t) = \frac{1}{10e^{t+4}}, \quad u(t) = v(t) = \frac{1}{20e^{t+4}} \quad and \quad v^* = \frac{1}{20e^4} < 1.$$

Condition:

$$\begin{aligned} \frac{u^*\widetilde{G}}{(1-v^*)} &\leq 2\frac{\frac{1}{20e^4}}{(1-\frac{1}{20e^4})\Gamma(\frac{5}{2})}\left(\frac{e^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{3}{2}} \approx 0.0082575 \\ &< 1, \end{aligned}$$

is satisfied with $T = e, a = 0$ and $\beta = \frac{3}{2}$. Hence all conditions of Theorem 4.5 are satisfied, it follows that the problem (4.20) has at least one solution on $J$.

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# Conclusion and Perspective

In this memory, we have studied the existence and uniqueness of the solution of implicit fractional differential equations with retarded and advanced arguments . The results are based on the fixed point theorems .
For this, we started with preliminaries where we recalled some basic tools of fractional calculus, and we stated some results on the compactness in metric space, the theorem of representation of the completely continuous operators as uniform limits of sequences of continuous operators of finite rank with the detailed proof of Banach, Schauder and Schaefer's fixed point theorem .

In the future, we plan to study the qualitative aspect of the solutions for the above mentioned problems in particular, we will look for the stability and controllability of the above cited problems.

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