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Contribution à l'étude de certains problèmes aux dérivées fractionnaires

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Dedication

This thesis is dedicated
to my Father who passed away,
to my Family and all my Friends,
especially my Mother and my Wife.

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Abstract :

Generalized Fractional Calculus appear in modeling of important various scientific fields such as the complex phenomena in mathematical, physics, engineering, chemistry, electricity and medicine.
The main objective of this thesis is to contribute the development of the theory of the existence and uniqueness of solutions of certain fractional differential equations such as: neutral differential equation, Langevin equation, hybrid differential equations involving different fractional derivatives as ψ-Caputo, Riesz-Caputo, Katugampola with local and nonlocal conditions in Banach spaces.
The results obtained in this work are based on the fixed point theory: Banach's contraction principle, Boyd-Wong, Scheafer, Krasnoselski"s, the technique of Nonlinear alternative of Leray-shauder, Dhage. We also establish the Ulam-Hyers stability results for some addressed problems. We have also provided an illustrative example to each of our considered problems for exhibit the effectiveness of our achieved results.
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Keywords : Fractional differential equations, Existence, uniqueness, Fixed point theorems, Ulam stability analysis.

Resumé

Le calcul fractionnaire généralisé joue un rôle important dans la modélisation des phénomènes complexes en mathématiques, physique, ingénierie, chimie, électricité et médecine. L'objectif principal de cette thèse est de contribuer à l'étude de l'existence et de l'unicité des solutions de certaines équations différentielles fractionnaires telles que : une équation différentielle neutre, Langevin équation, équations différentielles hybrides impliquant différents dérivés fractionnaires comme ψ-Caputo, Riesz-Caputo, Katugampola avec des conditions locales et non locales dans les espaces Banach. Les résultats obtenus dans ce travail sont basés sur la théorie des points fixes: Banach's principal contraction, Boyd-Wong, Scheafer, Krasnoselski, la technique de Nonlinear alternative de Leray-shauder, Dhage. Nous établissons également les résultats de stabilité d'Ulam-Hyers pour certains problèmes abordés. Nous concluons que les résultats obtenus par des exemples illustratifs.

Mots clés : Equations différentielle fractionnaire, Problemes aux limites, Existence, Unicité, Théorémes de point fixe, Ulam stabilité

الملخص

يلعب الحساب الكسري المعمم دورا هاما في نمذجة الظواهر المعقدة في الرياضيات والفيزياء والهندسة والكيمياء والكهرباء والطب، ...إلخ.
الهدف الرئيسي من هذه الأطروحة هو المساهمة في دراسة وجود ووحدانية الحلول لبعض المعادلات التفاضلية الكسرية مثل: معادلة أسية ، معادلة لانجفين، المعادلات التفاضلية الهجينة والتي تتضمن مشتقات كسرية مختلفة كمشتقة **ψ -كابوتو ، ريز-كابوتو ، كاتي غامبولا** تحت شروط محلية وغير محلية في فضاءات بناخ.

تستند النتائج التي تم الحصول عليها في هذا العمل إلى نظرية النقطة الثابتة: انكماش بناخ، بويد-ونغ، شيفر، كراسنوسلسكي، لوراي-شودر غير الخطية المتناوبة ونظرية داج في فضاءات بناخ الجبرية. كما ركزنا على تحليل الاستقرار لنتائجنا الرئيسية على اساس نظرية الاستقرار لأولام-هايرز. نلخص نتائجنا بإعطاء امثلة توضيحية لتبرير صحتها.

الكلمات المفتاحية: المعادلات التفاضلية الكسرية، الحساب الكسري، الوجود، الوحدانية، النقطة الثابتة، الاستقرار.

Notations

Ξ Xi.

ζ zeta.

α The integer part of the real number α.

$Re(\alpha)$ Real part of complex α.

$\mathbb{N}$ Set of Natural numbers 0,1,2,3....

$\mathbb{R}$ Set of Real numbers $(-\infty, \infty)$.

$\mathbb{R}^n$ Space of n-dimensional real vectors.

$\mathbb{C}$ Complex numbers.

$n!$ Factorial $(n), n \in \mathbb{N}$: The product of all the integers from1 to n.

C_k^n Coefficient binomial.

J Finite closed interval of the real axis $\mathbb{R}$.

$C(J)$ The space of all continuous functions from Ω into $\mathbb{R}$.

$C^n(J)$ Space of n time continuously. differentiable functions on Ω

$AC(J, \mathbb{E})$ Space of absolutely continuous functions on J.

$AC^n(J, \mathbb{E})$ Space of real-valued functions $f(t)$ which have continuous derivatives up to order $n-1$ on J.

$L^1(J)$ Space of Lebesgue integrable functions on Ω.

$L^p(J)$ Space of measurable functions u with $|u|^p$ belongs to $L^1(\Omega)$.

$L^\infty(J)$ space of functions u that are essentially bounded on Ω .

X_c^p Space of complex-valued Lebesgue measurable functions Ω.

$I^{\alpha,\psi}$ The fractional ψ-integral of order $\alpha > 0$.

$I_{a^+}^\alpha$ The Riemann-Liouville fractional integral of order $\alpha > 0$.

${}^\rho I_{a^+}^\alpha$ The Katugampola fractional integral of order $\alpha > 0$, $\rho > 0$.

$I^{\alpha,\delta}_{a^+,\eta}$ The Erdélyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$, $\alpha \in \mathbb{R}$.

$I^{\alpha,\psi}_{a^+}$ The ψ-Riemann-Liouville fractional integral of order $\alpha > 0$.

${}^{RL}D^{\alpha}_{a^+}$ The left-sided Riemann-Liouville fractional derivative of order $\alpha > 0$.

${}^{RL}D^{\alpha}_{b^-}$ The right-sided Riemann-Liouville fractional derivative of order $\alpha > 0$.

${}^{C}D^{\alpha}_{a^+}$ The left-sided Caputo fractional derivative of order $\alpha > 0$.

${}^{C}D^{\alpha}_{b^-}$ The right-sided Caputo fractional q-derivative of order $\alpha > 0$.

${}^{RC}_{0}D^{\alpha}_{T}$ The Riesz-Caputo fractional derivative of order $\alpha > 0$.

$D^{\rho,\alpha}_{a^+}$ The Katugampola fractional derivative of order $\alpha > 0$.

$D^{\alpha,\psi}_{a^+}$ The ψ-Caputo fractional derivative of order $\alpha > 0$.

$\Gamma(\alpha)$ Euler gamma function which is now denoted by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

$E_{\alpha}(z)$ The Its one parameter generalization, called the Mittag-Leffler function which is now denoted by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

$E_{\alpha,\beta}(z)$ A two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0.$$

FC Fractional calculus.

FD Fractional derivative.

FDE Fractional differential equation.

PDE Partial differential equation.

FI Fractional integral.

IVP Initial value problem.

BVP Boundary value problem.

FHDE Fractional hybrid differential equation.

FPT Fixed point theory.

UH Ulam-Hayer stability analysis.

FFPE Fractional Fokker-Plank equation

List of Publications

1. **N. Adjimi**, A. Boutiara, M. S. Abdo, M. Benbachir, Existence Results for Nonlinear Neutral Generalized Caputo Fractional Differential Equations. Journal of Pseudo-Differential Operators and Applications, Springer,12, No 25(2021). (published).

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3. **N. Adjimi**, M. Benbachir, Existence results for Langevin equation with Riesz-Caputo fractional derivative (accepted in Survey in Mathematics and its Applications).

4. **N. Adjimi**, M. Benbachir, K. Guerbati, Existence results for ψ-Caputo hybrid fractional integro-differential equations. Malaya journal of mathematik, 09(02)(2021),46-54.(published).

5. **N. Adjimi**, M. Benbachir, MS. Abdo, A. Boutiara, Analysis of a fractional boundary value problem involving Riesz-Caputo fractional derivative. Advances in the Theory of Nonlinear Analysis and its Applications, 6 (2022)N01,14-27. (published).

Contents

Introduction

0.1 History of fractional calculus

Since the 60s of the last century Fractional Calculus has got a remarkable progress and now it is recognized to be an important domain for scientists especially mathematicians. Fractional Calculus (FC) started in 1695 with the ideas of Gottfried Leibniz generated from letter which was written by Antoine Marquez-L'Hopital asking him "what would happen if the order of the derivative was a real number instead of an integer?". Leibniz responded: "It will lead to the paradox, from which beneficial consequences will one day be extracted". This exchange Between L'Hopital and Leibnitz is generally considered the beginning of fractional calculus. However, the actual development of fractional calculus was in 1832. When Joseph Liouville introduced what is now called the Riemann-Liouville definition of the fractional derivative. Many other definitions of fractional integrals and derivatives are based on the Riemann-Liouville integral, other definitions extend the notion based on the differences of the kernel functions.

0.2 List of mathematician's contributions to fractional calculus:

In this section we address a list of mathematicians, who have provided important contributions to fractional calculus.

P.S. Laplace (1812), proposed the idea of differentiation of non-integer order for the functions [61].

Liouville (1835) derived the formula of the fractional integral and fractional derivative respectively of the form (see [63])

$$D^{-\beta}g(x) = \frac{1}{(-1)^{\beta}\Gamma(\beta)}\int_0^{\infty} g(x+t)t^{\beta-1}, x \in \mathbb{R}, \quad Re(\beta) > 0 \tag{1}$$

and

$$D^{\beta}g(x) = \frac{1}{(-1)^{\beta}\Gamma(\beta)}\int_0^{\infty} \frac{d^n g(x+t)}{dx^n} t^{\beta-1}dt, x \in \mathbb{R}, \quad Re(\beta) > 0. \tag{2}$$

Riemann (1835) derived the formula of fractional integrals related with the Liouville fractional integral of the form (see [86])

$$D^{-\beta}g(x) = \frac{1}{\Gamma(\beta)}\int_a^{x} (x-t)^{\beta-1}g(t)dt, \quad Re(\beta) > 0, \tag{3}$$

where $Re(\beta) \in (n-1, n], n \in \mathbb{N}$ and $Re(\cdot)$ denotes the real part of complex number. Grünwald-Letnikov (1867-1868) introduce the operator called the Grünwald-Letnikov fractional operator of the form (see [64, 93])

$$D^{\beta} g(x) = \lim_{h \to 0} \frac{\Delta_h^{\beta} g(x)}{h^{\beta}}, \quad \beta > 0, h > 0, \tag{4}$$

where $\Delta_h^{\beta} g(x)$ is a difference of fractional order, given by

$$\Delta_h^{\beta} g(x) = \sum_{k=0}^{\infty} (-1)^k C_k^{\beta} g(x - kh). \tag{5}$$

Sonine (1872) introduced the Sonine fractional derivative, given as on the form (see [99, 100])

$$D^{\beta} g(x) = \frac{1}{\Gamma(\rho - \beta + 1)} \int_a^x \frac{dg(t)}{dt} (x-t)^{\rho - \beta} dt, \tag{6}$$

where $\mathrm{Re}(\rho) < \beta < \mathrm{Re}(\rho + 1)$, $\rho \in \mathbb{C}$.
Hadamard (1892) proposed the following fractional integral in the form (see [51])

$$I^{\beta} g(x) = \frac{x^{\beta}}{\Gamma(\beta)} \int_0^1 \frac{g(tx)}{(1-t)^{1-\beta}} dt, \quad Re(\beta) > 0. \tag{7}$$

Weyl (1917) derived the left-sided and right-sided of the Weyl fractional integrals in the form (see [109])

$$I_+^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^x \frac{g(t)}{(x-t)^{1-\beta}} dt, \quad 0 < \beta < 1, \tag{8}$$

$$I_-^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_x^{\infty} \frac{g(t)}{(x-t)^{1-\beta}} dt, \quad 0 < \beta < 1. \tag{9}$$

Marchaud (1927) introduced the Marchaud fractional derivatives of the form (see [71])

$$D^{\beta} g(x) = \frac{C}{\Gamma(\beta)} \int_x^{\infty} \frac{\Delta_t^l g(x)}{t^{1+\beta}} g(t) dt, \tag{10}$$

where $\Delta_t^l g(x)$ is the finite difference of order l, for $l > \beta$ and $l \in N$. When $\Delta_t^l g(x)$, it is called the Weyl type finite difference for $l = 1$ and $0 < \beta < 1$.
Hadamard (1927) introduced the Hadamard fractional integral, and the fractional derivative respectively was given by (see [71])

$$I_a^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{g(t)}{(\ln \frac{x}{t})^{1-\beta}} \frac{dt}{t}, \quad 0 < \beta < 1, \tag{11}$$

$$D_a^\beta g(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x \frac{g(x)-g(t)}{(\ln\frac{x}{t})^{\beta+1}} \frac{dt}{t}, \quad 0<\beta<1. \tag{12}$$

Hille and Tamarkin (1930) proposed the Abel type integral equation on the second kind (see [?])

$$g(x) = h(x) - \frac{\xi}{\Gamma(\beta)} \int_0^x \frac{h(t)}{(x-t)^{1-\beta}} dt, \quad 0<\beta<1, \xi \in \mathbb{C}, \tag{13}$$

with the solution (also called the Hille-Tamarkin fractional derivative, (see [?])

$$h(x) = \frac{d}{dx} \int_0^x E_\beta[\xi(x-t)^\beta] g(t) dt, \quad 0<\beta<1, \tag{14}$$

where $E_\beta[\xi(x-t)^\beta]$ is the Mittag-Leffler function, with one-parameter constant $\xi \in \mathbb{C}$ is defined as

$$E_\beta(\xi(x-t)^\beta) = \sum_{n=1}^{\infty} \frac{\xi(x-t)^{n\beta}}{\Gamma(n\beta+1)}, \quad \beta>0.$$

Love and young (1938) proposed the convergent fractional integral in the form (see [67, 68])

$$I_+^\beta g(x) = \frac{1}{\Gamma(\beta)} \lim_{m\to\infty} \int_0^m \frac{g(x-t)}{t^{1-\beta}} dt. \tag{15}$$

In 1939, Hille introduced the Hille fractional differential operator in the form (see [53])

$$R(\beta,\xi) g(x) = \frac{1}{\xi} \frac{d}{dx} \int_0^x E_\beta[\frac{(x-t)^\beta}{\xi}] f(t) dt, \quad \mathrm{Re}(\beta)>0, \xi \in \mathbb{R}^\star, \tag{16}$$

where $R(\beta,\xi)$ is the resolvent of ${}^{RL}I^\beta$ and $E_\beta[\frac{(x-t)^\beta}{\xi}]$ is the one parameter Mittag-Leffler function, with $\beta>0$.
Erdélyi-Kober (1940) proposed the fractional integrals and derivatives in the form (see [41])

$$I_{a+;\sigma,\eta}^\beta g(x) = \frac{\sigma t^{-\sigma(\beta+\eta)}}{\Gamma(\beta)} \int_a^x \frac{t^{\sigma(\eta+1)-1} g(s)}{(x^\sigma - s^\sigma)^{1-\beta}} ds, \tag{17}$$

and the fractional derivatives as

$$D_{a+;\sigma,\eta}^\beta g(x) = t^{-\sigma\eta} \left(\frac{1}{\sigma t^{\sigma-1}} D\right)^k t^{\sigma(\beta+\eta)} \left(I_{a+;\sigma,\eta+\beta}^\beta g\right)(x), \tag{18}$$

where $\beta>0, \sigma>0, \eta \in \mathbb{R}$.

Cossar (1941) reported the Cossar fractional derivative in the form (see [30])

$$D_{+}^{\beta}g(x) = -\frac{1}{\Gamma(1-\beta)} \lim_{m\to\infty} \frac{d}{dx} \int_{0}^{m} \frac{g(t)}{(t-x)^{\beta}} dt, \quad \beta > 0. \tag{19}$$

Riesz(1949) defined the fractional calculus based on the Fourier's work, which is called the Riesz fractional calculus in the form (see[87])

$${}^{RZ}I_{R}^{\beta}g(x) = \frac{1}{2\Gamma(\beta)\cos(\pi\beta/2)} \int_{-\infty}^{\infty} \frac{g(s)}{|s-x|^{1-\beta}} ds, \quad \mathrm{Re}(\beta) > 0, \tag{20}$$

and

$${}^{RZ}D_{R}^{\beta}g(x) = \frac{1}{2\Gamma(n-\beta)\cos(\pi\beta/2)} \frac{d^{n}}{dx^{n}} \int_{-\infty}^{\infty} \frac{g(s)}{|s-x|^{\beta-n+1}} ds. \tag{21}$$

Hille and Phillips (1957) introduced the integral in the form (see [54])

$$I_{\xi}^{\beta}g(t) = \frac{\xi^{\beta+1}}{\Gamma(1+\beta)} \int_{0}^{\infty} t^{\beta} e^{\xi(x-t)} g(t) dt. \tag{22}$$

Chen (1961) introduce the Chen fractional integrals and derivative respectively of the form (see [32])

$$I^{\beta}g(x) = \frac{1}{\Gamma(\beta)} \int_{a}^{x} |x-t|^{\beta-1} g(t) dt, \quad x > a, \mathrm{Re}(\beta) > 0, \tag{23}$$

$$D^{\beta}g(x) = \frac{1}{\Gamma(1-\beta)} \int_{a}^{x} \frac{1}{|x-t|^{\beta}} \frac{dg(t)}{dt} dt, \quad x > a, \mathrm{Re}(\beta) > 0, \tag{24}$$

where $\mathrm{Re}(\beta) \in (n-1, n], n \in \mathbb{N}$ and $\mathrm{Re}(\cdot)$ denotes the real part of complex number. Srivastava (1964) proposed the fractional integral in the kernel of the confluent hypergeometric function was given as (see [102])

$$I^{\beta}g(x) = \int_{0}^{x} \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} {}_{1}F_{1}(\alpha; \beta; x-t) g(t) dt, \quad \mathrm{Re}(\beta) > 0, \tag{25}$$

where ${}_{1}F_{1}(\alpha; \beta; x-t)$ is called the confluent hypergeometric function of the first kind, on the form

$${}_{1}F_{1}(a, b; z) = \sum_{n=0}^{\infty} \frac{(a)_{n}}{(b)_{n}} \frac{z^{n}}{n!},$$

is defined for $|z| < 1$ and α, β assumed arbitrarily real or complex values and $b \in \mathbb{Z}^{+}$.

Cooke (1965) proposed the Cooke fractional operator in the form (see [31])

$$I_{a;\rho}^{\nu,\beta} g(x) = \begin{cases} \frac{2x^{-2(\nu+\beta)}}{\Gamma(\beta)} \int_a^b \frac{t^{2(\nu+1)-1}}{(x^2-t^2)^{1-\beta}} g(t) t^{2\nu-1} dt, & \beta > 0, \\ g(x), \beta = 0, \\ \frac{2x^{-2(\nu+\beta)-1}}{\Gamma(\beta+1)} \frac{d}{dx} \int_a^b (x^2-t^2)^{\beta} t^{2\nu+1} g(t) dt, & 0 < \beta < 1. \end{cases} \tag{26}$$

Saxena (1967) introduced the Saxena fractional integral within the kernel of the Gauss hypergeometric function, is defined by (see [95])

$$I_a^{\beta} g(x) = \frac{x^{-\sigma-1}}{\Gamma(\beta)} \int_0^x {}_2F_1(1-\beta, \alpha+m; \alpha; t/x) g(t) t^{\sigma} dt, \quad \mathrm{Re}(\beta) > 0, \tag{27}$$

where ${}_2F_1(1-\beta, \alpha+m; \alpha; t/x)$ is the Gauss hypergeometric function is defined as

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

is defined for $|z| < 1$ and α, β assume arbitrary real or complex values and $c \in \mathbb{Z}^+$. Kalisch (1967) proposed the left-sided and the right-sided of the Kalisch fractional derivative of purely imaginary order β, where $\beta = i\theta$, respectively in the form (see [63])

$$I_{a^+}^{i\theta} g(x) = \frac{1}{\Gamma(1+i\theta)} \int_a^x (x-t)^{i\theta} g(t) dt, \tag{28}$$

and

$$I_{b^-}^{i\theta} g(x) = \frac{1}{\Gamma(1+i\theta)} \int_x^b (t-x)^{i\theta} g(t) dt. \tag{29}$$

Caputo (1967) introduced the Caputo fractional derivative in the form (see [**?**])

$$D^{\beta} g(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{1}{(x-t)^{\beta}} g^{(n)}(t) dt, \quad x > 0, \mathrm{Re}(\beta) > 0. \tag{30}$$

Dzherbashyan (1967) proposed the Dzherbashyan fractional integral used the generalization of Hadamarod's idea and gave the fractional integral in the form (see [38])

$$I^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_0^1 \frac{g(xz)}{(-lnz)^{1-\beta}} dz, \quad \mathrm{Re}(\beta) > 0. \tag{31}$$

Srivastava (1968) proposed the Srivastava fractional operator which is related to the generalized Whittakar transform in the form (see [103])

$$R_{\xi,\beta,n}^{-} g(x) = \frac{n}{\Gamma(\beta)} x^{\xi} \int_x^{\infty} (z^n - x^n)^{\beta-1} z^{-\xi-n\beta+n-1} f(z) dz, \quad x > 0. \tag{32}$$

where $g \in L_p(0,\infty)$, $\frac{1}{p}+\frac{1}{q}=1, 0<p<\infty, \beta>0, \xi>\frac{1}{p}$.
Dzhrbashyan and Nersesyan (1968) proposed the Dzhrbashyan-Nersesyan fractional derivative in the form (see [39])

$$D^{\beta}g(x)=\frac{1}{\Gamma(n-\beta)}\int_0^{\infty}\frac{1}{(x-t)^{n-\beta}}g^{n}(t)dt, \quad \mathrm{Re}(\beta). \tag{33}$$

Osler (1970) introduced Osler the fractional integral in the form (see [103])

$$I^{\beta}_{a;k}g(x)=\frac{1}{\Gamma(\beta)}\int_a^{x}\frac{g(t)}{(k(x)-k(t))^{1-\beta}}k^{(1)}(t)dt, \quad 0<\beta<1, \tag{34}$$

and the fractional derivative of the form

$$D^{\beta}_{a;k}g(x)=\frac{1}{\Gamma(1-\beta)}\frac{g(t)}{(k(x)-k(t))^{\beta}}\frac{\beta}{\Gamma(1-\beta)}\int_a^{x}\frac{g(x)-g(t)}{(k(x)-k(t))^{1+\beta}}k^{(1)}dt, \tag{35}$$

where $x>0, 0<\alpha<1$, $k\in C(I), k^{(1)}(t)\neq 0$.
Love (1971) considered the Love fractional integral and fractional derivative of purely imaginary order respectively as (see [69])

$$D^{i\beta}_{a}g(x)=\frac{1}{\Gamma(1-i\beta)}\frac{d}{dx}\int_a^{x}\frac{g(t)}{(x-t)^{i\beta}}dt, \tag{36}$$

$$I^{i\beta}_{a}g(x)=\frac{1}{\Gamma(i\beta)}\int_0^{\infty}\frac{g(t)}{(x-t)^{1-i\beta}}dt. \tag{37}$$

Rafal'son (1971) introduced the Rafal'son type Bessel fractional integration and derivative respectively in the form (see [85])

$$I^{\beta}_{-}g(x)=\frac{1}{\Gamma(\beta)}\int_x^{\infty}(x-t)^{\beta-1}e^{x-t}g(t)dt, \quad 0<\beta<1, \tag{38}$$

$$D^{\beta}_{+}g(x)=\frac{1}{\Gamma(\beta)}\int_x^{\infty}(x-t)^{\beta-1}e^{x-t}g^{(\beta)}(t)dt, \quad 0<\beta<1. \tag{39}$$

Prabhakar (1972) introduced the Prabhakar type Humbert fractional integral in the form (see [83])

$$I^{\beta}_{a}g(x)=\int_0^{x}\frac{(x-t)^{\beta-1}}{\Gamma(\beta)}\Theta_1\left(\beta,b,c;1-\frac{t}{x},\xi(x-t)\right)g(t)dt, \tag{40}$$

where $\Theta_1\left(\beta,b,c;1-\frac{t}{x},\xi(x-t)\right)$is two variable hypergeometrique function or the

Humbert function is defined as

$$\Theta_1\left(\beta, b, c; 1-\frac{t}{x}, \xi(x-t)\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\beta)_{n+m}(b)_m(1-\frac{t}{x})^m(\xi(x-t))^n}{m!n!(c)_{m+n}},$$

where $(\beta)_n = \frac{\Gamma(\beta+n)}{\beta}, n = 0, 1, 2, ...,$ and β, b, c are parameters which assume real or complex values.

Sneddon (1975) introduced the Sneddon fractional integral of the form (see [98])

$$I_{a,\rho}^{\nu,\beta} g(x) = \frac{\rho x^{-\rho(\nu+\beta)}}{\Gamma(\beta)} \int_a^x \frac{t^{\rho(\nu+1)-1}}{(x^\rho - t^\rho)^{1-\beta}} g(t)dt. \tag{41}$$

where $\beta, \nu \in \mathbb{C}, \mathrm{Re}(\beta) > 0, \rho > 0, t > 0.$

Saigo (1978) introduced the Saigo type Gauss hypergeometric fractional integral operator in the form (see [63])

$$I_x^{\beta,\gamma,\nu} g(x) = \frac{x^{-\beta-\gamma}}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} {}_2F_1\left(\beta+\gamma, -\nu, \beta; 1-\frac{t}{x}\right) g(t)dt, \tag{42}$$

where ${}_2F_1\left(\beta+\gamma, -\nu, \beta; 1-\frac{t}{x}\right)$ is the Gauss hypergeometric function defined by

$${}_2F_1(\beta+\gamma, -\nu, \beta; 1-\frac{t}{x}) = \sum_{n=0}^{\infty} \frac{(\beta+\gamma)_n(-\nu)_n}{(\beta)_n} \frac{(1-\frac{t}{x})^n}{n!},$$

and $\beta, \gamma, \nu \in \mathbb{C}$.

Gearhart (1979) introduced the Rafal'son-Gearhart type Bessel fractional integral (see [48])

$$I_-^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} (x-t)^{\beta-1} e^{\xi(x-t)} g(t)dt, \quad 0 < \beta < 1. \tag{43}$$

Skornik (1980) reported the Skornik tempered fractional integral and derivative respectively of the form (see [96, 97])

$$I_a^{\beta} g(x) = e^{\frac{-x^2}{4}} \int_a^x \frac{(x-t)^{\beta-1}}{\Gamma(\beta)} e^{\frac{t^2}{4}} g(t)dt, \quad 0 < \beta < 1, \tag{44}$$

and

$$D_a^{\beta} g(x) = \frac{e^{\frac{-x^2}{4}}}{\Gamma(1-\beta)} \frac{d^n}{dx^n} \int_a^x \frac{e^{\frac{t^2}{4}}}{(x-t)^{-\beta}} g(t)dt, \quad 0 < \beta < 1, \tag{45}$$

and

$$D_a^{\beta} I_a^{\beta} g(x) = g(x). \tag{46}$$

Peschanskii (1989) introduced the fractional integral operator involving the curvilinear convolution type in the form (see [84])

$$I_x^{\beta,\gamma,\nu} g(x) = \frac{1}{2\pi i} \int_\Gamma {}_2F_1\left(1,1;1+\beta;\frac{t}{x}\right) g(t)\frac{dt}{t}, \qquad \beta,\gamma,\nu \in \mathbb{C}, \tag{47}$$

where ${}_2F_1\left(1,1;1+\beta;\frac{t}{x}\right)$ is the Gauss hypergeometric function.
Samko and Ross (1993) reported the variable-order fractional integral given by (see [94])

$$I_a^{\beta(x)} g(x) = \frac{1}{\Gamma(\beta(x))} \int_a^x (x-t)^{-\beta(x)} g(t)dt, \quad \beta(x) > 0, \tag{48}$$

and the variable-order fractional derivative given as

$$D_a^{\beta(x)} g(x) = \frac{1}{\Gamma(1-\beta(x))} \frac{d}{dx} \int_a^x (x-t)^{-1-\beta(x)} g(t)dt, \quad 0 < \beta(x) < 1. \tag{49}$$

Hilfer (2000) introduced the Hilfer fractional derivative by (see [55])

$$D^{\beta,\gamma} g(x) = I_0^{\gamma(1-\beta)} \cdot D^1 \cdot I_0^{(1-\gamma)(1-\beta)} g(x), \tag{50}$$

where $0 < \alpha < 1, 0 \leq \beta \leq 1, g \in L^1(\mathbb{R}^+)$ and $D^{(1)} g(x) = \frac{dg(x)}{dx}$, $I^{\gamma(1-\beta)}$ and $I^{(1-\gamma)(1-\beta)}$ are the Riemann-Liouville fractional integrals.
Coimbra (2003) introduced the variable-order fractional integral in the form (see [29])

$$D^{\beta(x)} g(x) = \frac{1}{\Gamma(1-\beta(x))} \int_0^x \frac{1}{(x-t)^{\beta(x)}} \frac{dg(t)}{dt} dt + \frac{g(0^+ - g(0^-)}{\Gamma(1-\beta(x))(x-t)^{\beta(x)}}, \tag{51}$$

where $0 < \beta(x) < 1$.
Kilbas, Saigo and Saxena (2004)introduced the following general fractional derivative defined by (see [57])

$$D_+^\beta g(x) = \frac{d^n}{dx^n} \int_a^x (s-t)^{\mu+n-\nu-1} E_{\beta,\mu+n-\nu}^{\varphi}(\varpi(x-t)^\beta) g(t)dt, \tag{52}$$

where $\beta, \mu, \nu, \varpi \in \mathbb{C}, \mathrm{Re}(\beta) \in (n-1,n], n \in \mathbb{N}$ and $E_{\beta,\mu+n-\nu}^{\varphi}(\varpi(x-t)^\beta)$ is two parameter generalized Mittag-Leffler function with parameters $\alpha, \beta > 0$ defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n\alpha+\beta}.$$

Agrawal (2007) developed the fractional derivative and integral operators in terms of the Riesz fractional derivative, in the form (see [10])

$$I^{\beta}g(x)=\frac{1}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_a^b\frac{g(t)}{|x-t|^{n-\beta}}dt,\quad \mathrm{Re}(\beta)\in(n-1,n],\tag{53}$$

$$D^{\beta}g(x)=\frac{1}{\Gamma(n-\beta)}\int_a^b\frac{1}{|x-t|^{\beta}}\frac{d^ng(t)}{dt^n}dt,\quad \mathrm{Re}(\beta)\in(n-1,n],\tag{54}$$

and

$$I^{\beta}g(x)=\frac{1}{2\Gamma(\beta)}\int_a^b|x-t|^{\beta-1}g(t)dt,\quad 0<\beta<1.\tag{55}$$

Gajda and Magdziarz (2010) introduced the fractional derivative for Fokker Planck equation (FFPE) in the form (see [45])

$$D_{+}^{\beta}g(x)=\frac{d}{dx}\int_0^x M(x-t)g(t)dt,\quad \beta>0.\tag{56}$$

where the memory kernel $M(t)$ is defined via its Laplace transform denoted by $M(p)$, is

$$M(p)=\frac{1}{(p+\xi)^{\beta}+\xi^{\beta}}.\tag{57}$$

Garra et al.(2014) introduced the fractional derivative of the form (see [47])

$$D_{a^+}^{\beta,\varphi,\mu,\omega}g(x)=\int_a^x(t-\mu)^{\mu-1}E_{\beta,\mu}^{\varphi}(\omega(x-t)^{\beta})g^{(n)}(t)dt,\quad \mathrm{Re}(\mu)>0\tag{58}$$

where $\beta,\mu,\varphi,\omega\in\mathbb{C}$, and $\mathrm{Re}(\beta)\in(n-1,n]$, $n\in\mathbb{N}$, $\mathrm{Re}(\mu)>0, g\in AC^n[0,b], 0<t<b<\infty$ and $E_{\beta,\mu}^{\varphi}(\omega(x-t)^{\beta})$ is two generalized Mittag-Leffler function.
Caputo and Fabrizio (2015) introduced the Caputo-Fabrizio derivative in the form (see [25])

$$D_{+}^{\beta}g(x)=\frac{M(\beta)}{(1-\beta)}\int_a^x\exp\left(-\frac{\beta}{(1-\beta)}(x-t)\right)g^{(1)}(t)dt,\tag{59}$$

where $M(\beta)$ is a normalization function such that $M(0)=M(1)=1$, $0<\beta<1, g\in H^1(a,b), b>a, a<x<b$.
Zayernouri, Ainsworth and Karniadakis (2015) proposed the fractional derivatives in the form (see [115])

$$D_{+}^{\beta}g(x)=\frac{e^{\xi x}}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x(x-t)^{-\beta-1}e^{-\xi t}g(t)dt,\quad x>a,\tag{60}$$

and

$$D_{-}^{\beta}g(x)=\frac{e^{\xi x}}{\Gamma(1-\beta)}\frac{d}{dx}\int_x^b(x-t)^{-1-\beta}e^{-\xi t}g(t)dt,\quad x<a,\tag{61}$$

where $0 < \beta < 1, \xi \geq 0$.
Yang, Srivastava and Machado (2015) proposed the fractional derivative with the exponential function by (see [114])

$$D_x^{\beta} g(x) = \frac{(2-\beta)M(\beta)}{2(1-\beta)} \frac{d}{dx} \int_a^x \exp\left(-\frac{\beta}{(1-\beta)}(x-t)\right) g(t)dt. \tag{62}$$

where $0 < \beta < 1, M(\beta)$ is a normalization function such that $M(0) = M(1) = 1$, $g \in H^1(a,b), b > a, a < x < b$.
Sabzikar, Meerschaert and Chen (2015) introduced the fractional integrals and derivatives respectively of the form (see [91, 22, 26])

$$I_{-}^{(\beta)} g(x) = \frac{1}{\Gamma(\beta)} \int_x^{\infty} (x-t)^{\beta-1} e^{-\xi(x-t)} g(t)dt, \tag{63}$$

$$D_{+}^{(\beta)} g(x) = \frac{1}{\Gamma(\beta)} \int_x^{\infty} (x-t)^{\beta-1} e^{-\xi(x-t)} g^{(\beta)}(t)dt, \tag{64}$$

where $0 < \beta < 1, \xi \geq 0$.
Atangana and Baleanu (2016) proposed the Atangana-Baleanu fractional derivative with the Mittag-Leffler function in the form (see [18])

$$D_x^{\beta} g(x) = \frac{M(\beta)}{(1-\beta)} \int_a^x E_{\beta}\left(-\frac{\beta}{(1-\beta)}(x-t)^{\beta}\right) \frac{dg(t)}{dt} dt. \tag{65}$$

where $0 < \beta < 1, M(\beta)$ is a normalization function such that $M(0) = M(1) = 1$, $g \in H^1(a,b), b > a, a < x < b$, and $E_{\beta}\left(-\frac{\beta}{(1-\beta)}(x-t)^{\beta}\right)$ is one generalized Mittag-Leffler function.
Yang (2016) proposed the Yang fractional derivatives and variable fractional order as (see [111])

$$D_x^{\beta} g(x) = \frac{1+\beta^2}{\sqrt{\pi^{\beta}(1-\beta)}} \int_a^x \exp\left(-\frac{\beta}{(1-\beta)}(x-t)^{2\beta}\right) \frac{dg(t)}{dt} dt, \tag{66}$$

where $\beta \geq 0$,$g \in AC[a,b]$ and $a < x < b$.

$$D_x^{\beta} g(x) = \frac{M(\beta(x))}{(1-\beta(x))} \int_a^x \exp\left(-(x-t)^{\beta(x)}\right) \frac{dg(t)}{dt} dt, \tag{67}$$

and

$$D_x^{\beta(x)} g(x) = \frac{1}{\Gamma(1-\beta(x))} \int_a^x E_{\beta(x)}\left(-(x-t)^{\beta(x)}\right) \frac{dg(t)}{dt} dt, \tag{68}$$

where $E_{\beta(x)}\left(-(x-t)^{\beta(x)}\right)$ is the Mittag-Leffler function with one-parameter variable

$0 < \beta(x) < 1$.
Li and Deng (2016) proposed the Li-Deng fractional derivatives in the form (see [62]).

$$D_{+}^{\beta}g(x) = \frac{e^{-\xi x}}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_a^x \frac{g(t)}{(x-t)^{\beta+1}}e^{\xi t}dt, \tag{69}$$

and

$$D_{-}^{\beta}g(x) = \frac{(-1)^n e^{-\xi x}}{\Gamma(n-\beta)}\frac{d^n}{dx^n}\int_x^b \frac{g(t)}{(x-t)^{\beta+1}}e^{\xi t}dt, \tag{70}$$

where $\mathrm{Re}(\beta) \in (n-1, n], n \in \mathbb{N}$ and for any $\xi \geq 0$.
Torres (2017) introduced the Torres fractional derivatives in the form (see [107])

$$D_{+}^{\beta}g(x) = \xi^{\beta}g(x) + \frac{\beta}{\Gamma(1-\beta)}\int_{-\infty}^x \frac{g(x)-g(t)}{(x-t)^{\beta+1}}e^{-\xi(x-t)}dt, \tag{71}$$

and

$$D_{-}^{\beta}g(x) = \xi^{\beta}g(x) + \frac{\beta}{\Gamma(1-\beta)}\int_x^{\infty} \frac{g(x)-g(t)}{(x-t)^{\beta+1}}e^{-\xi(x-t)}dt, \tag{72}$$

where $0 < \beta < 1$ and for any $\xi \geq 0$.
Sun, Hao, Zhang and Baleanu (2017) proposed the fractional derivative in the form (see [104])

$$D_x^{\beta}g(x) = \frac{\Gamma(1+\beta)}{(1-\beta)^{\frac{1}{\beta}}}\int_a^x \exp\left(-\frac{\beta}{1-\beta}(x-t)^{\beta}\right)\frac{dg(t)}{dt}dt, 0 < \beta < 1. \tag{73}$$

Yang and Machado (2017) introduced the Yang-Machado variable-order fractional derivative with the another function by (see [113])

$$D_{a^+}^{\beta(x),\varphi}g(x) = \frac{1}{\Gamma(1-\beta(t))}\int_a^x \frac{g_{\varphi}^{(1)}(t)}{(\varphi(x)-\varphi(t))^{\beta(x)}}dt, \tag{74}$$

where $0 < \beta(x) < 1$, and $g, \varphi \in c^1[a,b], \varphi'^{(1)} \neq 0$.
Dehghan, Abbaszadeh and Deng (2017) presented the fractional derivative in the form (see [?])

$$D_{+}^{\beta}g(x) = \frac{1}{\Gamma(\beta-1)}\int_0^x (x-t)^{\beta+1}e^{-\xi(x-t)}\frac{d^{\beta}g(t)}{d|t|^{\beta}}dt, \tag{75}$$

where $\frac{d^{\beta}g(t)}{d|t|^{\beta}}$ is the Riesz fractional derivative, with $1 < \beta \leq 2, \xi > 0$.
Yang, Machado and Baleanu (2017) proposed the fractional derivatives in the form (see [112])

$$D_{+}^{\beta}g(x) = \int_a^t E_{\beta,\nu}^{\varphi,\phi}(-(t-s)^{\beta})\frac{dg(s)}{ds}ds, \tag{76}$$

where $E^{\varphi;\phi}_{\beta,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\varphi)_{n\phi}}{\Gamma(n\beta+\nu)} \frac{z^n}{n+1}$, with $\varphi, \beta, \varphi, \phi, \nu \in \mathbb{C}$, and $\mathrm{Re}(\beta) > 0$, $\max(0, \mathrm{Re}(\beta)) \in \mathbb{R}^+_0, n \in \mathbb{N}$.

Yang, Gao, Machado and Baleanu (2017) proposed (see [112])

$$D^{\beta}_{x} g(x) = \frac{\beta M(\beta)}{(1-\beta)} \int_a^x \sin c\left(-\frac{\beta(x-t)}{1-\beta}\right) \frac{dg(t)}{dt} dt, \tag{77}$$

where $0 < \beta < 1, M(\beta)$ is a normalization function such that $M(0) = M(1) = 1$, $g \in H^1(a,b), b > a, a < x < b.$,

$$\sin c(x) = \frac{\sin(\pi x)}{\pi x}, x \in \mathbb{R}. \tag{78}$$

Almeida (2017) based on the Liouville-Sonine-Caputo fractional derivative, Almeida defined the Liouville-Sonine-Caputo fractional derivative with respect to another function in the form

$$\begin{aligned} {}_{LSC}D^{\beta}_{a+,h} g(x) &= \left(I^{n-\beta;\varphi}_{a^+}\right) g^{(n)}(x) \\ &= I^{n-\beta;\varphi}_{a^+} \left(\frac{1}{\varphi'(x)\frac{d}{dx}}\right)^n g(x) \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^x \frac{\varphi^{(1)}(s)}{(\varphi(x)-\varphi(s))^{\beta-n+1}} \left(\frac{1}{\varphi^{(1)(s)}} \frac{d}{ds}\right)^n g^n(s) ds, \end{aligned} \tag{79}$$

where $g, \varphi \in C^n(I), \varphi'(x) \neq 0, \beta > 0, n = [\beta] + 1$.

Sousa and de Oliveira (2018) introduced the Sousa-Oliveira fractional derivative by (see [101])

$$^{RL}D^{\beta(t)}_{a^+} g(x) = \frac{M(\beta(t))}{1-\beta(t)} \left(\frac{1}{\psi'(t)} \frac{d}{dx}\right) \int_a^x \psi'(t) \mathbb{H}^{\beta(t);\psi}_{\gamma;\delta}(x,t) g(t) dt, \tag{80}$$

and

$$^{C}D^{\beta(t)}_{a^+} g(x) = \frac{M(\beta(t))}{1-\beta(t)} \int_a^x \psi'(t) \mathbb{H}^{\beta(t);\psi}_{\gamma;\delta}(x,t) g'(t) dt, \tag{81}$$

where

$$\mathbb{H}^{\beta(t);\psi}_{\gamma;\delta}(x,t) = \mathbb{E}\left[\frac{-\beta(t)(\psi(x)-\psi(t))^{\gamma}}{1-\beta(t)}\right],$$

and $0 < \beta(t) < 1$, $0 < \gamma, \delta < 1$, $\mathbb{E}(\cdot)$ is a Mittag-Leffler function, which is considered uniformly convergent on the interval $[a,b] = I$, $M(\beta(t))$ is a normalization function such that $M(0) = M(1) = 1$, and $\psi(\cdot)$ is a positive function and increasing monotone, such that $\psi(t)' \neq 0$.

0.3 Some published books

The evolution and applications of the fractional derivative fractional calculus have been analyzed in several books and survey papers. Therefore, it is important to collect as many up-to-date information as possible today. With the aim of highlighting key documents and events in the field of fractional calculus and the extent of their contribution and coverage of a large variety of applications in the real world from 1974 until the current year 2021.
The first monograph, published a book devoted to fractional calculus in 1974. This collaboration between a chemist (Oldham) and a mathematician (Spanier) in treating problems of mass and heat transfer in terms of the so-called semi-derivatives and semi-integrals, clearly manifested the origin of a new area for FC based both on physical intuition and mathematical versatility. In 1987, the most important book of S. Samko, A. Kilbas and O. Marichev, referred to now as "encyclopedia" of FC, appeared first in Russian, and later with an English edition in [1993].
I would like to refered to some books devoted to fractional calculus (and its applications) from Oldham and Spanier (1974), Samko, Kilbas and Marichev (1987-1993)[92], Miller and Ross (1993)[73], Kiryakova (1994)[59], Podlubny(1999)[82], Sabatier, Agrawal and Machado (2007)[90], Diethelm (2010) [36], Tarasov (2011) [105], Baleanu, Machado and Luo (2011) [36], Machado, Kiryakova and Mainardi(2011) [70], Baleanu, Diethelm, Scalas and Trujillo (2012) [19], Abbas, Benchohra and N'Guérékata (2012)[1], Carpinteri and Mainardi (2014), Gu. Xueke and Fenghui(2015)[49], George A. Anastassiou, Ioannis K. Argyros(2018), Piotr Ostalczyk, Dominik Sankowski, Jacek Nowakowski (2019), Xiao-Jun Yang(2019)[110], G.A. Anastassiou (2021)[15].

0.4 The objective and Motivation

The boundary value problems (BVPs) acquainted by FDE have been broadly concentrated throughout the most recent years. Especially, the investigation of solutions of FDEs is the key and critical subject of applied mathematics research. Many interesting and fascinating results have been considered with respect to the existence, uniqueness, and stability of solutions via some fixed point theorems [3, 4, 5]. the generalized fractional calculus has played an important role in modeling the complex phenomena with the power law behaviors in mathematical physics and engineering. For the details of the history of the generalized fractional calculus, readers refer to the results [58, 59, 70, 73, 79, 88].
In any case, the majority of the considered problems have been treated in the frame of FDs of Riemann-Liouville, or Caputo types . In order to enrich the work on

fractional BVPs involving generalized FD and generalized FI boundary conditions further, we study the existence and uniqueness of solutions for the generalized fractional differential equations.

In the context of this study, we have organized this thesis as follows:

Chapter 1, contains fundamental concepts of nonlinear analysis, generalized fractional calculus such as ψ-Riemenn-Liouville fractional integrals, ψ-Caputo fractional derivative, Katugampola fractional calculus and Riesz-Caputo fractional calculus. We also describe a number of fixed-point theorems used to establish the existence results for the proposed problems. Included among the fixed-point theorems recognized by their names are Banach's contraction principal, Boyd and Wong, Krasnoselskii's, Schiefer, Dhage, Leray-Schauder nonlinear alternative.

In Chapter 2, is devoted to study the existence and uniqueness of solutions for a nonlinear neutral ψ-Caputo type FDE with ψ-Riemann-Liouville FI boundary conditions of the form:

$$\begin{cases} {}^{C}D_{0^+}^{\xi;\psi}\left[{}^{C}D_{0^+}^{\zeta;\psi}\varkappa(\tau) - \mathcal{Q}(\tau,\varkappa(\tau))\right] = \mathcal{F}(\tau,\varkappa(\tau)), \quad \tau \in \mathrm{J} := [0,T], \\ \\ \varkappa(\chi) = 0, \quad I_{0^+}^{\gamma;\psi}\varkappa(T) = 0, \quad \chi \in (0,T), \end{cases} \tag{82}$$

where ${}^{C}D_{a^+}^{\sigma;\psi}$ is the ψ-Caputo fractional derivative of order $\sigma \in \{\xi,\zeta\} \subseteq (0,1]$, $I_{0^+}^{\gamma;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\gamma > 0$, and $\mathcal{F},\mathcal{Q} : \mathrm{J}\times\mathbb{R}\to\mathbb{R}$ are given functions.

The objective of Chapter 3 is to investigate the existence of solution for the following Katugampola fractional differential equation equipped with Erdélyi-Kober fractional integral boundary conditions of the form:

$$\begin{cases} D^{\rho,\alpha}u(t) + h(t,u(t)) = 0, \quad 0<t<T, \\ u(0) = 0, \\ u'(T) = \lambda I_{\eta}^{\gamma,\delta}u'(\xi), \qquad 0<\xi<T, \end{cases} \tag{83}$$

where $1 < \alpha < 2$, $\rho > 0$, $\delta > 0$, $\eta > 0$, λ, $\gamma \in \mathbb{R}$, and $h : [0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We establish some existence and uniqueness results for the given problems by means of classical fixed-point theorems.Finally, the main result is strengthened the through examples.

In Chapter 4, we examine existence and uniqueness of solutions for nonlinear Langevin equation involving Riesz-Caputo fractional derivatives, with a class of anti-periodic boundary conditions of the form:

$$\begin{cases} {}_0^{RC}D_T^{\alpha}({}_0^{RC}D_T^{\beta}+\chi)x(t)=f(t,x(t)) \text{ ,} 0<t<T, \\ x(0)+x(T)=0 \text{ , } x'(0)+x'(T)=0, \end{cases} \tag{84}$$

where ${}^{RC}D^{\alpha}$ and ${}^{RC}D^{\beta}$ are the Riesz-Caputo fractional derivatives of order $1<\alpha\leq 2$ and $0<\beta\leq 1$, $\chi\in\mathbb{R}$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function with respected to its both variables, t and x.

Our results obtained by using a variety of fixed point theorems as Banach, Schaefer and Krasnoselskii's fixed point theorems. Three examples are given to illustrate main results.

In Chapter 5, is concerned with the existence of solutions for ψ-Caputo hybrid fractional integro-differential equations of the form

$$\begin{cases} {}^{c}D_{a^+}^{\nu;\psi}\left[\frac{z(\tau)-\sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,z(\tau))}{\mathbb{G}(\tau,z(\tau))}\right]=\mathbb{H}(\tau,z(\tau)),\tau\in \mathrm{J}=[a,b], \\ z(a)=0, \end{cases} \tag{85}$$

where ${}^{c}\mathbb{D}_{a^+}^{\nu;\psi}$ is the ψ-Caputo fractional derivative of order $\nu\in(0,1]$, $\mathbb{I}_{a^+}^{\theta;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\theta>0,\theta\in\{\sigma_1,\sigma_2,\ldots,\sigma_m\},\sigma_k>0,k=1,2,\ldots,m$. $\mathbb{G}\in C(\mathrm{J}\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $\mathbb{F}_k,\mathbb{H}\in C(\mathrm{J}\times\mathbb{R},\mathbb{R})$,$(k=1,2,\ldots,m)$.

Existence and uniqueness results for the given problems are obtained with use an hybrid fixed point theorem for a sum of three operators due to Dhage for proving the main results. Also, the main result is strengthened an example. Finally, general conclusion and future research are given.

Chapter 1

Preliminaries

In this chapter we introduce preliminary facts that will be used in the remainder of this thesis. We give here the essential notions of functional space, the concepts of the fractional derivatives and fractional integrals, and the generalized fractional calculus. Methods of nonlinear analysis for boundary values problem will be discussed. Also we give some theorems of fixed point, that will need in our work.
A first we introduce the following function spaces.

1.1 Functional Space

1.1.1 Space of Continuous Functions

Let $J = [a, b](-\infty < a < b < +\infty)$ be a finite closed interval of the real axis $\mathbb{R} = (-\infty, +\infty)$.

Definition 1. *Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions $f : J \to \mathbb{R}$, equipped with the norm*

$$\|f\|_{\infty} = \sup_{t\in J} |f(t)|.$$

Analogously, $C^n(J, \mathbb{R})$ is the Banach space of functions $f : J \to \mathbb{R}$, where f is n time continuously differentiable on J.

$$\|f\|_{C^n} = \sum_{k=0}^{n} \|f^k\|_C = \sum_{k=0}^{n} \max_{t\in J} \|f^k(t)\|, \quad n \in \mathbb{N}.$$

1.1.2 L^p Spaces

Definition 2. *Denote by $L^1(J,\mathbb{R})$ the Banach space of functions f Lebesgues integrable with the norm*

$$\|f\|_{L^1} = \int_a^b |f(t)|dt.$$

We denote by $L^p(J,\mathbb{R})$ the space of Lebesgue complex-valued measurable functions f on J for which $\|f\|_{L^p} < 1$, endowed with the norme (see

$$\|f\|_{L^p} = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}, (1 < p < \infty).$$

In particular, if $p = \infty$, we denote by $L^\infty(J,\mathbb{R})$ the space of all functions u that are essentially bounded on J with essential supremum

$$\|f\|_{L^\infty} = ess\sup_{t\in J} |f(t)| = \inf\{c > 0 : |f(t)| \le c \quad for\ a.e.t\}.$$

1.1.3 $X_c^p(a,b)$ Space

Definition 3. *We denote by $X_c^p(J,\mathbb{R})$, $(c \in \mathbb{R}, 1 \le p \le \infty)$ the space of Lebesgue complex-valued measurable functions f on J for which $\|f\|_{L^p} < 1$, endowed with the norme (see*

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < \infty.$$

In particular, if $c \in \mathbb{R}, p = \infty$, we have

$$\|f\|_{X_c^\infty} = ess\sup_{t\in J} |t^c f(t)| = \inf\{c > 0 : |t^c f(t)| \le c \quad for\ a.e.t\}.$$

In particular, when $c = 1/p$, the space $X_c^p(a,b)$ coincides with the $L^p(a,b)$-space: $X_{1/p}^p = L^p(a,b)$.

Remark 4. *[17] Let $p, c, T \in \mathbb{R}_+^\star$ be such that $p \ge 1$, $c > 0$ and $T \le (pc)^{\frac{1}{pc}}$. One can easily see that $\forall f \in C[0,T]$*

$$\|f\|_{X_c^p} = \left(\int_0^T |s^c f(s)|^p \frac{ds}{s}\right)^{\frac{1}{P}} \le \left(\|f\|_c^p \int_0^T s^{pc-1} ds\right)^{\frac{1}{P}} = \frac{T^C}{(pc)^{\frac{1}{p}}}\|f\|_C,$$

and if $p = \infty$

$$\|f\|_{X_c^p} = ess\sup_{0\le t\le T} (t^c|f(t)|) \le T^c\|f\|_C,$$

which implies that $C[0,T] \hookrightarrow X_c^p[0,T]$, *and* $\|f\|_{X_c^p} \leq \|f\|_C$ *for all* $T \leq (pc)\frac{1}{pc}$.

1.1.4 Ths space of functions absouletly continuous $AC^n[a,b]$

Definition 5. *We denote by* $AC^n([a,b],\mathbb{R})$ *the space of real-valued functions* $f(t)$ *which have continuous derivatives up to order* $n-1$ *on* $[a,b]$ *such that* $f^{(n-1)} \in AC([a,b])$*(see [58, 92])*

$$AC^n([a,b]) = \left\{f : [a,b] \to \mathbb{R}, f^k \in C[a,b], k = 0...n-1, f^{(n-1)} \in AC([a,b])\right\}.$$

In particular, $AC^1[a,b] = AC[a,b]$.

A characterization of the functions of this space is given by the following.
In 1968, Kolmogorov and Fomin observed that the space $AC(\Omega)$ is in agreement with the space of the primitives of the Lebesgue summable functions (see [58, 92]):

$$f \in AC([a,b]) \Leftrightarrow \exists \phi \in L^1([a,b]) \text{ such that } f(t) = c + \int_a^t \phi(t),$$

where $\phi(t)$ is called the Kolmogorov-Fomin condition.
We remark that, if an absolutely continuous function $f(t)$ has a derivative $f^{(1)}(t) = \phi(t)$ almost everywhere on $[a,b]$, then there are $c = f(a)$ and $f(t) \in AC([a,b])$.

Lemma 6. *A function* $f(t) \in AC^n([a,b]), n \in \mathbb{N}^*$, *if and only if it is represented of the form*

$$f(t) = \frac{1}{(n-1)!}\int_a^t (t-u)^{n-1} f^{(n)}(u)du + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

1.1.5 Opérateurs compacts

Definition 7. *An operator* $T : E \to E$ *is called compact if the image of each bounded set* $\Omega \subset E$ *is relatively compact i.e (* $\overline{T(\Omega)}$ *is compact).* T *is called completely continuous operator if it is continuous and compact.*

1.1.6 The criteria for compactness for sets in the space of continuous functions $C([a,b])$

Theorem 8. *(Arzela-Ascoli theorem). A set* $\Omega \subset C([a,b],\mathbb{R})$ *is relatively compact in* $C([a;b],\mathbb{R})$ *if and only if the functions in* Ω *are uniformly bounded and equicontinuous on* $[a,b]$.

We recall that a family of continuous functions is uniformly bounded if there exists $M > 0$ such that

$$\|f\| = \max_{x\in[a,b]} |f(x)| \leq M, \quad f \in \Omega.$$

The family Ω is equicontinuous on $[a;b]$, if $\forall\epsilon > 0, \exists\delta > 0$ such that $\forall t_1, t_2 \in [a,b]$ and $\forall f \in \Omega$, we have

$$|t_1 - t_2| \leq \delta \Rightarrow |f(t_1) - f(t_2)| < \epsilon.$$

1.2 Basic fractional calculus

In this section we give the definitions and some properties of fractional integrals and fractional derivatives of different kinds, such as Riemann-Liouville, Caputo, Liouville, Riesz-Caputo, Katugampola, Erdély-kober, $\psi - Riemann - Liouville$.

1.2.1 The contribution for generalized fractional calculus and applications

The fractional calculus has found the important applications in fields of mathematics, science and applied engineering, including fluid flow, heat transfer, rheology, electrical circuit, networks, electromagnetic theory, control theory and probability, numerical analysis, economics and finance, engineering, physics, biology, , image denoising, cryptography, controls, etc. for instance, phenomena of physics [108], Applications in Engineering, Life and Social Sciences [20], Systems Decision and Control [15], Control and Optimization [23], Viscoelasticity [44], Financial Economics [42]. For the details of the history of the generalized fractional calculus, readers refer to the results [58, 59, 70, 73, 79, 88].

1.2.2 Riemann-Liouville fractional integrals

Definition 9. *The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ of a function $f \in L^1([a,b])$ are defined as (see [58, 79, 82, 88, 92]).*

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{ds}, \quad (t > a,\ \alpha > 0). \tag{1.1}$$

and

$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\mathrm{ds}, \quad (t < b,\ \alpha > 0). \tag{1.2}$$

where $\Gamma(\alpha)$ *is gamma function (see[58, 79]).*

Moreover, for $\alpha = 0$*, we set* $I^0_{a^+}f := f$*. That the Riemann-Liouville fractional integral coincides with the classical definition of* I^n_a *in the case* $n \in \mathbb{N}$*.*

Lemma 10 ([58])**.** *The following basic properties of the Riemann-Liouville integrals hold:*

1. *The integral operator* $I^{\alpha}_{a^+}$ *is linear;*
2. *The semigroup property of the fractional integration operator* $I^{\alpha}_{a^+}$ *is given by the following result*

$$I^{\alpha}_{a^+}(I^{\beta}_{a^+}f(t)) = I^{\alpha+\beta}_{a^+}f(t), \quad \alpha, \beta > 0, \tag{1.3}$$

holds at every point if $f \in C([a, b])$ *and holds almost everywhere if* $f \in L^1([a, b])$*,*

3. *Commutativity*

$$I^{\alpha}_{a^+}(I^{\beta}_{a^+}f(t)) = I^{\beta}_{a^+}(I^{\alpha}_{a^+}f(t)), \quad \alpha, \beta > 0; \tag{1.4}$$

4. *The fractional integration operator* $I^{\alpha}_{a^+}$ *is bounded in* $L^p[a, b]$ $(1 \le p \le \infty)$*;*

$$\|I^{\alpha}_{a^+}f\|_{L^p} \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p}. \tag{1.5}$$

1.2.3 Riemann-Liouville fractional derivatives

Definition 11. *The left-sided and the right-sided Riemann- Liouville fractional integrals of order* $\alpha \in \mathbb{C}$*,* $Re(\alpha) > 0$*, of a continuous function* $f : (0, \infty) \to \mathbb{R}$ *are defined as (see [79, 82, 88, 92]).*

$${}^{RL}D^{\alpha}_{a^+}f(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds,$$

and

$${}^{RL}D^{\alpha}_{b^-}f(t) = \left(-\frac{d}{dt}\right)^n (I^{n-\alpha}_{b^-}f)(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(s-t)^{\alpha-n+1}}ds,$$

where $Re(\alpha) \in (n-1, n], n \in \mathbb{N}$*.*

1.2.4 Caputo type fractional derivative

The Caputo (1967) fractional derivatives are closely related to the Riemann-Liouville derivatives.

Definition 12 ([58]). *For a function $f \in AC^n([a,b])$, the left sided and right sided Caputo fractional derivatives of order α are defined by (see [79, 82, 88, 92]).*

$$\begin{aligned} {}^cD^{\alpha}_{a^+} f(t) &= I^{n-\alpha}_{a^+} D^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} {}^cD^{\alpha}_{b^-} f(t) &= (-1)^n I^{n-\alpha}_{b^-} D^n f(t) \\ &= \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds, \end{aligned} \tag{1.7}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α. In particular, when $0 < \alpha < 1$ and $f(t) \in AC[a,b]$,

$${}^cD^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s) ds = I^{1-\alpha}_{a^+} D f(t), \tag{1.8}$$

and

$${}^cD^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b (s-t)^{-\alpha} f'(s) ds = -I^{1-\alpha}_{b^-} D f(t). \tag{1.9}$$

In the following, we give the proprieties of Caputo fractional derivative:

Lemma 13 ([58]). *The following basic properties of the Caputo fractional derivative hold:*

1. *The Caputo fractional derivative is linear.*

2. *Let $\alpha > 0$ and let $f(t) \in L_\infty$ or $f(t) \in C[a,b]$, if $\alpha \notin \mathbb{N}$*

$${}^CD^{\alpha}_{a^+} I^{\alpha}_{a^+} f(t) = f(t) \quad \textit{and} \quad {}^CD^{\alpha}_{b^-} I^{\alpha}_{b^-} f(t) = f(t). \tag{1.10}$$

3. *Let $\alpha > \beta > 0$, and $f \in L^1([a,b])$. Then we have:*

$${}^cD^{\beta}_{a^+} I^{\alpha}_{a^+} f(t) = I^{\alpha-\beta}_{a^+} f(t). \tag{1.11}$$

4. *If $f(t) \in AC^n[a,b]$ or $f(t) \in C^n[a,b]$, then*

$$I^{\alpha}_{a^+}\,{}^{C}D^{\alpha}_{a^+}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(t-a)^k, \tag{1.12}$$

and

$$I^{\alpha}_{b^-}\,{}^{C}D^{\alpha}_{b^-}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(-1)^k f^k(b)}{k!}(b-t)^k. \tag{1.13}$$

5. *In particular, if $0 < \alpha \leq 1$ and $f(t) \in AC[a,b]$ or $f(t) \in C[a,b]$, then*

$$I^{\alpha}_{a^+}\,{}^{C}D^{\alpha}_{a^+}f(t) = f(t) - f(a), \quad \textit{and} \quad I^{\alpha}_{b^-}\,{}^{C}D^{\alpha}_{b^-}f(t) = f(t) - f(b). \tag{1.14}$$

Example 14 ([58, 92]). *The Caputo derivative of the power function $(t-a)^{\beta-1}, \alpha > 0, \beta > 0$, $n = [\alpha]+1$, then the following relation hold*

$$ {}^{c}D^{\alpha}_{a^+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad (\beta > n), \tag{1.15}$$

$$ {}^{c}D^{\alpha}_{b^-}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}, \quad (\beta > n), \tag{1.16}$$

In particular,

$$ {}^{C}D^{\alpha}_{a^+}C = 0 \quad \textit{and} \quad {}^{C}D^{\alpha}_{b^-}C = 0. \tag{1.17}$$

The relation between the derivative of Caputo and that of Riemann-Liouville is given by following remark

Remark 15. *We note that if $f \in AC^n([a,b])$, then*

$$ {}^{RL}D^{\alpha}_{a^+}f(t) = {}^{c}D^{\alpha}_{a^+}f(t) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha}. \tag{1.18}$$

Clearly, we see that if $f^{(k)}(a) = 0$, for $k = 0, 1, \ldots, n-1$ then we have

$$ {}^{c}D^{\alpha}_{a^+}f(t) = {}^{RL}D^{\alpha}_{a^+}f(t). \tag{1.19}$$

1.2.5 Riesz-Caputo type fractional derivatives

In this section we present the definitions and some properties of the Riesz-Caputo type fractional integrals and fractional derivatives.

Definition 16. *([43, 58]) Riesz-Caputo derivative of order α, of a function $f \in$*

$C^n([0,T])$ is defined by

$$
\begin{aligned}
{}_0^{RC}D_T^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^T |t-u|^{n-\alpha-1} f^{(n)}(u)du \qquad (1.20)\\
&= \frac{1}{2}\left({}^{C}D_{0^+}^{\alpha} + (-1)^n\, {}^{C}D_T^{\alpha}\right) f(t),
\end{aligned}
$$

where ${}^{C}D_{0^+}^{\alpha}$ is the left Caputo derivative and ${}^{C}D_T^{\alpha}$ is right Caputo derivative.

Remark 17. *In particular if $f \in C^2([0,T])$ and $0 < \alpha \leq 1$, then*

$$
{}_0^{RC}D_T^{\alpha} f(t) = \frac{1}{2}\left({}^{C}D_{0^+}^{\alpha} - {}^{C}D_T^{\alpha}\right) f(t). \qquad (1.21)
$$

If $f \in C^2([0,1])$ and if $1 < \alpha \leq 2$, then

$$
{}_0^{RC}D_T^{\alpha} f(t) = \frac{1}{2}\left({}^{C}D_{0^+}^{\alpha} + {}^{C}D_T^{\alpha}\right) f(t). \qquad (1.22)
$$

Lemma 18. *([43, 58]) If $f(t) \in C^n([0,T])$,Then*

$$
I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)(t-0)^k}{k!}(t-0)^k, \qquad (1.23)
$$

and

$$
I_T^{\alpha}\, {}^{C}D_T^{\alpha} f(t) = (-1)^n \left[f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(T)}{k!}(T-t)^k \right]. \qquad (1.24)
$$

From the above definitions and lemmas, we have

$$
\begin{aligned}
{}_0I_T^{\alpha}\, {}_0^{RC}D_T^{\alpha} f(t) &= \frac{1}{2}\left(I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} + I_T^{\alpha}\, {}^{C}D_{0^+}^{\alpha}\right) f(t) \qquad (1.25)\\
&+ \frac{(-1)^n}{2}\left(I_{0^+}^{\alpha}\, {}^{C}D_T^{\alpha} + I_T^{\alpha}\, {}^{C}D_T^{\alpha}\right) f(t)\\
&= \frac{1}{2}\left(I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} + (-1)^n\, I_T^{\alpha}\, {}^{C}D_T^{\alpha}\right) f(t).
\end{aligned}
$$

In particular if $1 < \alpha \leq 2$ and $f \in C^2([0,T])$, then

$$
{}_0I_T^{\alpha}\, {}_0^{RC}D_T^{\alpha} f(t) = f(t) - \frac{1}{2}(f(0) + f(T)) - \frac{1}{2}f'(0)t + \frac{1}{2}f'(T)(T-t). \qquad (1.26)
$$

1.3 Generalized fractional integrals

In this section we present the definitions and some properties of the generalized type fractional integrals such as Katugampola fractional integrals, Erdélyi-kober and ψ-

Riemann-liouville fractionl integrals, ψ-Caputo derivative.
In the first we present the definitions and properties of Katugampola fractional integrals introduced by Katugampola in 2011.

1.3.1 The Katugampola fractional integral

Definition 19. *[76] The Katugampola fractional integral of order $\alpha > 0$ and $\rho > 0$ of a function $f(t)$ for all $0 < t < \infty$, is defined by*

$$I_{0^+}^{\rho,\alpha} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds, \quad t \in [0,T], \tag{1.27}$$

for $\rho > 0$. This integral is called left-sided integral.

Lemma 20. *[76] Let be the constants $\rho, q > 0$ and $p > 0$. Then the following formula holds:*

$$I^{\rho,q} t^p = \frac{\Gamma(\frac{p+\rho}{\rho})}{\Gamma(\frac{p+\rho q+\rho}{\rho})} \frac{t^{p+\rho q}}{\rho^q}. \tag{1.28}$$

Remark 21. *[76] The above definition (19) of Katugampola fractional integral corresponds to the Riemann-Liouville fractional integral of order $\alpha > 0$, when $\rho = 1$, while the famous Hadamard fractional integral follows for $\rho \to 0$; that is:*

$$\lim_{\rho \to 0} I_{0^+}^{\rho,\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds. \tag{1.29}$$

1.3.2 The Erdélyi-Kober fractional integrals

The conceptions of the Erdélyi-Kober type operators of fractional integration, as the extensions of the Riemann-Liouville left-sided and right-sided fractional integrals, are given as follows ([58]).

Definition 22. *[58, 92] The Erdélyi-Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$, of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by :*

$$I_\eta^{\gamma,\delta} h(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1}}{(t^\eta - s^\eta)^{1-\delta}} h(s)ds, \tag{1.30}$$

provided that the right-side is pointwise defined on $\mathbb{R}^+$.

Remark 23. *[58, 92] For $\eta = 1$ the above operator is reduced to the Kober operator*

$$I_1^{\gamma,\delta} h(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma}{(t-s)^{1-\delta}} h(s)ds.$$

That was introduced for the first time by Kober. For $\gamma = 0$, the Kober operator is reduced to the Riemann-Liouville fractional integral with a power weight

$$I_1^{0,\delta} h(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{h(s)}{(t-s)^{1-\delta}} ds, \quad \delta > 0. \tag{1.31}$$

Lemma 24. *[58, 92] Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then we have*

$$I_\eta^{\gamma,\delta} t^q = \frac{t^q \Gamma(\gamma + (\frac{q}{\eta}) + 1)}{\Gamma(\gamma + (\frac{q}{\eta}) + \delta + 1)}. \tag{1.32}$$

Theorem 25. *The operator $J_{a^+}^{\rho,\alpha}$ is linear and bounded from $C([a,b])$ to $C([a,b])$, then*

$$\|J_{a^+}^{\rho,\alpha} x\|_C \leq K_{\alpha,\rho} \|x\|_C, \tag{1.33}$$

with $K_{\alpha,\rho} = \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}(b^\rho - a^\rho)^\alpha$.

Proof. For any $x \in C[0,T]$; one has

$$\begin{aligned}\left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} x(s) ds\right| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \|x\|_C \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds, \\ &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} (b^\rho - a^\rho)^\alpha \|x\|_C.\end{aligned}$$

□

1.3.3 ψ-Riemann-Liouville type fractional integrals

In this section we present the definitions and some properties of the ψ-Riemann-Liouville type fractional integrals introduce by Almeida.

Definition 26 ([13, 15]). *For $\alpha > 0$, the left-sided ψ–Riemann-Liouville fractional integral of order α for an integrable function $f\colon [a,b] \longrightarrow \mathbb{R}$ with respect to another function $\psi\colon [a,b] \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in$ J is defined as follows*

$$I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) \mathrm{ds}, \tag{1.34}$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function (see [58, 79]).
The following semigroup property is valid for fractional integrals: if $\alpha, \beta > 0$, then

$$I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(t) = I_{a^+}^{\alpha+\beta;\psi}. \tag{1.35}$$

1.4 Generalized fractional derivatives

1.4.1 The generalized Katugampola fractional derivative

In this part we give the definitions of Katugampola fractional derivative, introduced by Katugampola in 2014.

Definition 27. *[76] The generalized fractional derivative of order $\alpha > 0$ corresponding to the Katugampola fractional integral is defined for any $0 < t < \infty$ by:*

$$\begin{aligned} D_{0^+}^{\rho,\alpha} f(t) &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \left(J_{0^+}^{\rho,n-\alpha} f\right)(t), \qquad (1.36)\\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{\alpha-n+1}} f(s)ds, \quad t \in [0,T], \end{aligned}$$

where $n = [\alpha] + 1$ and $\rho > 0$ (when the integral exists).

Remark 28. *[76] As a basic example, we quote for $\alpha, \rho > 0$ and $\mu > -\rho$*

$$D_{0^+}^{\rho,\alpha} t^\mu = \frac{\rho^{\alpha-1}\Gamma(1+\frac{\mu}{\rho})}{\Gamma(1-\alpha+\frac{\mu}{\rho})} t^{\mu-\alpha\rho}. \qquad (1.37)$$

1.4.2 The generalized ψ-Riemenn-Liouville fractional derivative

Definition 29 ([13, 15])**.** *Let $n \in \mathbb{N}$ and let $\psi, f \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in$ J. The left-sided ψ–Riemann–Liouville fractional derivative of a function f of order α is defined by*

$$\begin{aligned} D_{a^+}^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f(s)\mathrm{ds}, \end{aligned}$$

where $n = [\alpha] + 1$.

1.4.3 The generalized ψ-Caputo fractional derivative

In this section we present the definitions and some properties of the ψ-Caputo type fractional derivarive introduce by Almeida in 2017.

Definition 30 ([13])**.** *Let $n \in \mathbb{N}$ and let $\psi, f \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in$ J. The left-sided ψ-Caputo fractional* *derivative of f of order α is defined by*

$$^{c}D_{a^+}^{\alpha;\psi}f(t) \quad = \quad I_{a^+}^{n-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t),$$

where $n=[\alpha]+1$ for $\alpha \notin \mathbb{N}$, $n=\alpha$ for $\alpha \in \mathbb{N}$.
To simplify notation, we will use the abbreviated symbol

$$f_{\psi}^{[n]}(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t).$$

From the definition, it is clear that

$$^{c}D_{0^+}^{\alpha;\psi}f(t)=\begin{cases}\int_0^t \frac{\psi'(\psi(t)-\psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)}f_{\psi}^{[n]}(s)ds, if \alpha \notin \mathbb{N},\\ f_{\psi}^{[n]}(t), if \alpha \in \mathbb{N}.\end{cases} \tag{1.38}$$

We note that if $f \in C^n(\mathrm{J},\mathbb{R})$ the ψ- Caputo fractional derivative of order α of f is determined as

$$^{c}D_{0^+}^{\alpha;\psi}f(t)=D_{0^+}^{\alpha;\psi}\left[f(t)-\sum_{k=0}^{n-1}\frac{f_{\psi}^{[n]}(0)}{k!}(\psi(t)-\psi(0))^k\right].$$

Lemma 31 ([13])**.** . *Let $\alpha,\beta>0$,and $f \in L^1(\mathrm{J},\mathbb{R})$. Then*

$$I_{0^+}^{\alpha;\psi}I_{0^+}^{\beta;\psi}f(t)=I_{0^+}^{\alpha+\beta;\psi}f(t), a.e.t \in \mathrm{J}.$$

In particular, if $f \in C(\mathrm{J},\mathbb{R})$, then $I_{0^+}^{\alpha;\psi}I_{0^+}^{\beta;\psi}f(t)=I_{0^+}^{\alpha+\beta;\psi}f(t), t \in \mathrm{J}$.

Lemma 32 ([13])**.** . *Let $\alpha>0$,The following holds:*
If $f \in C(\mathrm{J},\mathbb{R})$, then

$$^{c}D_{0^+}^{\alpha;\psi}I_{0^+}^{\alpha;\psi}f(t)=f(t), t \in \mathrm{J}.$$

If $f \in C^n(\mathrm{J},\mathbb{R}), n-1<\alpha<n$. Then

$$I_{0^+}^{\alpha;\psi c}D_{0^+}^{\alpha;\psi}=f(t)-\sum_{k=0}^{n-1}\frac{f_{\psi}^{[n]}(0)}{k!}\left[\psi(t)-\psi(0)\right]^k, t \in \mathrm{J}.$$

Lemma 33 ([13])**.** *Let $\alpha,\beta>0$, and $f \in C([a,b],\mathbb{R})$. Then for each $t \in \mathrm{J}$ we have*

1. $^{c}D_{a^+}^{\alpha;\psi}I_{a^+}^{\alpha;\psi}f(t)=f(t)$,
2. $I_{a^+}^{\alpha;\psi c}D_{a^+}^{\alpha;\psi}f(t)=f(t)-f(a), \quad 0<\alpha\le 1$,
3. $I_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(a))^{\beta+\alpha-1}$,

4. ${}^{c}D_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\beta-\alpha-1}$,

5. ${}^{c}D_{a^+}^{\alpha;\psi}(\psi(t)-\psi(a))^{k} = 0,$ *for all* $k \in \{0,\ldots,n-1\}, n \in \mathbb{N}$.

1.5 Some Fixed point theorems and their applications

In fractional differential equations, it would be necessary to introduce the distinction between quantitative and qualitative information in finding the solution, if all fractional differential equations could be solved as easily, it would be unnecessary to introduce the distinction between quantitative and qualitative information concerning solutions. Most fractional differential equations, especially nonlinear equations, must be studied with one technique to obtain quantitative information (using numerical analysis), and by another technique to obtain qualitative information (nonlinear analysis).

Currently, the numerical analysis of fractional differential equations is an active field of research. Various numerical methods have been developed to solve nonlinear fractional differential equations, such as, Adomian Decomposition Method [6], New Iterative Method [33], predictorcorrector approach [37], Homotopy perturbation method, for detail see [35, 66, 14, 16, 19].

1.5.1 Nonlinear Analysis and fixed point theory

Nonlinear Analysis is a very broad subject a useful in the study of boundary value problems. The fundamental methods of nonlinear analysis and their efficient application to nonlinear boundary value problems for fractional differential equations such as, nonlinear operators (classes of nonlinear operators: compact, maximal monotone, pseudomonotone, generalized pseudomonotone), no smooth analysis, fixed point theory (Banach's fixed point theory, was created and demonstrated in the year 1922 by Stefan Banach (1892-1945), it guarantees the existence and uniqueness of solution), degree theory (presents degree theories: Brouwer's degree (1912), Leray-Schauder degree (1934) and degree of set-valued maps), variational principles and critical point theory, Morse theory, bifurcation theory, regularity theorems and maximum principles, and a spectrum of differential operators for detail see [75].

In the following we give a description to use and the application of certain theorems of fixed point and topological methods (topological degree) in the study existence of solution for nonlinear fractional differential and integral equations.

1.5.2 Application of fixed point theory

The most original and far-reaching the contributions made by Henri Poincaré to mathematics was his introduction of the use of topological or "qualitative" methods in the study of nonlinear problems in analysis [24].

Fixed point theory is important tool for solving problems arising in various branches of mathematical analysis such as in the existence of the theory of FDE and PDE, integral equations and inclusions, nonlinaer matrix equations, stochastic fractional differential equations, equilibrium problems, variational inequality problems, these problems can be solved by reducing them to an equivalent fixed point problem. As example for applications of the fixed point theory in several areas such as Optimal control theory, the approximation methods, economics to stochastic game theory [80, 81]

1.5.3 Classification of fixed point theory

There is a rough classification of fixed point theorems into three basic classes:

- (a) Metric fixed point theory.
- (b) Topological fixed point theory.
- (c) Order fixed point theorems,

Metric Fixed Point Theory

We include all characteristics geometric of spaces and/or the maps, with use of metric structures, including (Banach's fixed point theory, Boyd and Wong, Scheafer, Krasnoselskii's, Shauder,...).

Topological fixed point theory

These theories are fundamentally based on the topological structure of space. The first work given by Brouwer's (1912), in the case of infinite dimensional subsets of some function spaces. Brouwer's-Schauder (1934), extended Brouwer's theorem to the case the space is compact and convex subsets of a normed linear space, this theorem was extended to locally convex topological vector space by Tychonoff (1935).

Order fixed point theorems

This class belong all those fixed point results which exploit the order structure induced by a cone. Of course this classification is not strict and there are no clear boundaries separating the three classes see [78, 81].

When to apply fixed point and topological techniques No exact answer to the question however

- Topological methods describe qualitative information such as the upper and lower bounds of the solution values.
- The fixed-point theorem and local topological degree are closely connected. They were developing by Leray, schauder, Nirenberg, Cesari and others.
- We can prove the fixed-point theorem without using any "topological machinery".
- The topological degree it has an important advantage over the fixed-point theorem: it gives information about the number of distinct solutions, continuous families of solutions, and stability of solutions [27]. The concept of degree of mapping in all these forms is one of the most effective tools for studying the properties of existence and multiplicity of solutions of nonlinear equations.
- In the finite-dimensional case, we use the classical topological degree as they were explicitly formulated by Brouwer in 1912. In infinite dimensional its extension by Leray and Schauder [65] in 1934 to mappings in infinite-dimensional Banach spaces of the form $I - g$, with g compact.
- The fixed point and the topological methods should be regarded as a last resort or at least a later resort than analytical methods.

1.6 The Classical Fixed points theory

Fixed point theorems are the basic mathematical tools that help establish the existence of solutions of various kinds of equations. The fixed point method consists of transforming a given problem into a fixed point problem. The fixed points of the transformed problem are thus the solutions of the given problem. In this section, we recall the famous fixed point theorems that we will use to obtain varied existence results. We start with the definition of a fixed point theorems which are used throughout this thesis.

Definition 34. *Let f be an application of a set E in itself. We call fixed point of f any point $u \in E$ such that*

$$f(u) = u.$$

1.6.1 Banach's contraction principle

Banach's contraction principle, which guarantees the existence of a single fixed point of a contraction of a complete metric space with values in itself, is certainly the best known of the fixed point theorems. This theorem proved in 1922 by Stefan Banach is based essentially on the notions of Lipschitzian application and of contracting application.

Theorem 35. *[80, 46](Banach contraction principle)*
Let E be a complete metric space and let $T : E \to E$ be a contracting application, then T has a unique fixed point.

1.6.2 Schaefer's Fixed-Point Theorem

Lemma 36. *[80, 81] Let E be a Banach space. Assume that $T : E \to E$ is completely continuous operator and the set*

$$\Omega = \{x \in E : x = \mu T x, 0 < \mu < 1\},$$

is bounded. Then T has a fixed point in E.

1.6.3 Leray-Schauder Nonlinear Alternative

Theorem 37. *[46] (Leray-Schauder nonlinear alternative) Let K be a convex subset of a Banach space E, and let U be an open subset of K with $0 \in U$. Then every completely continuous map $N : \overline{U} \to K$ has at least one of the following two properties:*

1. *N has a fixed point in $\overline{U}$;*

2. *there is an $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = \lambda N x$.*

1.6.4 Boyd-Wong Nonlinear Contraction

Definition 38. *[21, 12] Assume that E is a Banach space and $T : E \to E$ is a mapping. If there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such* *that* $\psi(0) = 0$ *and* $\psi(\varepsilon) < \varepsilon$ *for all* $\varepsilon > 0$ *with the property:*

$$\|Tx - Ty\| \leq \psi(\|x - y\|), \forall x, y \in E,$$

then, we say that T *is a nonlinear contraction.*

Theorem 39. *(Boyd-Wong Contraction Principle)[21, 12] Suppose that* E *is a Banach space and* $T : E \to E$ *is a nonlinear contraction. Then* T *has a unique fixed point in* E*.*

1.6.5 Krasnoselskii's Fixed-Point Theorem

Theorem 40. *[80, 81] (Krasnoselskii's) Let* M *be a closed bounded, convex and nonempty subset of a Banach space* E*. Let* A, B *be two operators such that,*

(a) $Ax + By \in M$*, whenever* $x, y \in M$*,*

(b) A *is compact and continuous,*

(c) B *is a contraction mapping.*
Then there exists $z \in M$ *such that* $z = Az + Bz$*.*

1.6.6 Dhage's Fixed Point in Banach Algebra

Definition 41. *An algebra* $\mathcal{E}$ *is a vector space endowed with an internal composition law noted by* $(\cdot)$ *that is,*

$$\begin{cases} \mathcal{E} \times \mathcal{E} & \longrightarrow \mathcal{E} \\ (x, y) & \longrightarrow x \cdot y, \end{cases}$$

which is associative and bilinear.
A normed algebra is an algebra endowed with a norm satisfying the following property

$$\textit{for all}\, x, y \in \mathcal{E} / \ \|x \cdot y\| \leq \|x\| \|y\|.$$

A complete normed algebra is called a Banach algebra.

The following hybrid fixed point theorem for three operators in a Banach algebra $\mathcal{E}$ due to Dhage [77, 72] will be used to prove the existence result for the nonlocal boundary value problem.

Theorem 42. *[34] Let S be a closed convex , bounded and nonempty subset of a Banach algebra $\mathcal{E}$, and let $\mathcal{A}, \mathcal{C} : \mathcal{E} \longrightarrow \mathcal{E}$ and $\mathcal{B} : S \longrightarrow \mathcal{E}$ be three operators such that*

1. *$\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants δ and ξ , respectively;*
2. *$\mathcal{B}$ is compact and continuous,*
3. *$x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x \Rightarrow x \in S$ for all $y \in S$,*
4. *$\delta M + \xi < 1$ where $M = \big\|\mathcal{B}(S)\big\|$.*

Then the operator equation $\mathcal{A}x\mathcal{B}x + \mathcal{C}x = x$ has a solution in S.

Chapter 2

Existence Results for Nonlinear Neutral Generalized Caputo Fractional Differential Equations

2.1 Introduction

A differential equation is said to be neutral if the highest degree derivative of the unknown function appears with and without delay. Neutral differential equations are one of the most thoroughly studied classes of equations. It has many applications in technology and the natural sciences such as: the oscillatory behavior problems of neutral differential equations have a number of practical applications in the study of distributed networks containing lossless transmission lines that arise in high speed computers where lossless transmission lines are used to interconnect switching circuits, see [52, 74]. During the past few years there has been interest in many researchers to study the oscillatory behavior of this type of equations, see [7] Furthermore, many researchers are investigating the regularity and existence of solutions of nonlinear neutral fractional differential equations see [50, 8].
This chapter is devoted to proving some existence and uniqueness of solutions to a category of boundary value problems for a nonlinear neutral generalized Caputo fractional differential equation with generalized Riemann-Liouville integral boundary conditions. We apply a variety assort of fixed point theorems such as Krasnoselskii's and Banach. We also establish the Ulam-Hyers stability results for the addressed problem. Further, an example illustrate our results.
In order to enrich the work on fractional BVPs involving generalized FD and generalized FI boundary conditions further, we study the existence and uniqueness of

solutions for a nonlinear neutral ψ-Caputo type FDE with ψ-Riemann-Liouville FI boundary conditions of the form

$$\begin{cases} {}^{c}D_{0^{+}}^{\xi;\psi}\left[{}^{c}D_{0^{+}}^{\zeta;\psi}\varkappa(\tau)-\mathcal{Q}(\tau,\varkappa(\tau))\right]=\mathcal{F}(\tau,\varkappa(\tau)), \quad \tau\in \mathrm{J}:=[0,T], \\ \\ \varkappa(\chi)=0, \quad I_{0^{+}}^{\gamma;\psi}\varkappa(T)=0, \quad \chi\in(0,T), \end{cases} \tag{2.1}$$

where ${}^{c}D_{a^{+}}^{\sigma;\psi}$ is the ψ-Caputo fractional derivative of order $\sigma\in\{\xi,\zeta\}\subseteq(0,1)$, $I_{0^{+}}^{\gamma;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\gamma>0$, and $\mathcal{F},\mathcal{Q}:\mathrm{J}\times\mathbb{R}\to\mathbb{R}$ are given functions.

2.2 Main Results

We denote by $C(\mathrm{J},\mathbb{R})$ the Banach space of all continuous functions $\varkappa:\mathrm{J}\to\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by

$$\|\varkappa\|=sup\{|\varkappa(\tau)|:\tau\in[0,T]\}.$$

Before proceeding to the main results, we give the following lemma.

Lemma 43. *For given $\mathcal{F},\mathcal{Q}\in C$ and $0<\xi,\zeta\leq 1$, the solution of the boundary value problem*

$$\begin{cases} {}^{c}D_{0^{+}}^{\xi;\psi}\left[{}^{c}D_{0^{+}}^{\zeta;\psi}\varkappa(\tau)-\mathcal{Q}(\tau)\right]=\mathcal{F}(\tau), \quad \tau\in \mathrm{J}=[0,T], \\ \varkappa(\chi)=0, \quad I_{0^{+}}^{\gamma;\psi}\varkappa(T)=0, \quad \chi\in(0,T), \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} \varkappa(\tau) &= I_{0^{+}}^{\zeta;\psi}\mathcal{Q}(u)(\tau)+I_{0^{+}}^{\xi+\zeta;\psi}\mathcal{F}(u)(\tau) \\ &+\frac{(\psi(\tau)-\psi(0))^{\zeta}}{\Omega\Gamma(\zeta+1)}\Big[I_{0^{+}}^{\zeta+\gamma;\psi}\mathcal{Q}(u)(T)+I_{0^{+}}^{\xi+\zeta+\gamma;\psi}\mathcal{F}(u)(T) \\ &-\frac{M^{\gamma}}{\Gamma(\gamma+1)}\left(I_{0^{+}}^{\zeta;\psi}\mathcal{Q}(u)(\chi)+I_{0^{+}}^{\xi+\zeta;\psi}\mathcal{F}(u)(\chi)\right)\Big] \\ &+\frac{1}{\Omega}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\left(I_{0^{+}}^{\zeta;\psi}\mathcal{Q}(u)(\chi)+I_{0^{+}}^{\xi+\zeta;\psi}\mathcal{F}(u)(\chi)\right) \\ &-\frac{N^{\zeta}}{\Gamma(\zeta+1)}\left(I_{0^{+}}^{\zeta+\gamma;\psi}\mathcal{Q}(u)(T)+I_{0^{+}}^{\xi+\zeta+\gamma;\psi}\mathcal{F}(u)(T)\right)\Big], \end{aligned} \tag{2.3}$$

where

$$M = (\psi(T) - \psi(0)), \quad N = (\psi(\chi) - \psi(0)), \tag{2.4}$$

$$\Omega = \frac{M^{\zeta} N^{\gamma}}{\Gamma(\zeta+1)\Gamma(\gamma+1)} - \frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}. \tag{2.5}$$

Proof. using lemma 33, the general solution of the nonlinear fractional differential equation in (2.2) can be represented as

$$\varkappa(\tau) = I_{0^+}^{\zeta;\psi} \mathcal{Q}(u)(\tau) + I_{0^+}^{\xi+\zeta;\psi} \mathcal{F}(u)(\tau) + \frac{(\psi(\tau) - \psi(0))^{\zeta}}{\Gamma(\zeta+1)} c_0 + c_1, \tag{2.6}$$

where $c_0, c_1 \in \mathbb{R}$ are an arbitrary constants.
Applying the ψ-Riemann-Liouville integral of order γ to (2.6), we obtain

$$I_{0^+}^{\gamma;\psi} \varkappa(\tau) = I_{0^+}^{\zeta+\gamma;\psi} \mathcal{Q}(u)(\tau) + I_{0^+}^{\xi+\zeta+\gamma;\psi} \mathcal{F}(u)(\tau) + \frac{(\psi(\tau) - \psi(0))^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} c_0 + \frac{(\psi(\tau) - \psi(0))^{\gamma}}{\Gamma(\gamma+1)} c_1. \tag{2.7}$$

By using the boundary condition in (2.7) and the above value of $\varkappa(\tau)$ in (2.6), we have

$$I_{0^+}^{\zeta;\psi} \mathcal{Q}(u)(\chi) + I_{0^+}^{\xi+\zeta;\psi} \mathcal{F}(u)(\chi) + \frac{N^{\zeta}}{\Gamma(\zeta+1)} c_0 + c_1 = 0, \tag{2.8}$$

$$I_{0^+}^{\zeta+\gamma;\psi} \mathcal{Q}(u)(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi} \mathcal{F}(u)(T) + \frac{(\psi(T) - \psi(0))^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} c_0 + \frac{(\psi(T) - \psi(0))^{\gamma}}{\Gamma(\gamma+1)} c_1 = 0. \tag{2.9}$$

Solving the above system for c_0 and c_1, we find that

$$\begin{aligned} c_0 = \frac{1}{\Omega} \Big[& I_{0^+}^{\zeta+\gamma;\psi} \mathcal{Q}(u)(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi} \mathcal{F}(u)(T) \\ & - \frac{M^{\zeta}}{\Gamma(\gamma+1)} \left(I_{0^+}^{\zeta;\psi} \mathcal{Q}(u)(\chi) + I_{0^+}^{\xi+\zeta;\psi} \mathcal{F}(u)(\chi) \right) \Big], \\ c_1 = \frac{1}{\Omega} \Big[& \frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} \left(I_{0^+}^{\zeta;\psi} \mathcal{Q}(u)(\chi) + I_{0^+}^{\xi+\zeta;\psi} \mathcal{F}(u)(\chi) \right) \\ & - \frac{N^{\zeta}}{\Gamma(\gamma+1)} \left(I_{0^+}^{\zeta+\gamma;\psi} \mathcal{Q}(u)(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi} \mathcal{F}(u)(T) \right) \Big]. \end{aligned}$$

Finally, substituting the values of c_0 and c_1 in equation (2.6), we obtain the general solution of problem (2.2) which is (2.3). Converse is also true by using the direct computation. This completes the proof. $\square$

Also, we define the notations:

$$\theta_1 = \left\{ \frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)} + \frac{M^{\xi+2\zeta+\gamma}}{|\Omega|\Gamma(\zeta+1)\Gamma(\xi+\zeta+\gamma+1)} + \frac{M^{\xi+\zeta}N^{\xi+\zeta}}{|\Omega|\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\xi+\zeta+1)} + \frac{1}{|\Omega|}\left(\frac{M^{\xi+2\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)\Gamma(\xi+\zeta+1)} + \frac{M^{\xi+\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\xi+\zeta+\gamma+1)} \right) \right\} \tag{2.10}$$

$$\theta_2 = \left\{ \frac{M^{\zeta}}{\Gamma(\zeta+1)} + \frac{M^{2\zeta+\gamma}}{|\Omega\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{|\Omega|\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\zeta+1)} + \frac{1}{|\Omega|}\left(\frac{M^{2\zeta+\gamma}}{\Gamma(\xi+\zeta+1)\Gamma(\zeta+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} \right) \right\} \tag{2.11}$$

$$\varpi = \left[\frac{M^{\zeta}}{\Gamma(\zeta+1)} + \frac{M^{2\zeta+\gamma}}{|\Omega|\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\zeta+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{|\Omega|\Gamma(\zeta+\gamma+1)\Gamma(\zeta+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} \right] \tag{2.12}$$

In the sequel, the following assumptions will be considered fulfilled:

(C_1) The functions $\mathcal{F}, \mathcal{Q} : \mathrm{J} \times \mathbb{R} \to \mathbb{R}$ are continuous.

(C_2) There exist two constants $\mathcal{L}, \mathcal{K} > 0$ such that

$$|\mathcal{F}(\tau, \varkappa(\tau)) - \mathcal{F}(\tau, \overline{\varkappa}(\tau))| \le \mathcal{L}|\varkappa - \overline{\varkappa}|, \quad for \quad \tau \in \mathrm{J}\ , \varkappa, \overline{\varkappa} \in C(\mathrm{J}),$$

and

$$|\mathcal{Q}(\tau, \varkappa(\tau)) - \mathcal{Q}(\tau, \overline{\varkappa}(\tau))| \le \mathcal{K}|\varkappa - \overline{\varkappa}|, \quad for \quad \tau \in \mathrm{J}\ , \varkappa, \overline{\varkappa} \in C(\mathrm{J}).$$

(C_3) There exist two functions $p, q \in C(\mathrm{J}, \mathbb{R}_+)$ with bounds $\|p\|$, $\|q\|$, respectively such that:

$$|\mathcal{F}(\tau, \varkappa)| \le p(\tau) \quad \text{and} \quad |\mathcal{Q}(\tau, \varkappa)| \le q(\tau),$$

for all $\tau \in \mathrm{J}$ and $\varkappa \in C(\mathrm{J})$.

In the following subsections, we prove existence (uniqueness) results for the boundary value problem (2.1) by using a variety of fixed point theorems.

2.2.1 Existence result via Krasnoselskii"s fixed point theorem

Theorem 44. *Suppose* $(C_1) - (C_3)$ *hold. If* $\mathcal{K}\varpi < 1$. *Then the problem* (2.1) *has a least one solution defined on* J.

Proof. We consider $\mathcal{B}_\rho = \{\varkappa \in C(\mathrm{J}) : \|\varkappa\| \leq \rho\}$, be a closed, bounded, convex, and nonempty subset the Banach space $C(\mathrm{J}, \mathbb{R})$, where ρ is a fixed constant.
Choosing

$$\rho \geq \|p\|\theta_1 + \|q\|\theta_2.$$

We define the operator $\mathcal{H} : C(\mathrm{J}) \to C(\mathrm{J})$ as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, where

$$\begin{aligned}(\mathcal{H}_1\varkappa)(\tau) &= I_{0^+}^{\zeta;\psi}\mathcal{Q}(u,\varkappa(u))(\tau) \\ &+ \frac{(\psi(\tau)-\psi(0))^\zeta}{\Omega\Gamma(\zeta+1)}\left[I_{0^+}^{\zeta+\gamma;\psi}\mathcal{Q}(u,\varkappa(u))(\tau) - \frac{M^\gamma}{\Gamma(\gamma+1)}I_{0^+}^{\zeta;\psi}\mathcal{Q}(u,\varkappa(u))(\chi)\right] \\ &+ \frac{1}{\Omega}\left[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}I_{0^+}^{\zeta;\psi}\mathcal{Q}(u,\varkappa(u))(\chi) - \frac{N^\zeta}{\Gamma(\zeta+1)}I_{0^+}^{\zeta+\gamma;\psi}\mathcal{Q}(u,\varkappa(u))(T)\right],\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}(\mathcal{H}_2\varkappa)(\tau) &= I_{0^+}^{\xi+\zeta;\psi}\mathcal{F}(u,\varkappa(u))(\tau) \\ &+ \frac{(\psi(\tau)-\psi(0))^\zeta}{\Omega\Gamma(\zeta+1)}\left[I_{0^+}^{\xi+\zeta+\gamma;\psi}\mathcal{F}(u,\varkappa(u))(T) - \frac{M^\gamma}{\Gamma(\gamma+1)}I_{0^+}^{\xi+\zeta;\psi}\mathcal{F}(u,\varkappa(u))(\chi)\right] \\ &+ \frac{1}{\Omega}\left[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}I_{0^+}^{\xi+\zeta;\psi}\mathcal{F}(u,\varkappa(u))(\chi) - \frac{N^\zeta}{\Gamma(\zeta+1)}I_{0^+}^{\xi+\zeta+\gamma;\psi}\mathcal{F}(u,\varkappa(u))(T)\right].\end{aligned} \tag{2.14}$$

Now, we show that the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy the hypothesis of Krasnoselskii's theorem (40) in three steps.
The first step, we show that $\mathcal{H}_1\varkappa + \mathcal{H}_2\overline{\varkappa} \in \mathcal{B}_\rho$ for any $\varkappa, \overline{\varkappa} \in \mathcal{B}_\rho$, we have

$$\begin{aligned}\|\mathcal{H}_1\varkappa + \mathcal{H}_2\overline{\varkappa}\| &\leq I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))|(T) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\overline{\varkappa}(u))|(T) \\ &+ \frac{M^\zeta}{|\Omega|\Gamma(\zeta+1)}\Big[I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u))|(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{F}(u,\overline{\varkappa}(u))|(T) \\ &+ \frac{M^\gamma}{\Gamma(\gamma+1)}\Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))|(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\overline{\varkappa}(u))(\chi)\Big)\Big] \\ &+ \frac{1}{|\Omega|}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\overline{\varkappa}(u))(\chi)\Big) \\ &+ \frac{N^\zeta}{\Gamma(\zeta+1)}\Big(I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u))(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{F}(u,\overline{\varkappa}(u))(T)\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq \|p\| \left(\frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)} \right.\\
&+ \frac{M^{\xi+2\zeta+\gamma}}{|\Omega|\Gamma(\zeta+1)\Gamma(\xi+\zeta+\gamma+1)} + \frac{M^{\xi+\zeta}N^{\xi+\zeta}}{|\Omega|\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\xi+\zeta+1)}\\
&\left. + \frac{1}{|\Omega|} \left(\frac{M^{\xi+2\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)\Gamma(\xi+\zeta+1)} + \frac{M^{\xi+\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\xi+\zeta+\gamma+1)} \right) \right)\\
&+ \|q\| \left(\frac{M^{\zeta}}{\Gamma(\zeta+1)} + \frac{M^{2\zeta+\gamma}}{|\Omega\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{|\Omega|\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\zeta+1)} \right.\\
&\left. + \frac{1}{|\Omega|} \left(\frac{M^{2\zeta+\gamma}}{\Gamma(\xi+\zeta+1)\Gamma(\zeta+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} \right) \right)\\
&= \|p\|\theta_1 + \|q\|\theta_2 \leq \rho.
\end{aligned}
$$

Which implies that $\mathcal{H}_1\varkappa + \mathcal{H}_2\overline{\varkappa} \in \mathcal{B}_\rho$.
Next step, is related to the compactness and continuity of the operator $\mathcal{H}_2$.
Continuity of the function $\mathcal{F}$ implies that the operator $\mathcal{H}_2$ is continuous.
Also, $\mathcal{H}_2$ is uniformly bounded on $\mathcal{B}_\rho$ as

$$\|\mathcal{H}_2\| \leq \|q\|\theta_1.$$

Now, we prove the compactness of the operator $\mathcal{H}_2$.
Let $\tau_1, \tau_2 \in \mathrm{J}$, with $\tau_1 < \tau_2$ and $\varkappa \in \mathcal{B}_\rho$. Then we obtain

$$
\begin{aligned}
&|(\mathcal{H}_2\varkappa)(\tau_2) - (\mathcal{H}_2\varkappa)(\tau_1)|\\
&\leq \frac{\|p\|}{\Gamma(\xi+\zeta+1)} \left[2|\psi(\tau_2) - \psi(\tau_1)|^{\xi+\zeta} + |(\psi(\tau_2) - \psi(0))^{\xi+\zeta} - (\psi(\tau_1) - \psi(0))^{\xi+\zeta}| \right]\\
&+ \frac{\|p\||(\psi(\tau_2) - \psi(0))^{\zeta} - (\psi(\tau_1) - \psi(0))^{\zeta}|}{|\Omega|\Gamma(\zeta+1)} \left[\frac{M^{\xi+\zeta+\gamma}}{\Gamma(\xi+\zeta+\gamma+1)} + \frac{M^{\gamma}N^{\xi+\zeta}}{\Gamma(\gamma+1)\Gamma(\xi+\zeta+1)} \right],
\end{aligned}
$$

which is independent of $\varkappa$ and tends to zeros as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{H}_2$ is equicontinuous. So $\mathcal{H}_2$ is relatively compact on $\mathcal{B}_\rho$. Hence by the Arzela-Ascoli, $\mathcal{H}_2$ is compact on $\mathcal{B}_\rho$.

Finally, we show that the operator $\mathcal{H}_1$ is a contraction. By using assumption (C_1),

$$\begin{aligned}
\|\mathcal{H}_1\varkappa - \mathcal{H}_1\overline{\varkappa}\| &\leq I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T) \\
&+ \frac{M^{\zeta}}{|\Omega|\Gamma(\zeta+1)}\Big[I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T) \\
&+ \frac{M^{\gamma}}{\Gamma(\gamma+1)}I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(\chi)\Big] \\
&+ \frac{1}{|\Omega|}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T)(\chi) \\
&+ \frac{N^{\zeta}}{\Gamma(\zeta+1)}I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T)(T)\Big] \\
&\leq \mathcal{K}\Big[\frac{M^{\zeta}}{\Gamma(\zeta+1)} + \frac{M^{2\zeta+\gamma}}{|\Omega|\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\zeta+1)} \\
&+ \frac{M^{\zeta+\gamma}N^{\zeta}}{|\Omega|\Gamma(\zeta+\gamma+1)\Gamma(\zeta+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)}\Big]\|\varkappa - \overline{\varkappa}\| \\
&= \mathcal{K}\varpi\|\varkappa - \overline{\varkappa}\|.
\end{aligned}$$

Thus all the assumptions of Krasnoselskii"s FPT are satisfied. So, Theorem 44 shows that (2.1) has at least one solution on J. The proof is finished. $\square$

2.2.2 Existence and uniqueness Result

Here we prove the existence and uniqueness result for the problem (2.1) and by using the Banach's contraction mapping principle.

Theorem 45. *If the conditions (C_1) and (C_2) hold, then the problem* (2.1) *has a unique solution on* J*, if* $\Delta := (\mathcal{L}\theta_1 + \mathcal{K}\theta_2) < 1$.

Proof. Let us fix $\mathcal{F}_0 = \sup_{\tau\in[0,T]}|\mathcal{F}(\tau,0)|$, $\mathcal{Q}_0 = \sup_{\tau\in[0,T]}|\mathcal{Q}(\tau,0)|$,
and choose $r \geq \frac{\mathcal{F}_0\theta_1+\mathcal{Q}_0\theta_2}{1-\mathcal{L}\theta_1-\mathcal{K}\theta_2}$. In the first step, we show that $\mathcal{H}\mathcal{B}_r \subset \mathcal{B}_r$, we take

$\varkappa \in \mathcal{B}_r = \{\varkappa \in C(\mathrm{J}) : \|\varkappa\| \leq r\}$ so that

$$\begin{aligned}
\|(\mathcal{H}\varkappa)(\tau)\| \leq & \sup_{\tau\in[0,T]} \Big\{ I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))|(\tau) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u))|(\tau) \\
& + \frac{|\psi(\tau)-\psi(0)|^{\zeta}}{|\Omega|\Gamma(\zeta+1)} \Big[I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u))|(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u))|(T) \\
& + \frac{M^{\gamma}}{\Gamma(\gamma+1)} \Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))|(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u))(\chi) \Big) \Big] \\
& + \frac{1}{|\Omega|} \Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} \Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))|(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u))|(\chi) \Big) \\
& + \frac{N^{\zeta}}{\Gamma(\zeta+1)} \Big(I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u))|(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{F}(u,\varkappa(u))|(T) \Big) \Big] \Big\} \\
\leq & \sup_{\tau\in[0,T]} \Big\{ I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,0)| + |\mathcal{Q}(u,0)|)(T) \\
& + \mathbb{I}_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,0)| + |\mathcal{F}(u,0)|)(T) \\
& + \frac{|\psi(\tau)-\psi(0)|^{\zeta}}{|\Omega|\Gamma(\zeta+1)} \Big[I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,0)| + |\mathcal{Q}(u,0)|)(T) \\
& + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,0)| + |\mathcal{F}(u,0)|)(T) \\
& + \frac{M^{\gamma}}{\Gamma(\gamma+1)} \Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,0)| + |\mathcal{Q}(u,0)|)(T)(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u))(\chi) \Big) \Big] \\
& + \frac{1}{|\Omega|} \Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} \Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u))|(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,0)| + |\mathcal{F}(u,0)|)(\chi) \Big) \\
& + \frac{N^{\zeta}}{\Gamma(\zeta+1)} \Big(I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,0)| + |\mathcal{Q}(u,0)|)(T) \\
& + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,0)| + |\mathcal{F}(u,0)|)(T) \Big) \Big] \Big\} \\
\leq & (\mathcal{L}\|\varkappa\| + \mathcal{F}_0) \Big\{ \frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)} + \frac{M^{\xi+2\zeta+\gamma}}{|\Omega\Gamma(\zeta+1)\Gamma(\xi+\zeta+\gamma+1)} \\
& + \frac{M^{\xi+\zeta}N^{\xi+\zeta}}{|\Omega|\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\xi+\zeta+1)} \\
& + \frac{1}{|\Omega|} \Big(\frac{M^{\xi+2\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)\Gamma(\xi+\zeta+1)} + \frac{M^{\xi+\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\xi+\zeta+\gamma+1)} \Big) \Big\} \\
& + (\mathcal{K}\|\varkappa\| + \mathcal{Q}_0) \Big\{ \frac{M^{\zeta}}{\Gamma(\zeta+1)} + \frac{M^{2\zeta+\gamma}}{|\Omega\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{|\Omega|\Gamma(\zeta+1)\Gamma(\gamma+1)\Gamma(\zeta+1)} \\
& + \frac{1}{|\Omega|} \Big(\frac{M^{2\zeta+\gamma}}{\Gamma(\xi+\zeta+1)\Gamma(\zeta+1)} + \frac{M^{\zeta+\gamma}N^{\zeta}}{\Gamma(\zeta+1)\Gamma(\zeta+\gamma+1)} \Big) \Big\} \\
= & (\mathcal{L}r + \mathcal{F}_0)\theta_1 + (\mathcal{K}r + \mathcal{Q}_0)\theta_2 \leq r,
\end{aligned}$$

which implies that $\mathcal{H}\mathcal{B}_r \subset \mathcal{B}_r$.

Next, we let $\varkappa, \overline{\varkappa} \in C(\mathrm{J})$. Then, for $\tau \in \mathrm{J}$, we have

$$
\begin{aligned}
&\|(\mathcal{H}\varkappa)(\tau) - (\mathcal{H}\overline{\varkappa})(\tau)\| \\
&\leq \sup_{\tau\in[0,T]} \Big\{ I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(\tau) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,\overline{\varkappa}(u))|(\tau) \\
&+ \frac{|\psi(\tau)-\psi(0)|^{\zeta}}{|\Omega|\Gamma(\zeta+1)} \Big[I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T) \\
&+ \frac{M^{\gamma}}{\Gamma(\gamma+1)} \Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(\chi) \Big) \Big] \\
&+ \frac{1}{|\Omega|} \Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} \Big(I_{0^+}^{\zeta;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(\chi) + I_{0^+}^{\xi+\zeta;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,\overline{\varkappa}(u))|(\chi) \Big) \\
&+ \frac{N^{\zeta}}{\Gamma(\zeta+1)} \Big(I_{0^+}^{\zeta+\gamma;\psi}|\mathcal{Q}(u,\varkappa(u)) - \mathcal{Q}(u,\overline{\varkappa}(u))|(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}|\mathcal{F}(u,\varkappa(u)) - \mathcal{F}(u,\overline{\varkappa}(u))|(T) \Big) \Big] \Big\} \\
&\leq \sup_{\tau\in[0,T]} \Big\{ \frac{M^{\zeta}}{\Gamma(\zeta+1)}\mathcal{K}\|\varkappa-\overline{\varkappa}\| + \frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)}\mathcal{L}\|\varkappa-\overline{\varkappa}\| \\
&+ \frac{M^{\zeta}}{|\Omega|\Gamma(\zeta+1)} \Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\mathcal{K}\|\varkappa-\overline{\varkappa}\| + \frac{M^{\xi+\zeta+\gamma}}{\Gamma(\xi+\zeta+\gamma+1)}\mathcal{L}\|\varkappa-\overline{\varkappa}\| \\
&+ \frac{M^{\gamma}}{\Gamma(\gamma+1)} \Big(\frac{M^{\zeta}}{\Gamma(\zeta+1)}\mathcal{K}\|\varkappa-\overline{\varkappa}\| + \frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)}\mathcal{L}\|\varkappa-\overline{\varkappa}\| \Big) \Big] \\
&+ \frac{1}{|\Omega|} \Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)} \Big(\frac{M^{\zeta}}{\Gamma(\zeta+1)}\mathcal{K}\|\varkappa-\overline{\varkappa}\| + \frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)}\mathcal{L}\|\varkappa-\overline{\varkappa}\| \Big) \\
&+ \frac{N^{\zeta}}{\Gamma(\zeta+1)} \Big(\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\mathcal{K}\|\varkappa-\overline{\varkappa}\| + \frac{M^{\xi+\zeta+\gamma}}{\Gamma(\xi+\zeta+\gamma+1)}\mathcal{L}\|\varkappa-\overline{\varkappa}\| \Big) \Big] \Big\} \\
&= (\mathcal{L}\theta_1 + \mathcal{K}\theta_2)\|\varkappa-\overline{\varkappa}\|.
\end{aligned}
$$

As $(\mathcal{L}\theta_1 + \mathcal{K}\theta_2) < 1$, the operator $\mathcal{H}$ is a contraction. So, the problem (2.1) has a unique solution on J. □

2.3 UH stability analysis

In this section, we study the Ulam stability, and we adopt the definitions in ([89, 2]) of the UH and generalized UH stability of the problem (2.1). Let $\varepsilon > 0$. We consider the following inequality:

$$\left| D_{a^+}^{\xi;\psi} \left[D_{a^+}^{\zeta;\psi} \widetilde{\varkappa}(\tau) - \mathcal{Q}(\tau, \widetilde{\varkappa}(\tau)) \right] - \mathcal{F}(\tau, \widetilde{\varkappa}(\tau)) \right| \leq \varepsilon, \quad \tau \in \mathrm{J}. \tag{2.15}$$

Definition 46. *The equation (2.1) is UH stable if there exists a real number $c_{\mathcal{F}} > 0$ such that, for each $\varepsilon > 0$ and for each solution $\widetilde{\varkappa} \in C(\mathrm{J})$ of inequality(2.15) there* *exists a solution* $\varkappa \in C(\mathrm{J})$ *of (2.1) with*

$$|\widetilde{\varkappa}(\tau) - \varkappa(\tau)| \leq \varepsilon c_{\mathcal{F}}, \quad \tau \in \mathrm{J}.$$

Definition 47. *The equation (2.1) is generalized UH stable if there exists* $C_{\mathcal{F}} : \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+)$ *with* $C_{\mathcal{F}}(0) = 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\tilde{\varkappa} \in C(\mathrm{J})$ *of inequality* (2.15), *there exists a solution* $\varkappa \in C(\mathrm{J})$ *of (5.1) with*

$$|\widetilde{\varkappa}(\tau) - \varkappa(\tau)| \leq C_{\mathcal{F}}(\varepsilon), \quad \tau \in \mathrm{J}.$$

Lemma 48. *A function* $\widetilde{\varkappa} \in C(\mathrm{J})$ *is a solution of inequality (2.15) if and only if there exists a function* $\sigma \in C(\mathrm{J})$ *(which depends on solution* $\tilde{\varkappa}$ *) such that*
1.$|\sigma(\tau)| \leq \varepsilon, \tau \in \mathrm{J}$.
2.$D_{a^+}^{\xi;\psi}\left[D_{a^+}^{\zeta;\psi}\widetilde{\varkappa}(\tau) - \mathcal{Q}(\tau, \widetilde{\varkappa}(\tau))\right] = \mathcal{F}(\tau, \widetilde{\varkappa}(\tau)) + \sigma(\tau), \quad \tau \in \mathrm{J}.$

Now, we discuss the UH stability of solution to the problem (2.1).

Theorem 49. *Suppose that the assumption* (C_2) *is fulfilled. Then the problem (2.1) is UH stable on* J *and consequently generalized UH stable provided that* $\Delta < 1$.

Proof. Let $\varepsilon > 0$ and let $\widetilde{\varkappa} \in C(\mathrm{J})$ be a function which satisfies the inequality (2.15) and let $\varkappa \in C(\mathrm{J})$ the unique solution of the following problem

$$\begin{cases} {}^{c}D_{0^+}^{\xi;\psi}\left[{}^{c}D_{0^+}^{\zeta;\psi}\varkappa(\tau) - \mathcal{Q}(\tau, \varkappa(\tau))\right] = \mathcal{F}(\tau, \varkappa(\tau)), \quad \tau \in \mathrm{J} := [0, T], \\ \\ \varkappa(\chi) = 0, \quad I_{0^+}^{\gamma;\psi}\varkappa(T) = 0, \quad \chi \in (0, T). \end{cases} \tag{2.16}$$

By Lemma 43, we have

$$\begin{aligned} \varkappa(\tau) = {} & I_{0^+}^{\zeta;\psi}\mathcal{Q}(u)(\tau) + I_{0^+}^{\xi+\zeta;\psi}\mathcal{F}(u)(\tau) \\ & + \frac{(\psi(\tau) - \psi(0))^{\zeta}}{\Omega\Gamma(\zeta+1)}\left[I_{0^+}^{\zeta+\gamma;\psi}\mathcal{Q}(u)(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}\mathcal{F}(u)(T)\right. \\ & \left. - \frac{M^{\gamma}}{\Gamma(\gamma+1)}\left(I_{0^+}^{\zeta;\psi}\mathcal{Q}(u)(\chi) + I_{0^+}^{\xi+\zeta;\psi}\mathcal{F}(u)(\chi)\right)\right] \\ & + \frac{1}{\Omega}\left[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\left(I_{0^+}^{\zeta;\psi}\mathcal{Q}(u)(\chi) + I_{0^+}^{\xi+\zeta;\psi}\mathcal{F}(u)(\chi)\right)\right. \\ & \left. - \frac{N^{\zeta}}{\Gamma(\zeta+1)}\left(I_{0^+}^{\zeta+\gamma;\psi}\mathcal{Q}(u)(T) + I_{0^+}^{\xi+\zeta+\gamma;\psi}\mathcal{F}(u)(T)\right)\right], \end{aligned} \tag{2.17}$$

Indeed, by Remark 48, we conclude that

$$\begin{cases} D_{a^+}^{\xi;\psi}\left[D_{a^+}^{\zeta;\psi}\widetilde{\varkappa}(\tau)-\mathcal{Q}(\tau,\widetilde{\varkappa}(\tau))\right]=\mathcal{F}(\tau,\widetilde{\varkappa}(\tau))+\sigma(\tau), \quad \tau\in \mathrm{J}=[0,T], \\ \\ \widetilde{\varkappa}(\chi)=0, \quad I_{0^+}^{\gamma;\psi}\widetilde{\varkappa}(T)=0, \quad \chi\in(0,T). \end{cases} \tag{2.18}$$

Again by Lemma 43, we have

$$\begin{aligned}
\widetilde{\varkappa}(\tau) &= I_{0^+}^{\zeta;\psi}\widetilde{\mathcal{Q}}(\widetilde{u})(\tau)+I_{0^+}^{\xi+\zeta;\psi}\widetilde{\mathcal{F}}(\widetilde{u})(\tau)\\
&+\frac{(\psi(\tau)-\psi(0))^{\zeta}}{\Omega\Gamma(\zeta+1)}\Big[I_{0^+}^{\zeta+\gamma;\psi}\widetilde{\mathcal{Q}}(\widetilde{u})(T)+I_{0^+}^{\xi+\zeta+\gamma;\psi}\widetilde{\mathcal{F}}(\widetilde{u})(T)\\
&-\frac{M^{\gamma}}{\Gamma(\gamma+1)}\Big(I_{0^+}^{\zeta;\psi}\widetilde{\mathcal{Q}}(\widetilde{u})(\chi)+I_{0^+}^{\xi+\zeta;\psi}\widetilde{\mathcal{F}}(\widetilde{u})(\chi)\Big)\Big]\\
&+\frac{1}{\Omega}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\Big(I_{0^+}^{\zeta;\psi}\widetilde{\mathcal{Q}}(\widetilde{u})(\chi)+I_{0^+}^{\xi+\zeta;\psi}\widetilde{\mathcal{F}}(\widetilde{u})(\chi)\Big)\\
&-\frac{N^{\zeta}}{\Gamma(\zeta+1)}\Big(I_{0^+}^{\zeta+\gamma;\psi}\widetilde{\mathcal{Q}}(\widetilde{u})(T)+I_{0^+}^{\xi+\zeta+\gamma;\psi}\widetilde{\mathcal{F}}(\widetilde{u})(T)\Big)\Big]\\
&+I_{0^+}^{\xi+\zeta;\psi}\sigma(\tau)+\frac{(\psi(\tau)-\psi(0))^{\zeta}}{\Omega\Gamma(\zeta+1)}\Big[I_{0^+}^{\xi+\zeta+\gamma;\psi}\sigma(T)-\frac{M^{\gamma}}{\Gamma(\gamma+1)}\Big(I_{0^+}^{\xi+\zeta;\psi}\sigma(\chi)\Big)\Big]\\
&+\frac{1}{\Omega}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\Big(I_{0^+}^{\xi+\zeta;\psi}\sigma(\chi)\Big)-\frac{N^{\zeta}}{\Gamma(\zeta+1)}\Big(I_{0^+}^{\xi+\zeta+\gamma;\psi}\sigma(T)\Big)\Big]
\end{aligned}$$

On the other hand, we have, for each $\tau\in \mathrm{J}$

$$\begin{aligned}
|\widetilde{\varkappa}(\tau)-\varkappa(\tau)| &\le I_{0^+}^{\zeta;\psi}\left|\widetilde{\mathcal{Q}}(\widetilde{u})(\tau)-\mathcal{Q}(u)(\tau)\right|+I_{0^+}^{\xi+\zeta;\psi}\left|\widetilde{\mathcal{F}}(\widetilde{u})(\tau)-\mathcal{F}(u)(\tau)\right|\\
&+\frac{(\psi(\tau)-\psi(0))^{\zeta}}{\Omega\Gamma(\zeta+1)}\Big[I_{0^+}^{\zeta+\gamma;\psi}\left|\widetilde{\mathcal{Q}}(\widetilde{u})(T)-\mathcal{Q}(u)(T)\right|+I_{0^+}^{\xi+\zeta+\gamma;\psi}\left|\widetilde{\mathcal{F}}(\widetilde{u})(T)-\mathcal{F}(u)(T)\right|\\
&-\frac{M^{\gamma}}{\Gamma(\gamma+1)}\Big(I_{0^+}^{\zeta;\psi}\left|\widetilde{\mathcal{Q}}(\widetilde{u})(\chi)-\mathcal{Q}(u)(\chi)\right|+I_{0^+}^{\xi+\zeta;\psi}\left|\widetilde{\mathcal{F}}(\widetilde{u})(\chi)-\mathcal{F}(u)(\chi)\right|\Big)\Big]\\
&+\frac{1}{\Omega}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\Big(I_{0^+}^{\zeta;\psi}\left|\widetilde{\mathcal{Q}}(\widetilde{u})(\chi)-\mathcal{Q}(u)(\chi)\right|+I_{0^+}^{\xi+\zeta;\psi}\left|\widetilde{\mathcal{F}}(\widetilde{u})(\chi)-\mathcal{F}(u)(\chi)\right|\Big)\\
&+\frac{N^{\zeta}}{\Gamma(\zeta+1)}\Big(I_{0^+}^{\zeta+\gamma;\psi}\left|\widetilde{\mathcal{Q}}(\widetilde{u})(T)-\mathcal{Q}(u)(T)\right|+I_{0^+}^{\xi+\zeta+\gamma;\psi}\left|\widetilde{\mathcal{F}}(\widetilde{u})(T)-\mathcal{F}(u)(T)\right|\Big)\Big]\\
&+I_{0^+}^{\xi+\zeta;\psi}\sigma(\tau)+\frac{(\psi(\tau)-\psi(0))^{\zeta}}{\Omega\Gamma(\zeta+1)}\Big[I_{0^+}^{\xi+\zeta+\gamma;\psi}|\sigma(T)|-\frac{M^{\gamma}}{\Gamma(\gamma+1)}\Big(I_{0^+}^{\xi+\zeta;\psi}|\sigma(\chi)|\Big)\Big]\\
&+\frac{1}{\Omega}\Big[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\Big(I_{0^+}^{\xi+\zeta;\psi}|\sigma(\chi)|\Big)-\frac{N^{\zeta}}{\Gamma(\zeta+1)}\Big(I_{0^+}^{\xi+\zeta+\gamma;\psi}|\sigma(T)|\Big)\Big]
\end{aligned}$$

Hence. Using part (i) of Remark 48 and (C_1) we can get

$$\begin{aligned}|\widetilde{\varkappa}(\tau)-\varkappa(\tau)| \leq & \left\{\frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)}+\frac{M^{\zeta}}{|\Omega|\Gamma(\zeta+1)}\left[\frac{M^{\xi+\zeta+\gamma}}{\Gamma(\xi+\zeta+\gamma+1)}+\frac{M^{\gamma}}{\Gamma(\gamma+1)}\left(\frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)}\right)\right]\right. \\ & \left.+\frac{1}{|\Omega|}\left[\frac{M^{\zeta+\gamma}}{\Gamma(\zeta+\gamma+1)}\left(\frac{M^{\xi+\zeta}}{\Gamma(\xi+\zeta+1)}\right)+\frac{N^{\zeta}}{\Gamma(\zeta+1)}\left(\frac{M^{\xi+\zeta+\gamma}}{\Gamma(\xi+\zeta+\gamma+1)}\right)\right]\right\}\varepsilon \\ & +\Delta\|\widetilde{\varkappa}-\varkappa\| := \Xi\varepsilon+\Delta\|\widetilde{\varkappa}-\varkappa\|\end{aligned}$$

In consequence. It follows that

$$\|\widetilde{\varkappa}-\varkappa\|_{\infty} \leq \frac{\Xi\varepsilon}{(1-\Delta)}$$

If we let $C_{\mathcal{F}}=\Xi$, then, the UH stability condition is satisfied. More generally, for $C_{\mathcal{F}}(\varepsilon)=\frac{\Xi\varepsilon}{(1-\Delta)}$; $C_{\mathcal{F}}(0)=0$ the generalized UH stability condition is also satisfied. This completes the proof. □

2.4 Application

This section is devoted to the illustration of the results derived in the last section.

Example 50. *Consider the following BVP:*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2};e^{\tau}}\left[{}^{C}D_{0^+}^{\frac{3}{4};e^{\tau}}\varkappa(\tau)-\frac{\sin|\varkappa(\tau)|}{(\tau+50)}\right]=\frac{1}{e^{\tau}+9}\left(1+\frac{|\varkappa(\tau)|}{1+|\varkappa(\tau)|}\right), \tau\in[0,1] \\ \varkappa(0)=0, \quad I_{0^+}^{\frac{3}{4};e^{\tau}}\varkappa(1)=\varkappa_0, \end{cases} \tag{2.19}$$

In this case we take

$$\begin{aligned}\xi=\frac{1}{2}, \zeta=\frac{3}{4}, T=1, \psi(\tau)=e^{\tau}, \\ \mathcal{F}(\tau,\varkappa)=\frac{1}{e^{\tau}+9}\left(1+\frac{|\varkappa(\tau)|}{1+|\varkappa(\tau)|}\right), \\ \mathcal{Q}(\tau,\varkappa)=\frac{\sin|\varkappa(\tau)|}{(\tau+50)}.\end{aligned}$$

It is clear that assumptions (C_1) and (C_2) of the Theorem 45 is satisfied. On the other hand, for any $\tau\in[0,1], \varkappa, y\in\mathbb{R}$ we have

$$|\mathcal{F}(\tau,\varkappa)-\mathcal{F}(\tau,y)| \leq \frac{1}{10}|\varkappa-y|,$$

$$|\mathcal{Q}(\tau,\varkappa)-\mathcal{Q}(\tau,y)| \leq \frac{1}{50}|\varkappa-y|.$$

Hence condition (C_2) *holds with* $\mathcal{L}=\frac{1}{10}$ *and* $\mathcal{K}=\frac{1}{50}$. *Also by simple calculations, we find that* $\Delta=0.5342<1$. *Then by Theorem 45, the BVP (2.19) has a unique solution on* $[0,1]$. *Moreover, Theorem 49 implies that the problem (2.1) is HU stable and generalized HU stable.*

Example 51. *Consider the following BVP: In this case we take*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{2}{3};2^\tau}\left[{}^{C}D_{0^+}^{\frac{1}{3};2^\tau}\varkappa(\tau)-\mathcal{Q}(\tau,\varkappa(\tau))\right]=\mathcal{F}(\tau,\varkappa(\tau)), \tau\in[0,1], \\ \varkappa(0)=0, \quad I_{0^+}^{\frac{3}{4};2^\tau}\varkappa(1)=\varkappa_0, \end{cases} \tag{2.20}$$

with

$$\mathcal{F}(\tau,\varkappa(\tau))=\frac{1}{3(\tau+2)^2}\left(\tau+\sqrt{1+\tau^2}\right), \tag{2.21}$$

$$\mathcal{Q}(\tau,\varkappa(\tau))=\frac{1}{8}+\frac{\sin\sqrt{|\tau(\tau)|}}{24}, \tag{2.22}$$

It is clear that assumptions (C_2) *of the Theorem 45 are satisfied. On the other hand, for any* $\tau\in[0,1],\varkappa\in\mathbb{R}$ *we have*

$$|\mathcal{F}(\tau,\varkappa)-\mathcal{F}(\tau,y)| \leq \frac{1}{12}|\varkappa-y|,$$

$$|\mathcal{Q}(\tau,\varkappa)-\mathcal{Q}(\tau,y)| \leq \frac{1}{24}|\varkappa-y|.$$

Hence condition (C_2) *holds with* $\mathcal{L}=\frac{1}{12}$ *and* $\mathcal{K}=\frac{1}{24}$. *We shall check that condition in Theorem 45 is satisfied. Indeed* $\Delta=0.2014<1$.

To explain Theorem 44, let us take $\mathcal{F}(\tau,\varkappa)$ *given by (2.21). Clearly, the conditions* $(C_1)-(C_3)$ *holds with* $\|p\|=\frac{1}{12}$ *and* $\|q\|=\frac{1}{24}$. *In addition,* $\mathcal{K}\varpi\approx 0.0723<1$. *Hence, all hypotheses of Theorem 70 are satisfied. So, the problem (2.20) has an existence of a solution on* $[0,1]$.

2.5 Concluding Remarks

In this chapter, we have given some results of the existence and uniqueness of solutions for BVPs of nonlinear neutral FDEs involving the generalized Caputo FD and the generalized Riemann-Liouville FI boundary conditions. As a first step, the BVP is turned to a fixed point problem. Based on this, the existence results are established via the Krasnoselskii's and Banach's fixed point theorems. On other hand discusses the Ulam-Hyers stability result of the considered problem. We give an example to justify the theoretical results.

We confirm that the results of this work are novel and generalize some previous works. For example, by taking $\psi(\tau) = \tau$ in the obtained results, which can be considered a special case studied by [11]. In addition to this, when taking different values for the function ψ, our studied problem covers many problems that contain classical operators, which are incorporated into the operators used in our study.

Chapter 3

Existence results for Langevin equation with Riesz-Caputo fractional derivative

3.1 Introduction

The Langevin equation is a stochastic differential equations which describes the Brownian motion, it was first formulated in the work of Paul Langevin 1908. This Scientist prepared a detailed and accurate of Brownian motion. In physics the Langevin equation is utilized as a modeling of physics phenomena such that: study of the random motion of a small particle in a fluid due to collisions with the surrounding molecules in thermal motion, analyzing the stock market, photo electron. Therefore, the generalized of the Langevin equation can be used to formulate many various problems featuring molecular motion in condensed matter, for example complex systems. An important characteristic of the generalized Langevin equation is that it involves an aftereffect function, which is named a memory function. As examples for applications of nonlinear Langevin equation, one many refer to modeling the financial market (SPW), Fractional Langevin equation to describe anomalous diffusion [60], fractional Brownian motion (JHMR), single-file diffusion [40] and applications to stochastic Problems in Physics, Chemistry and Electrical Engineering [28].

The most current work related to fractional differential equations of the Caputo derivative which are unilateral factors unfortunately only reflect the influence of past and future memory. The Riesz-Caputo derivative is a two-sided fractional operator, including the right and left derivative, which can reflect both past and future memory effects. This function is particular for partial modeling on a finite body. Some recent applications of this derivative concern abnormal diffusion.

Many researchers studied existence and uniqueness solution for fractional nonlinear Langevin equation with fractional derivatives ([56, 106, 9]).
The objective of this chapter is to develop the existence and uniqueness of solutions for nonlinear Langevin equation involving Riesz-Caputo fractional derivatives, with a class of anti-periodic boundary conditions of the form:

$$\begin{cases} {}_0^{RC}D_T^{\alpha}({}_0^{RC}D_T^{\beta}+\chi)x(t)=f(t,x(t)) \ ,0<t<T, \\ x(0)+x(T)=0 \ , \ x'(0)+x'(T)=0, \end{cases} \tag{3.1}$$

where ${}^{RC}D^{\alpha}$ is the Riesz-Caputo fractional derivatives of order $1<\alpha\leq 2$ and $0<\beta\leq 1$, $\chi\in\mathbb{R}$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function with respected to its both variables, t and x. We aim to establish an existence and uniqueness result of the problem 3.1.via Banach , Schaefer (Theorem 36) and Krasnoselskii's FPT (Theorem 40). Three examples are given to illustrate main results.
Let $\alpha>0$, and $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and $n=[\alpha]$, where $[\cdot]$ is The integer part of the real number α.

3.2 Existence of solutions

Before proved the existence of solution for Langevin fractional differential equations with Riesz-Caputo derivative, we first shall present and prove the following lemma.

Lemma 52. *Let $g\in C([0,T],\mathbb{R})$ and $x\in C^2([0,T],\mathbb{R})$. Then the problem*

$$\begin{cases} {}_0^{RC}D_T^{\alpha}({}_0^{RC}D_T^{\beta}+\chi)x(t)=g(t), \ 0<t<T, \\ x(0)+x(T)=0 \ , \ x'(0)+x'(T)=0, \end{cases} \tag{3.2}$$

is equivalent to the integral equation given by

$$\begin{aligned} x(t) &= \frac{-\chi}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds - \frac{\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}x(s)ds \\ &+ \frac{\chi^2 T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^{\beta})}\int_0^T (T-s)^{\beta-2}x(s)ds \\ &- \frac{\chi T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})}\int_0^T (T-s)^{\alpha+\beta-2}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}g(s)ds \end{aligned} \tag{3.3}$$

Also, we define the notations:

$$\begin{aligned}\phi_1 = &\frac{2|\chi|T^{\beta}}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^{\beta}|} \\ &+ \frac{|\chi| L_1 T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} + \frac{2L_1 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},\end{aligned} \tag{3.4}$$

$$k_1 = \left(\frac{2|\chi|T^{\beta}}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^{\beta}|}\right), \tag{3.5}$$

$$k_2 = \left[\frac{|\chi| T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^{\beta}|} + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right], \tag{3.6}$$

Proof. Applying the integral operator ${}_0I_T^{\alpha}$ to both sides of (3.2)and by using Lemma(18), we get

$${}_0^{RC}D_T^{\beta}x(t) + \chi x(t) + \frac{{}_0^{RC}D_T^{\beta}x'(T)T}{2} + \frac{1}{2}\chi x'(t)T =_0 I_T^{\alpha} g(t). \tag{3.7}$$

Applying the integral operator ${}_0I_T^{\beta}$ to the both sided of (3.7)and using the Lemma (18), we obtain

$$x(t) = \frac{1}{2}(x(0)+x(T)) - \chi_0 I_T^{\beta}x(t) - \frac{1}{2}{}_0 I_T^{\beta}x'(T)T +_0 I_T^{\alpha+\beta}g(t). \tag{3.8}$$

Rewriting equation (3.8) under the form

$$\begin{aligned}x(t) = &\frac{-\chi}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds - \frac{\chi}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}x(s)ds \\ &- \frac{\chi T x'(T)}{2\Gamma(\beta+1)}t^{\beta} + \frac{\chi T x'(T)}{2\Gamma(\beta+1)}(T-t)^{\beta} + \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}g(s)ds.\end{aligned} \tag{3.9}$$

Then taking the derivative of (3.9), we get

$$\begin{aligned}x'(t) = &\frac{-\chi}{\Gamma(\beta-1)}\int_0^t (t-s)^{\beta-2}x(s)ds + \frac{\chi}{\Gamma(\beta-1)}\int_t^T (s-t)^{\beta-2}x(s)ds \\ &- \frac{\chi T x'(T)}{2\Gamma(\beta)}t^{\beta-1} - \frac{\chi T x'(T)}{2\Gamma(\beta)}(T-t)^{\beta-1} \\ &+ \frac{\alpha+\beta-1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-2}g(s)ds - \frac{\alpha+\beta-1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-2}g(s)ds.\end{aligned}$$

Using the boundary conditions of (3.1), we deduce

$$\begin{aligned} x'(T) = & \frac{-2\chi(\beta-1)}{2\Gamma(\beta)+\chi T^{\beta}} \int_0^T (T-s)^{\beta-2} x(s)ds \\ & + \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\alpha+\beta-2} g(s)ds. \end{aligned} \tag{3.10}$$

Substituting the value of (3.10) in (3.9), we obtain (3.3). The proof is complete. □

By lemma(52), we define an operator $H : C([0,T]) \to C([0,T])$, associated to (3.1)

$$\begin{aligned} (Hx)(t) = & \frac{-\chi}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)ds - \frac{\chi}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} x(s)ds \\ & + \frac{\chi^2 T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\beta-2} x(s)ds \\ & - \frac{\chi T(t^{\beta}-(T-t)^{\beta})}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^{\beta})} \int_0^T (T-s)^{\alpha+\beta-2} f(s,x(s))ds \\ & + \frac{1}{\Gamma(\alpha+\beta)}{}_0^t (t-s)^{\alpha+\beta-1} f(s,x(s))ds \\ & + \frac{1}{\Gamma(\alpha+\beta)} \int_t^T (s-t)^{\alpha+\beta-1} f(s,x(s))ds. \end{aligned} \tag{3.11}$$

In the next, we obtain some existence and uniqueness results or the boundary value problem(3.1) by using a variety of fixed point theorems.

3.2.1 Existence and uniqueness result via Banach fixed point theorem

Theorem 53. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(H_1) *There exists a constant $L_1 > 0$ such that*

$$|f(t,x) - f(t,y)| \leq L_1 |x-y|,$$

for each $t \in [0,T]$ and $x, y \in \mathbb{R}$.
Then the boundary value problem (84) has a unique solution on $[0,T]$ if

$$\phi_1 < 1,$$

where ϕ_1 is defined by (3.4).

Proof. By using operator H, which is defined by (3.11), we have

$$
\begin{aligned}
|(Hx)(t)-(Hy)(t)| &\le \frac{|\chi|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)-y(s)|ds + \frac{|\chi|}{\Gamma(\beta)}\int_t^T (s-t)^{\beta-1}|x(s)-y(s)|ds \\
&+ \frac{\chi^2 T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\beta-1)(|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\beta-2}|x(s)-y(s)|ds \\
&+ \frac{|\chi|T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s))-f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))-f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}|f(s,x(s)-f(s,y(s))|ds, \\
&\le \frac{|\chi|T^\beta}{\Gamma(\beta+1)}\|x-y\| + \frac{|\chi|T^\beta}{\Gamma(\beta+1)}\|x-y\| + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta-1)(|2\Gamma(\beta)+\chi T^\beta|}\|x-y\| \\
&+ \frac{L_1|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\|x-y\| + \frac{L_1 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|x-y\| + \frac{LT^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|x-y\| \\
&\le \left\{\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi|T^\beta} + \frac{|\chi|L_1 T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|} + \frac{2L_1T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\}\|x-y\|, \\
&\le \phi_1\|x-y\|,
\end{aligned}
$$

for any $x, y \in C([0,T])$, and for each $t \in [0,T]$. Thus implies that $\|Hx - Hy\| \le \phi_1\|x-y\|$. As $\phi_1 < 1$, the operator $H : C([0,T]) \to C([0,T])$ is a contraction mapping. Consequently, the boundary value problem (3.1) has a unique solution on $[0,T]$. □

3.2.2 Existence result via Scheafer fixed point theorem

Theorem 54. *Assume that there exists a positive constant $L_2 > 0$, such that $|f(t,x)| \le L_2$, for $t \in [0,T], x \in \mathbb{R}$. Then the boundary value problem (3.1) has at least one solution on $[0,T]$.*

Proof. **Step 1**: We show that the operator H defined by (3.11) is completely continuous. Observe that the continuity of H follows from the continuity of f.
For a positive constant d, let

$$B_d = \{x \in C([0,T]) : \|x\| \le d\},$$

be a closed bounded subset in $C([0,T])$.
Step 2: H maps bounded sets into bounded sets in $C([0,T])$.

For each $x \in B_d$ and $t \in [0,T]$, we have

$$\begin{aligned}
|(Hx)(t)| \leq{}& \frac{|\chi|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)|ds + \frac{|\chi|}{\Gamma(\beta)}\int_t^T (t-s)^{\beta-1}|x(s)|ds \\
&+ \frac{\chi^2 T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\beta-2}|x(s)|ds \\
&+ \frac{|\chi|T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}|f(s,x(s))|ds \\
\leq{}& \frac{2|\chi|T^\beta}{\Gamma(\beta+1)}\|x(s)\| + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\|x(s)\| \\
&+ \frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|}L_2 + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}L_2 := K.
\end{aligned}$$

Then K is a constant and $\|H(x)\| \leq K$, which implies that H maps bounded sets into bounded sets in $C([0,T])$.

Step 3: H maps bounded sets into equicontinuous sets of $C([0,T])$ (H is completely continuous). Let $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, and $x \in B_d$. Then we have

$$\begin{aligned}
&|(Hx)(t_2)-(Hx)(t_1)| \leq \frac{|\chi|}{\Gamma(\beta)}\int_0^{t_1} \left|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}\right| |x(s)|ds \\
&+ \frac{|\chi|}{\Gamma(\beta)}\int_{t_1}^{t_2} |(t_2-s)^{\beta-1}-(s-t_1)^{\beta-1}||x(s)|ds \\
&+ \frac{|\chi|}{\Gamma(\beta)}\int_{t_2}^{T} \left|(s-t_2)^{\beta-1}-(s-t_1)^{\beta-1}\right| |x(s)|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1} \left|(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}\right| |f(s,x(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2} \left|(t_2-s)^{\alpha+\beta-1}-(s-t_1)^{\alpha+\beta-1}\right| |f(s,x(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_{t_2}^{T} \left|(s-t_2)^{\alpha+\beta-1}-(s-t_1)^{\alpha+\beta-1}\right| f(s,x(s))|ds \\
&+ \frac{\chi^2 T\left(|(T-t_2)^\beta-(T-t_1)^\beta|+|t_2^\beta-t_1^\beta|\right)}{\beta\Gamma(\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\beta-1}|x(s)|ds \\
&+ \frac{|\chi|T\left(|(T-t_2)^\beta-(T-t_1)^\beta|+|t_2^\beta-t_1^\beta|\right)}{\beta\Gamma(\alpha+\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s))|ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|\chi|\rho}{\Gamma(\beta+1)}\left|(t_2-t_1)^\beta+(t_2^\beta-t_1^\beta)\right|+\frac{|\chi|\rho}{\Gamma(\beta+1)}\left|(t_2-t_1)^\beta+(t_2-t_1)^\beta\right| \\
&+\frac{|\chi|\rho}{\Gamma(\beta+1)}\left|(t_2-t_1)^\beta-(t_1-t_2)^\beta\right| \\
&+\frac{L_2}{\Gamma(\alpha+\beta+1)}\left|(t_2-t_1)^{\alpha+\beta}+(t_2-t_1)^{\alpha+\beta}\right| \\
&+\frac{L_2}{\Gamma(\alpha+\beta+1)}\left|(t_2-t_1)^{\alpha+\beta}-(t_{1-2})^{\alpha+\beta}\right| \\
&+\frac{L_2}{\Gamma(\alpha+\beta+1)}\left|(T-t_2)^{\alpha+\beta}-(T-t_1)^{\alpha+\beta}-(t_2-t_1)^{\alpha+\beta}\right| \\
&+\frac{\chi^2T^\beta\left|(T-t_1)^\beta-(T-t_2)^\beta|+(t_2^\beta-t_1^\beta)\right|}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|} \\
&+\frac{|\chi|T^{\alpha+\beta}\left|(T-t_2)^\beta-(T-t_1)^\beta+(t_2^\beta-t_1^\beta)\right|}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|}
\end{aligned}$$

As $t_2-t_1\to 0$, the right-hind side of the above inequality tends to zeros independently of $x\in B_d$. That means H is equicontinuous and by Arzela-Ascoli theorem the operator $H:C([0,T])\to C([0,T])$ is completely continuous.
Step 4: Finally, we consider the set V defined by:

$$V=\{x\in C([0,T])/x=\mu Hx, 0<\mu<1\},$$

and show that V is bounded.
For $x\in V$and $t\in[0,T]$, we have

$$\begin{aligned}
|x(t)| &\leq \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)}+\frac{\chi^2T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\|x(s)\| \\
&+\frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|}L_2+\frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}L_2 \\
&\leq \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)}+\frac{\chi^2T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)d \\
&+\left(\frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|}+\frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)L_2.
\end{aligned}$$

Consequently

$$\|x(t)\|\leq(k_1d+k_2L_2)=G.$$

Then

$$\|x\|\leq G.$$

Therefore, V is bounded. Hence, by theorem (54), the boundary value problem (3.1) has at least one solution on $[0, T]$. □

3.2.3 Existence result via Krasnoselskii's fixed point theorem

Theorem 55. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose that condition (H_1) holds.*
In addition, we assume that the function f satisfies the assumptions:

(H_3) *There exists a nonnegative function $\Omega \in C([0, T], \mathbb{R}^+)$ such that $|f(t, x(t))| \leq \Omega(t)$ for any $(t, x) \in [0, T] \times \mathbb{R}$.*

(H_4) *$L_1 k_2 < 1$, where k_2 is defined by (3.6).*

Then the boundary value problem (3.1) has a least one solution in $[0, T]$.

Proof. We first define two new operators H_1 and H_2 by:

$$
\begin{aligned}
(H_1 x)(t) &= \frac{-\chi}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s) ds - \frac{\chi}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} x(s) ds \\
&\quad + \frac{\chi^2 T(t^\beta - (T-t)^\beta)}{\beta\Gamma(\beta-1)(2\Gamma(\beta) + \chi T^\beta)} \int_0^T (T-s)^{\beta-2} x(s) ds, \qquad (3.12) \\
(H_2 x)(t) &= -\frac{\chi T(t^\beta - (T-t)^\beta)}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta) + \chi T^\beta)} \int_0^T (T-s)^{\alpha+\beta-2} f(s, x(s)) ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s, x(s)) ds \\
&\quad + \frac{1}{\Gamma(\alpha+\beta)} \int_t^T (s-t)^{\alpha+\beta-1} f(s, x(s)) ds, t \in [0, T]. \qquad (3.13)
\end{aligned}
$$

We consider a closed, bounded, convex and nonempty subset of Banach space defined by $C([0, T])$ as $B_\rho = \{x \in C([0, T]), \|x\| \leq \rho\}$,with $sup_{t\in[0,T]}|\Omega(t)| = \|\Omega\|$. We take

$$
\rho \geq \frac{k_2 \|\Omega\|}{(1 - k_1)},
$$

where $k_1 < 1$ and k_1, k_2 are given by (3.5) and (3.6) respectively.

Now, we show that $H_1x + H_2y \in B_\rho$, indeed for any $x, y \in B_\rho$, we have

$$\begin{aligned}
|H_1x(t) + H_2y(t)| \leq{}& \frac{|\chi|}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}|x(s)|ds + \frac{|\chi|}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1}|x(s)|ds \\
&+ \frac{\chi^2 T(t^\beta - (T-t)^\beta)}{\beta\Gamma(\beta-1)(2\Gamma(\beta)+\chi T^\beta)} \int_0^T (T-s)^{\beta-2}|x(s)|ds \\
&+ \frac{\chi T(t^\beta - (T-t)^\beta)}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^\beta)} \int_0^T (T-s)^{\alpha+\beta-2}|f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}|f(s,y(s))|ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_t^T (s-t)^{\alpha+\beta-1}|f(s,y(s))|ds \\
\leq{}& \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\|x(s)\| \\
&+ \left(\frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|} + \frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)\|\Omega\| \\
={}& k_1\rho + k_2\|\Omega\| \leq \rho.
\end{aligned}$$

which implies that $\|H_1x + H_2y\| \leq \rho$. This shows that $H_1x + H_2y \in B_\rho$.
The next step is related to the compactness and continuity of the operator H_1.
Continuity of f implies that the operator H_1 is continuous, also H_1 is uniformly bounded on B_ρ as

$$\begin{aligned}
\|(H_1x)(t)\| \leq{}& \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\|x(s)\| \\
\leq{}& \left(\frac{2|\chi|T^\beta}{\Gamma(\beta+1)} + \frac{\chi^2 T^{2\beta}}{\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\right)\rho \\
={}& k_1\rho.
\end{aligned}$$

Now we will prove the compactness of the operator H_1.
For $t_1, t_2 \in [0,T]$, $t_1 < t_2$ we have

$$\begin{aligned}
|(H_1x)(t_2) - (H_1x)(t_1)| \leq{}& \frac{|\chi|}{\Gamma(\beta)} \int_0^{t_1} |(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}||x(s)|ds \\
&+ \frac{|\chi|}{\Gamma(\beta)} \int_{t_2}^T |(s-t_2)^{\beta-1} - (s-t_1)^{\beta-1}||x(s)|ds \\
&+ \frac{|\chi|}{\Gamma(\beta)} \int_{t_1}^{t_2} |(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}||x(s)|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\chi^2 T\left(|(T-t_2)^\beta-(T-t_1)^\beta|+|t_2^\beta-t_1^\beta|\right)}{\beta\Gamma(\beta-1)|2\Gamma(\beta)+\chi T^\beta|}\int_0^T (T-s)^{\beta-2}|x(s)|ds\\
&\leq \frac{|\chi|}{\Gamma(\beta+1)}|(t_2^\beta-t_1^\beta)-(t_2-t_1)^\beta|\|x(s)\|\\
&+\frac{|\chi|}{\Gamma(\beta+1)}|(T-t_2)^\beta-(T-t_1)^\beta+(t_2-t_1)^\beta|\|x(s)\|\\
&\frac{|\chi|}{\Gamma(\beta+1)}|(t_2-t_1)^\beta-(t_1-t_2)^\beta|\|x(s)\|\\
&+\frac{\chi^2 T\left(|(T-t_1)^\beta-(T-t_2)^\beta|+(t_2^\beta-t_1^\beta)\right)|}{\beta\Gamma(\beta+1)|2\Gamma(\beta)+\chi T^\beta|}\|x(s)\|,
\end{aligned}
$$

we see that the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$, as $t_2 \to t_1$. Thus H_1 is equicontinuous, so H_1 is relatively compact on B_ρ. Therefore, by the conclusion of the Arzela-Ascoli theorem, the operator H_1 is continuous and compact on B_ρ.

Now, we prove that H_2 is contraction mapping.

Let $x, y \in C([0,T])$, and for each $t \in [0,T]$, we have

$$
\begin{aligned}
|H_2x(t)-H_2y(t)| &\leq \frac{|\chi|T|t^\beta-(T-t)^\beta|}{\beta\Gamma(\alpha+\beta-1)(2\Gamma(\beta)+\chi T^\beta)}\int_0^T (T-s)^{\alpha+\beta-2}|f(s,x(s))-f(s,y(s))|ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|f(s,x(s))-f(s,y(s))|ds\\
&+\frac{1}{\Gamma(\alpha+\beta)}\int_t^T (s-t)^{\alpha+\beta-1}|f(s,x(s))-f(s,y(s))|ds\\
&\leq \left(\frac{|\chi|T^{\alpha+2\beta}}{\beta\Gamma(\alpha+\beta)|2\Gamma(\beta)+\chi T^\beta|}+\frac{2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right)L_1\|x-y\|\\
&= k_2L_1\|x-y\|.
\end{aligned}
$$

Thus all the assumptions of theorem (55) are satisfied. So the boundary value problem (3.1) has at least one solution on $[0,T]$. □

3.3 Examples

Example 56. *Consider the following nonlinear Langevin equation with Riesz-Caputo fractional derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{3}{2}}({}_0^{RC}D_T^{\frac{1}{3}} + \frac{1}{7})x(t) = \frac{\pi|x(t)|}{|x(t)|+5}\frac{\cos^2 t}{(\pi+t^2)} \ ,0 < t < 1, \\ x(0) + x(T) = 0 \ , \ x'(0) + x'(T) = 0, \end{cases} \tag{3.14}$$

Here, $\alpha = \frac{3}{2}, \beta = \frac{1}{3}, \chi = \frac{1}{7}, L_1 = \frac{1}{5\pi}$. *Moreover,*

$$f(t,x) = \frac{\pi|x(t)|}{|x(t)|+5}\frac{\cos^2 t}{(\pi+t^2)}.$$

Hence, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{5\pi}\|x-y\|.$$

The condition (H_1) *is satisfied with* $L_1 = \frac{1}{5\pi}, \phi = 0,4032$. *So* $\phi_1 < 1$.
Then by using theorem (53) the boundary value problem (3.14) has a unique solution on $[0,1]$.

Example 57. *Consider the following nonlinear Langevin equation with Riesz-Caputo fractional derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{11}{6}}({}_0^{RC}D_T^{\frac{1}{5}} + \frac{1}{9})x(t) = \frac{\cos^2 t+1}{3}\sin^{-1}\left(\frac{|x|}{|x|+1}\right) \ ,0 < t < \frac{1}{2}, \\ x(0) + x(T) = 0 \ , \ x'(0) + x'(T) = 0, \end{cases} \tag{3.15}$$

Here, $\alpha = \frac{11}{6}, \beta = \frac{1}{5}, \chi = \frac{1}{9}, L_1 = \frac{2}{3}, \phi_1 = 0,3781$. *Also we have* $|f(t,x)| \leq \frac{\pi}{3}$, *with* $L_2 = \frac{\pi}{3}$. *Clearly the hypothesis of theorem (54)is satisfied. Thus boundary value problem (3.15)admits at least a solution on* $[0, \frac{1}{2}]$.

Example 58. *Consider the following nonlinear Langevin equation with Riesz-Caputo fractional derivative:*

$$\begin{cases} {}_0^{RC}D_T^{\frac{9}{5}}({}_0^{RC}D_T^{\frac{1}{2}} + \frac{1}{12})x(t) = \frac{1}{\sqrt{t^2+121}}\left(\frac{|x|}{(1+|x|)}\right) + \frac{e^t}{2} \ ,0 < t < 2, \\ x(0) + x(T) = 0 \ , \ x'(0) + x'(T) = 0, \end{cases} \tag{3.16}$$

Here, $\alpha = \frac{9}{5}, \beta = \frac{1}{2}, \chi = \frac{1}{12}$,

$$f(t,x) = \frac{1}{\sqrt{t^2+121}}\frac{|x|}{(1+|x|)} + \frac{e^t}{2}.$$

Moreover,

$$|f(t,x)| \leq \frac{1}{\sqrt{t^2+121}} + \frac{e^t}{2}.$$

Hence, we have

$$|f(t,x) - f(t,y)| \leq \frac{1}{11}\|x-y\|,$$

with $L_1 = \frac{1}{11}, k_2 = 0,3584, L_1 k_2 = 0,0326$. *So* $L_1 k_2 < 1$.
Then by using theorem (55) the boundary value problem (3.16) has at least a solution on $[0,2]$.

Chapter 4

Katugampola fractional differential equation with Erdélyi-Kober integral boundary conditions

4.1 Introduction

This chapter investigates the following Katugampola fractional differential equation with Erdélyi-Kober fractional integral boundary conditions:

$$\begin{cases} D^{\rho,\alpha}u(t) + h(t, u(t)) = 0, & 0 < t < T, \\ u(0) = 0, & \\ u'(T) = \lambda I_{\eta}^{\gamma,\delta} u'(\xi), & 0 < \xi < T, \end{cases} \tag{4.1}$$

where $D^{\rho,\alpha}$ is the Katugampola derivative of order $1 < \alpha < 2$, $\rho > 0$ and $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_{\eta}^{\gamma,\delta}$ denotes Erdélyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$, λ, $\gamma \in \mathbb{R}$. Some new existence and uniqueness results are obtained using nonlinear's contraction principle and Krasnoselskii's and Leray-Schauder's fixed point theorems. Four examples are given in the last section to illustrate the obtained results.

4.2 Existence of solutions

For the existence of solutions for the problem (4.1), we need the following auxiliary lemma.

Lemma 59. *[?] Let $\alpha, \rho > 0$, if $u \in C[0, T]$, then we have the following properties.*

(i) *The fractional differential equation* $D^{\rho,\alpha}_{0^+}u(t) = 0$ *admits a solution defined by:*

$$u(t) = c_0 + c_1 t^{\rho} + c_2 t^{2\rho} + ... + c_n t^{(n-1)\rho},$$

where $c_i \in \mathbb{R}$, *with* $i = 0, 1, 2, ..., n, n = [\alpha] + 1$.

(ii) *Let* $\alpha > 0$, *then*

$$I^{\rho,\alpha} D^{\rho,\alpha}_{0^+} u(t) = u(t) + c_0 + c_1 t^{\rho} + c_2 t^{2\rho} + ... + c_n t^{(n-1)\rho},$$

where $c_i \in \mathbb{R}$ *and* $n = [\alpha] + 1$.

Lemma 60. *Let* $1 < \alpha < 2$ *and* $\lambda \in \mathbb{R}$. *A function* $u \in C([0, T], \mathbb{R})$ *is a solution of nonlinear Katugampola fractional integral equation*

$$u(t) = \frac{-T^{\rho-1}t^{\rho}}{A} I^{\rho,\alpha-1} g(T) + \frac{t^{\rho}\lambda}{A} I^{\gamma,\delta}_{\eta} \xi^{\rho-1} I^{\rho,\alpha-1} g(\xi) - I^{\rho,\alpha} g(t), \tag{4.2}$$

if and only if u is a solution of the Katugampola fractional differential equation with Erdélyi-Kober fractional integral conditions

$$\begin{cases} D^{\rho,\alpha}u(t) + g(t) = 0, & 0 < t < T, \\ u(0) = 0, & \\ u'(T) = \lambda I^{\gamma,\delta}_{\eta} u'(\xi), & 0 < \xi < T. \end{cases} \tag{4.3}$$

Proof. Applying Lemma (59) to equation (4.3), we obtain

$$u(t) = -c_0 - c_1 t^{\rho} - I^{\rho,\alpha} g(t), \tag{4.4}$$

with $c_0, c_1 \in \mathbb{R}$. The condition $u(0) = 0$ implies that $c_0 = 0$.
Thus

$$u'(t) = -\rho c_1 t^{\rho-1} - t^{\rho-1} I^{\rho,\alpha-1} g(t) \tag{4.5}$$

Combining the Erdélyi-Kober fractional integral with (4.5), we get

$$\lambda I^{\gamma,\delta}_{\eta} u'(\xi) = -\rho c_1 \lambda \xi^{\rho-1} \frac{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} - \lambda I^{\gamma,\delta}_{\eta} \xi^{\rho-1} I^{\rho,\alpha-1} g(\xi).$$

$$\begin{aligned} u'(T) &= -\rho c_1 T^{\rho-1} - T^{\rho-1} I^{\rho,\alpha-1} g(T), \\ &= -\rho c_1 \lambda \xi^{\rho-1} \frac{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} - \lambda I^{\gamma,\delta}_{\eta} \xi^{\rho-1} I^{\rho,\alpha-1} g(\xi). \end{aligned}$$

Solving the above equation for c_1 and choosing

$$A = \rho\lambda\xi^{\rho-1}\frac{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} - \rho T^{\rho-1},$$

we obtain

$$c_1 = \frac{T^{\rho-1}}{A} I^{\rho,\alpha-1} g(T) - \frac{\lambda}{A} I_\eta^{\gamma,\delta} \xi^{\rho-1} I^{\rho,\alpha-1} g(\xi).$$

Substituting the constant c_1 into (4.2), we find (4.2). □

Also, we consider the notations:

$$\phi = \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho(\alpha+1)-1} + \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1} + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha}, \tag{4.6}$$

$$\Omega_1 = \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1}. \tag{4.7}$$

In the following section, we investigate existence and uniqueness results for the boundary value problem (4.1).

4.2.1 Existence and Uniqueness Result via Banach's Fixed Point Theorem

We defined the operator $H : C([0,T]) \to C([0,T])$ associated to the problem (4.1) as

$$(Hu)(x) = -\frac{T^{\rho-1}t^\rho}{A} I^{\rho,\alpha-1} h(s,u(s))(T) + \frac{t^\rho\lambda}{A} I_\eta^{\gamma,\delta} \xi^{\rho-1} I^{\rho,\alpha-1} h(s,u(s))(\xi) - I^{\rho,\alpha} h(s,u(s))(t). \tag{4.8}$$

We use the following expressions:

$$I^{\rho,\alpha} h(r,u(r))(\xi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\xi (\xi^\rho - r^\rho)^{\alpha-1} r^{\rho-1} h(r,u(r))dr,$$

$$I_\eta^{\gamma,\delta} I^{\rho,\alpha} h(r,u(r))(\xi) = \frac{\eta\xi^{-\eta(\gamma+\delta)}\rho^{1-\alpha}}{\Gamma(\alpha)\Gamma(\delta)} \int_0^\xi \int_0^r \frac{r^{\eta\gamma+\eta-1}(r^\rho - t^\rho)^{\alpha-1} t^{\rho-1}}{(\xi^\eta - r^\eta)^{1-\delta}} h(t,u(t))dtdr,$$

where $\xi \in [0,T]$.

Theorem 61. *Let $h : [0,T] \to \mathbb{R}$ be a continuous function. Assume that:*

(H_1) *there exists a positive constant* L *such that*

$$|h(t,u) - h(t,v)| \leq L\|u - v\|,$$

for each $t \in [0,T]$ *and* $u, v \in \mathbb{R}$.

(H_2) $L\phi < 1$, *where* ϕ *is defined by 4.6.*

Then the boundary value problem (4.1) has a unique solution on $[0,T]$.

Proof. By using the operator H defined by the formula (4.8) and applying the Banach contraction mapping principle, we will show that the operator H has a unique fixed point.

For any $u, v \in C([0,T])$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
|Hu(t) - Hv(t)| \leq & \frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(T) \\
& + \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(\xi) \\
& + I^{\rho,\alpha}|h(s,u(s)) - h(s,v(s))|(t), \\
& \leq \frac{L\|u-v\|T^{2\rho-1}}{A} I^{\rho,\alpha-1}(1)(T) \\
& + \frac{L\|u-v\|T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}(1)(\xi) \\
& + L\|u-v\| I^{\rho,\alpha}(1)(T), \\
& \leq L\|u-v\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho(\alpha+1)-1} + \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1} \right. \\
& \left. + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha} \right\}, \\
& = L\phi\|u-v\|.
\end{aligned}$$

This implies that $\|Hu - Hv\| \leq L\phi\|u - v\|$ because $L\phi < 1$.

The operator $H : C([0,T]) \to C([0,T])$ is a contraction mapping, therefore, we deduce by Banach's contraction principle mapping, that the operator H has a fixed point which is the unique solution of problem (4.1) on $[0,T]$. □

4.2.2 Existence and Uniqueness Result via Boy-Wong Fixed Point Theorem

Theorem 62. *[12] Let* $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that the following condition holds:*

(H_3) $|h(t,u)-h(t,v)| \leq k(t)\frac{\|u-v\|}{B+\|u-v\|}$, *for* $t \in [0,T]$, *where* $k : [0,T] \to \mathbb{R}^+$ *is a given function.*

Then the problem (4.1) has a unique solution on $[0,T]$.

Proof. Let us define the continuous and nondecreasing function, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\begin{cases} \varphi(r) = \frac{Br}{B+r}, & \forall r > 0, \\ \varphi(0) = 0, & \varphi(r) < r, \end{cases}$$

where $B := \frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}k(T) + \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}k(\xi) + I^{\rho,\alpha}k(T)$.

For any $u, v \in C([0,T])$ and for each $t \in [0,T]$, one has

$$\begin{aligned}
|Hu(t) - Hv(t)| &\leq \frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(T) \\
&+ \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(\xi) \\
&+ I^{\rho,\alpha}|h(s,u(s)) - h(s,v(s))|(t), \\
&\leq \frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}\left(k(s)\frac{\|u-v\|}{B+\|u-v\|}\right)(T) \\
&+ \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}\left(k(s)\frac{\|u-v\|}{B+\|u-v\|}\right)(\xi) \\
&+ I^{\rho,\alpha}\left(k(s)\frac{\|u-v\|}{B+\|u-v\|}\right)(T), \\
&\leq \frac{\varphi(\|u-v\|)}{B}\left\{\frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}k(T) + \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}k(\xi) + J^{\rho,\alpha}k(T)\right\}. \\
&= \varphi(\|u-v\|).
\end{aligned}$$

This implies that $\|Tu - Tv\| \leq \varphi(\|u-v\|)$. Therefore T is a nonlinear contractions. Hence by Theorem(62) the operator T has a fixed point which is solution of the problem (4.1), which completes the proof. □

4.2.3 Existence Result via Krassnoselskii's Fixed Point Theorem

Theorem 63. *Let* $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and suppose that the condition (*H_1*) holds and the function* h *satisfies the assumptions:*

(H_4) *There exists a nonnegative function* $\Theta \in (C[0,T], \mathbb{R})$ *such that* $|h(t,u(t))| \leq \Theta(t)$ *for any* $(t,u) \in [0,T] \times \mathbb{R}$,

(H_5) $L\Omega_1 < 1$, where Ω_1 is defined by (4.7).

Then the boundary value problem (4.1) has a least one solution in $[0,T]$.

Proof. We first define two new operators T_1 and T_2 as

$$(T_1u)(t) = -\frac{T^{\rho-1}t^{\rho}}{A} I^{\rho,\alpha-1}h(s,u(s))(T), \tag{4.9}$$

$$(T_2u)(t) = \frac{t^{\rho}\lambda}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}h(s,u(s))(\xi) - I^{\rho,\alpha}h(s,u(s))(t), \quad t\in[0,T]. \tag{4.10}$$

Then we consider a closed, bounded, convex and nonempty subset of the Banach space X as $B_d = \{u \in C([0,T]), \|u\| \leq d\}$ with, $\|\Theta\|\phi \leq d$, where ϕ is defined by (4.6).

Now for any $u, v \in B_d$, we have

$$\begin{aligned}
|T_1u(t) + T_2v(t)| &\leq \frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}|h(s,u(s))|(T) \\
&+ \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}|h(s,u(s))|(\xi) \\
&+ I^{\rho,\alpha}|h(s,u(s))|(T) \\
&\leq \frac{T^{2\rho-1}\|\Theta\|}{A} I^{\rho,\alpha-1}(1)(T) \\
&+ \frac{T^{\rho}|\lambda|\|\Theta\|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1}I^{\rho,\alpha-1}(1)(\xi) \\
&+ \|\Theta\| I^{\rho,\alpha}(1)(T) \\
&\leq \|\Theta\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho(\alpha+1)-1} + \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1} \right. \\
&\left. + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha} \right\} \\
&= \|\Theta\|\phi \leq d.
\end{aligned}$$

Therefore, it's clear that $\|T_1u(t) + T_2v(t)\| \leq d$. Hence $T_1u(t) + T_2v(t) \in B_d$.
The next step concerns the compactness and continuity of the operator T_1. Continuity of h implies that the operator T_1 is continuous and uniformly bounded on B_d as

$$\|T_1\| \leq \|\Theta\| \frac{\rho^{1-\alpha}T^{\rho(\alpha+1)-1}}{\Gamma(\alpha)}.$$

Now we prove the compactness of the operator T_1. For $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$|T_1u(t_2) - T_1u(t_1)| \leq \|\Theta\| \frac{\rho^{1-\alpha}T^{\rho\alpha-1}}{\Gamma(\alpha)} |t_2^\alpha - t_1^\alpha|,$$

which is independent of u and tends to zero when $t_2 - t_1 \to 0$. Thus T_1 is equicontinuous. By Arzela-Ascoli theorem, T_1 is compact on B_d.
Now, we prove that T_2 is a contraction mapping.
For $u, v \in C([0,T])$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
|T_2u(t) - T_2v(t)| &\leq I^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(T) \\
&+ \frac{T^\rho|\lambda|}{A} I_\eta^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}|h(s,u(s)) - h(s,v(s))|(\xi) \\
&\leq L\|u-v\| I^{\rho,\alpha-1}(1)(T) \\
&+ \frac{T^\rho|\lambda| \leq L\|u-v\|}{A} I_\eta^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}(1)(\xi) \\
&\leq \frac{\rho^{-\alpha}L\|u-v\|}{A\Gamma(\alpha+1)} T^{\rho\alpha} + L\|u-v\| \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1}, \\
&= L\|u-v\| \left\{ \frac{\rho^{-\alpha}}{A\Gamma(\alpha)} T^{\rho\alpha} + \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1} \right\},
\end{aligned}$$

which implies that $\|T_2u(t) - T_2v(t)\| \leq L\Omega_1\|u-v\|$. As $L\Omega_1\|u-v\| < 1$, the operator T_2 is a contraction. Thus all the assumption of Theorem (40) are satisfied. So this implies that the problem (4.1) has at least one solution on $[0,T]$. □

4.2.4 Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 64. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuous function. Assume that*

(H_6) *There exist a nonnegative function $z \in C([0,T], \mathbb{R})$ and a continuous nondecreasing function $\Theta : [0,\infty) \to [0,\infty)$ such that $|f(t,u)| \leq z(t)\Theta(\|u\|)$, for all $(t,u) \in [0,T] \times \mathbb{R}$,*

(H_7) *There exists a constant $N > 0$ such that*

$$\frac{N}{\phi\|z\|\Theta(N)} > 1,$$

where ϕ is defined as in (4.6). Then the problem (4.1) has a least one solution on $[0,T]$.

Proof. Let $B_R = \{u \in C([0,T])/\|u\| \leq R\}$ be a closed bounded subset in $C([0,T],\mathbb{R})$. Let H be the operator defined by (4.8). As a first step, we show that the operator H maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. Then for $t \in [0,T]$, we have

$$\begin{aligned}
|Hu(t)| &\leq \frac{T^{2\rho-1}}{A} I^{\rho,\alpha-1}|h(s,u(s))|(T) \\
&+ \frac{T^{\rho}|\lambda|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}|h(s,u(s))|(\xi) \\
&+ I^{\rho,\alpha}|h(s,u(s))|(T), \\
&\leq \frac{T^{\rho\alpha-1}\Theta(\|u\|)}{A} I^{\rho,\alpha-1} z(s)(T) \\
&+ \frac{T^{\rho}|\lambda|\Theta(\|u\|)}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1} z(s)(\xi) \\
&+ \Theta(\|u\|) I^{\rho,\alpha} z(s)(T), \\
&\leq \frac{T^{2\rho-1}\Theta(\|u\|)}{A} I^{\rho,\alpha-1}\|z\|(T) \\
&+ \frac{T^{\rho}|\lambda|\Theta(\|u\|)}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}\|z\|(\xi) \\
&+ \Theta(\|u\|) I^{\rho,\alpha}\|z\|(T), \\
&\leq \Theta(\|u\|)\|z\| \left\{ \frac{\rho^{1-\alpha}}{A\Gamma(\alpha)} T^{\rho(\alpha+1)-1} + \frac{\rho^{1-\alpha}|\lambda|\Gamma(\gamma + (\frac{\rho-1}{\eta}) + 1)}{A\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho-1}{\eta}) + \delta + 1)} T^{\rho(\alpha+1)-1} + \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)} T^{\rho\alpha} \right\}, \\
&= \phi\Theta(\|u\|)\|z\|.
\end{aligned}$$

Consequently, $\|Hu(t)\| \leq \phi\Theta(\|u\|)\|z\|$. Next, we show that the map $H : C([0,T]) \to C([0,T])$ is completely continuous. Therefore, we will prove that the operator H maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Indeed let $t_1, t_2 \in [0,T]$, with $t_1 < t_2$ and $u \in B_R$, then we have

$$\begin{aligned}
|Hu(t_2) - Hu(t_1)| &\leq \frac{T^{\rho-1}|t_1^{\rho} - t_2^{\rho}|}{A} I^{\rho,\alpha-1}|h(s,u(s))|(T) + |I^{\rho,\alpha}h(s,u(s))(t_2) - I^{\rho,\alpha}h(s,u(s))(t_1)| \\
&+ \frac{|\lambda||t_1^{\rho} - t_2^{\rho}|}{A} I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}|h(s,u(s))|(\xi) \\
&\leq \frac{T^{\rho-1}|t_1^{\rho} - t_2^{\rho}|}{A} \Theta(\|u\|) z(s) I^{\rho,\alpha-1}(1)(T) + \Theta(\|u\|) z(s) |I^{\rho,\alpha}(1)(t_2) - I^{\rho,\alpha}(1)(t_1)| \\
&+ \frac{|\lambda||t_1^{\rho} - t_2^{\rho}|}{A} \Theta(\|u\|) z(s) I_{\eta}^{\gamma,\delta}\xi^{\rho-1} I^{\rho,\alpha-1}(1)(\xi)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{|t_1^\rho - t_2^\rho|}{A}\Theta(\|u\|)\|z\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}T^{\rho\alpha-1} + \Theta(\|u\|)\|z\|\frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}|(t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)})| \\
&+ \frac{|\lambda||t_1^\rho - t_2^\rho|}{A}\Theta(\|u\|)\|z\|\frac{\rho^{1-\alpha}\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + 1)}{\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + \delta + 1)}T^{\rho\alpha-1}, \\
&\leq \frac{|t_1^\rho - t_2^\rho|}{A}\Theta(R)\|z\|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}T^{\rho\alpha-1} + \Theta(R)\|z\|\frac{\rho^{-\alpha}}{\Gamma(\alpha)}|(t_2^{\rho(\alpha-1)} - t_1^{\rho(\alpha-1)})| \\
&+ \frac{|\lambda||t_1^\rho - t_2^\rho|}{A}\Theta(R)\|z\|\frac{\rho^{1-\alpha}\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + 1)}{\Gamma(\alpha)\Gamma(\gamma + (\frac{\rho\alpha-1}{\eta}) + \delta + 1)}T^{\rho\alpha-1}.
\end{aligned}
$$

It is clear that the right-hand side of above inequality tends to zero independently of $u \in B_R$ as $t_2 - t_1 \to 0$. Therefore by the Ascoli-Arzela theorem, the operator $H : C([0,T]) \to C([0,T])$ is completely continuous.
In the last step we show that the operator H has a fixed point. Let u be a solution of $H(u) = u$, then for each $t \in [0,T]$,

$$\|Hu\| = \|u\| \leq \phi\|z\|\Theta(\|u\|),$$

which implies that

$$\frac{\|u\|}{\phi\|z\|\Theta(\|u\|)} \leq 1.$$

From (H_7), there exists $N > 0$ such that $\|u\| \neq N$. Let us set $G = \{u \in C([0,T]) : \|u| < N\}$.
Then the operator $H : \overline{G} \to C([0,T])$ is continuous and completely continuous. Consequently, there doesn't exist any $u \in \partial G$ such that $u = \mu Hu$ for some $\mu \in (0,1)$. Assume that there exists $u \in \partial G$ such that $u = \mu Hu$ for some $\mu \in (0,1)$. Then

$$\|u\| = \|\mu Hu\| \leq \|Hu\| \leq \phi\|z\|\Theta(\|u\|),$$

$$\frac{\|u\|}{\phi\|z\|\Theta(\|u\|)} \leq 1.$$

This contradicts $\frac{\|u\|}{\phi\|z\|\Theta(\|u\|)} > 1$. Consequently, by nonlinear alternative Leray-Schauder principal, we conclude, that H has a fixed point $u \in \overline{G}$, which is a solution of problem (4.1), this completes the proof. □

4.3 Examples

Example 65. *Consider the following nonlinear Katugampola fractional differential equation with Erdélyi-Kober fractional integral conditions:*

$$\begin{cases} D^{1,\frac{3}{2}}u(t) = \left(\frac{|u|}{|u|+1}\right)\frac{e^{-\sin t}}{\pi+2} + \frac{1}{2}\ ,\ t \in [0,1], \\ u(0) = 0\ , \\ u'(1) = \frac{3}{5} I_{\frac{1}{5}}^{\frac{3}{4},\frac{\sqrt{2}}{2}} u'(\frac{1}{2}). \end{cases} \tag{4.11}$$

Here, $\alpha = \frac{3}{2}$, $\rho = 1$, $\gamma = \frac{3}{4}$, $\eta = \frac{1}{5}$, $\delta = \frac{\sqrt{2}}{2}$, $\xi = \frac{1}{2}$, $\lambda = \frac{3}{5}$.

$$f(t,u) = \left(\frac{|u|}{|u|+1}\right)\frac{e^{-\sin t}}{\pi+2} + \frac{1}{2}.$$

Hence, we have

$$|f(t,u) - f(t,v)| \leq \frac{1}{\pi+2}\|u-v\|.$$

Assumption (H_1) is satisfied with $L = \frac{1}{\pi+2}$. *Using the given value, we get* $\phi = 3,3839$. *Therefore* $L\phi = 0,6581 < 1$, *which implies that assumption (H_2) holds. Using theorem (61), we deduce that the boundary value problem (4.11) has a unique solution on* $[0,1]$.

Example 66. *Consider the following nonlinear Katugampola fractional differential equation with Erdélyi-Kober fractional integral conditions:*

$$\begin{cases} D^{1,\frac{7}{4}}u(t) = \frac{t^2}{\pi\sqrt{t^2+9}}\left(\frac{|u|}{|u|+5}\right) + \frac{e^t+t}{2}\ ,\ t \in [0,\frac{1}{2}], \\ u(0) = 0\ , \\ u'(\frac{1}{2}) = \frac{8}{3} I_{\frac{12}{7}}^{\frac{\sqrt{5}}{3},\frac{1}{\sqrt{6}}} u'(\frac{3}{11})\ , \end{cases} \tag{4.12}$$

Here, $\alpha = \frac{7}{4}$, $\rho = 1$, $\gamma = \frac{\sqrt{5}}{3}$, $\eta = \frac{12}{7}$, $\delta = \frac{1}{\sqrt{6}}$, $\xi = \frac{3}{11}$, $\lambda = \frac{8}{3}$, $B = 0,0245$ *and*

$$f(t,u) = \frac{t^2}{\pi\sqrt{t^2+9}}\left(\frac{|u|}{|u|+5}\right) + \frac{e^t+t}{2}.$$

Choosing $k(t) = \frac{t^2}{3\pi}$, *we get*

$$|f(t,u) - f(t,v)| \leq \frac{t^2}{3\pi}\left(\frac{|u-v|}{0,0751+|u-v|}\right).$$

Clearly, all the assumptions of Theorem (62) are satisfied, witch implies that the *problem (4.12) has at least one solution on* $[0, \frac{1}{2}]$.

Example 67. *Consider the following nonlinear Katugampola fractional differential equation with Erdélyi-Kober fractional integral conditions:*

$$\begin{cases} D^{2,\frac{9}{5}}u(t) = \sin\left(\frac{|u|}{|u|+1}\right)\frac{e^{-2t}}{3(\pi+7)} + \frac{t^2+1}{2},\ t \in [0,2] \\ u(0) = 0, \\ u'(2) = \frac{3}{8} I_{\frac{7}{3}}^{\frac{\sqrt{2}}{2},\frac{1}{4}} u'(\frac{3}{2}). \end{cases} \tag{4.13}$$

Here, $\alpha = \frac{9}{5}$, $\rho = 2$, $\gamma = \frac{\sqrt{2}}{2}, \eta = \frac{7}{3}$, $\delta = \frac{1}{4}$, $\xi = \frac{3}{2}$, $\lambda = \frac{3}{8}$, *and*

$$|f(t,u) - f(t,v)| \leq \frac{1}{3(\pi+7)}|u-v|,$$

$$|f(t,u)| \leq \frac{e^{-2t}}{3(\pi+7)} + \frac{t^2+1}{2},$$

with $L = \frac{1}{3(\pi+7)}$, $\phi = 8,4101$, $L\phi = 0,2764$, $\Omega_1 = 3,8622$, $L\Omega_1 = 0,1269 < 1$.

Again, the hypothesis of Theorem (63) are satisfied and, as a consequence, the problem (4.13) has at least one solution on $[0, 2]$.

Example 68. *Consider the following nonlinear Katugampola fractional differential equation with Erdélyi-Kober fractional integral conditions :*

$$\begin{cases} D^{3,\frac{11}{6}}u(t) = \left(\frac{u^2(t)}{|u|+1} + 1\right)\left(\frac{\sqrt{t}+1}{8}\right),\ t \in [0, \frac{9}{16}], \\ u(0) = 0, \\ u'(\frac{9}{16}) = \frac{5}{11} I_{\frac{1}{9}}^{\frac{4}{13},\frac{1}{5}} u'(\frac{3}{7}). \end{cases} \tag{4.14}$$

Here $\alpha = \frac{11}{6}$, $\rho = 3$, $\gamma = \frac{4}{13}$, $\eta = \frac{1}{9}$, $\delta = \frac{1}{5}$, $\xi = \frac{3}{8}$,$\lambda = \frac{5}{11}$.

Moreover

$$|f(t,u)| = \left|\left(\frac{u^2(t)}{|u|+1} + 1\right)\left(\frac{\sqrt{t}+1}{8}\right)\right| \leq \frac{\sqrt{t}+1}{8}(|u|+1).$$

We choose $z(t) = \frac{\sqrt{t}+1}{8}$ *and* $\Theta(\|u\|) = \|u\| + 1$. *We have* $\|z\| = \frac{7}{32}$ *and* $\phi = 0,0115$. *Now, we need to show that there exists* $N > 0$ *such that*

$$\frac{N}{\Theta(N)\|z\|\phi} > 1,$$

and such $N > 0$ *exists if* $1 - \|z\|\phi > 0$. *A straightforward calculus give* $\|z\|\phi = 0,0025 < 1$, *assumption* H_7 *is satisfied. Hence using Theorem (64), the boundary* *value problem (4.14) has at least one solution on* $[0, \frac{9}{16}]$.

4.4 Conclusion

In this chapter with the help of standard fixed point theorems type, we obtained conditions for existence of at least one solution of a Katugampola fractional differential equation with Erdélyi-Kober fractional integral boundary conditions. In the future it seems interesting to obtain sufficient conditions to ensure Ulam-Hyers and Ulam-Hyers-Rassias stabilities.

Chapter 5

Existence results for generalized Caputo hybrid fractional integro-differential equations

5.1 Introduction

The class of fractional differential equations includes an unknown derivative and a nonlinear hybrid function. The importance of hybrid fractional differential equations is that they involve different dynamical systems in mathematics and applied physics, for example, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows [72, 77].
In this chapter,we prove the existence of solutions for hybrid fractional integro-differential equations involving ψ-Caputo derivative of the form

$$\begin{cases} {}^{c}D_{a^+}^{\nu;\psi}\left[\frac{z(\tau)-\sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau,z(\tau))}{\mathbb{G}(\tau,z(\tau))}\right] = \mathbb{H}(\tau,z(\tau)), \tau \in \mathrm{J}=[a,b], \\ z(a)=0, \end{cases} \tag{5.1}$$

where ${}^{c}D_{a^+}^{\nu;\psi}$ is the ψ-Caputo fractional derivative of order $\nu \in (0,1], I_{a^+}^{\theta;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $\theta > 0, \theta \in \{\sigma_1,\sigma_2,\ldots,\sigma_m\}, \sigma_k > 0, k=1,2,\ldots,m$. $\mathbb{G} \in C(\mathrm{J}\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $\mathbb{F}_k, \mathbb{H} \in C(\mathrm{J}\times\mathbb{R},\mathbb{R})$,$(k=1,2,\ldots,m)$.
We use an hybrid fixed point theorem for a sum of three operators due to Dhage for proving the main results. An example is provided to illustrate main results.
We denote by $C([a,b],\mathbb{R})$ the Banach space of all continuous functions z from $[a,b]$

into $\mathbb{R}$ with the supremum norm

$$\|z\|_C = \sup_{\tau\in[a,b]} |z(\tau)|,$$

and the multiplication in $C(\mathrm{J})$ by

$$(zy)(\tau) = z(\tau)y(\tau).$$

Clearly, $C(\mathrm{J})$ is a Banach algebra with respect to the supremum norm and multiplication in it.

5.2 Existence Theorem

In this section,we establish an existence result for the problem (5.1). Firstly, we need the following lemma.

Lemma 69. *Let $\nu \in (0,1]$ be fixed and functions $\mathbb{F}_i, (i = 1,\cdots,n), \mathbb{G}, \mathbb{H}$ satisfy problem (5.1). Then the function $z \in C([a,b],\mathbb{R})$ is a solution of the hybrid fractional integro-differential problem (5.1) if and only if it satisfies the integral equation*

$$z(\tau) = \mathbb{G}(\tau, z(\tau)) \left[\mathbb{M}_\psi + I_{a^+}^{\nu;\psi}\mathbb{H}(\tau, z(\tau))\right] + \sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(\tau, z(\tau)), \quad \tau \in [a,b], \tag{5.2}$$

where

$$\mathbb{M}_\psi = \frac{-\sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi}\mathbb{F}_k(a,0)}{\mathbb{G}(a,0)} \tag{5.3}$$

For the proof of Lemma 69, it is useful to refer to [72, 77].

Theorem 70. *Assume that:*

(H_1) *Let the functions $\mathbb{G} : \mathrm{J}\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ and, $\mathbb{F}_k, \mathbb{H} : \mathrm{J}\times\mathbb{R} \to \mathbb{R}$, $k = 0,1,2,...,m$ are continuous*

(H_2) *There exists two positive functions $\mathbb{L}_{\mathbb{F}_k}, \mathbb{L}_{\mathbb{G}}$, $k = 0,1,2,...,m$ with bounds $\|\mathbb{L}_{\mathbb{F}_k}\|$ and $\|\mathbb{L}_{\mathbb{G}}\|$, $k = 0,1,2,...,m$, respectively, such that*

$$|\mathbb{F}_k(\tau, z(\tau)) - \mathbb{F}_k(\tau, \overline{z}(\tau))| \leq \mathbb{L}_{\mathbb{F}_k}(\tau)|z - \overline{z}|, \quad k = 0,1,2,...,m, \tag{5.4}$$

and

$$|\mathbb{G}(\tau, z(\tau)) - \mathbb{G}(\tau, \overline{z}(\tau))| \leq \mathbb{L}_{\mathbb{G}}(\tau)|z - \overline{z}|, \tag{5.5}$$

for all $(\tau, z, \overline{z}) \in \mathrm{J} \times \mathbb{R} \times \mathbb{R}$.

(H_3) *There exist a function* $p \in C(\mathrm{J}, \mathbb{R}_+)$ *and a continuous nondecreasing function* $\Omega : [0, \infty) \to (0, \infty)$ *such that*

$$|\mathbb{H}(\tau, z(\tau))| \leq p(\tau)\Omega(|z|), \tag{5.6}$$

for all $\tau \in \mathrm{J}$ *and* $z \in \mathbb{R}$.

(H_4) *There exists a number* $r > 0$ *such that*

$$r \geq \frac{\mathbb{G}^*\Lambda + \ell_{\psi}^{\sigma_k}\mathbb{F}_k^*}{1 - \|\mathbb{L}_{\mathbb{G}}\|\Lambda - \ell_{\psi}^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|}, \tag{5.7}$$

and

$$\|\mathbb{L}_{\mathbb{G}}\|\Lambda + l_{\psi}^{\sigma_k}\mathbb{F}_k^* < 1, \tag{5.8}$$

where $\mathbb{F}_k^* = sup_{\tau\in \mathrm{J}}|\mathbb{F}_k(\tau, 0)|$, *and* $\mathbb{G}^* = sup_{\tau\in \mathrm{J}}|\mathbb{G}(\tau, 0)|$, $k = 0, 1, 2, ..., m$, *and*

$$\Lambda = |\mathbb{M}_{\psi}| + \Omega(r)\|p\|l_{\psi}^{\nu}, \tag{5.9}$$

Then hybrid fractional integro-differential problem (5.1) has a least one solution defined on J.

Proof. In order to use Dhage's fixed-point theorem to prove our main result, we define a subset $\mathbb{S}_r$ of $C(\mathrm{J})$ by

$$\mathbb{S}_r = \{z \in C(\mathrm{J}) : \|z\|_C \leq r\},$$

with r is a constant defined by hypothesis H_4.
Notice that $\mathbb{S}_r$ is closed, convex and bounded subset of $C(\mathrm{J})$. Define three operators $\mathbb{A}, \mathbb{C} : C(\mathrm{J}) \longrightarrow C(\mathrm{J})$ and $\mathbb{B} : \mathbb{S}_r \longrightarrow C(\mathrm{J})$ by

$$\begin{cases} \mathbb{A}z(\tau) &= \mathbb{G}(\tau, z(\tau)), \\ \mathbb{B}z(\tau) &= \mathbb{M}_{\psi} + I_{a^+}^{\nu;\psi}\mathbb{H}(\tau, z(\tau)), \end{cases} \quad \tau \in \mathrm{J},$$

and

$$\mathbb{C}z(\tau) = \sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi} \mathbb{F}_k(\tau, z(\tau)), \quad \tau \in \mathrm{J}.$$

Then (5.2) in operator form becomes

$$z(\tau) = \mathbb{A}z(\tau)\mathbb{B}z(\tau) + \mathbb{C}z(\tau), \quad \tau \in \mathrm{J}.$$

We shall prove that the operators $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ satisfy the conditions of Theorem 42. For the sake of clarity, we split the proof into a sequence of steps.

Step 1: First, we show that $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian on $C(\mathrm{J})$. Let $z, \bar{z} \in C(\mathrm{J})$. then by (H2), for $\tau \in [a, b]$, we have

$$\begin{aligned} |\mathbb{A}z(\tau) - \mathbb{A}\bar{z}(\tau)| &= |\mathbb{G}(\tau, z_(\tau)) - \mathbb{G}(\tau, \bar{z}(\tau))| \\ &\leq \mathbb{L}_{\mathbb{G}}(\tau)\|z(\tau) - \bar{z}(\tau)\|_{\mathfrak{C}}. \end{aligned}$$

Taking supremum over $\tau \in [a, b]$, we obtain

$$\|\mathbb{A}z - \mathbb{A}\bar{z}\|_C \leq \|\mathbb{L}_{\mathbb{G}}\|\|z(\tau) - \bar{z}(\tau)\|_{\mathfrak{C}},$$

for all $z, \bar{z} \in C(\mathrm{J})$. Therefore, $\mathbb{A}$ is a Lipschitzian on $C(\mathrm{J})$ with Lipschitz constant $\mathbb{L}_{\mathbb{G}}$. Also, for any $z, \bar{z} \in C(\mathrm{J})$, we have

$$\begin{aligned} |\mathbb{C}z(\tau) - \mathbb{C}\bar{z}(\tau)| &\leq \sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi} |\mathbb{F}_k(\tau, z(\tau)) - \mathbb{F}_k(\tau, \bar{z}(\tau))| \\ &\leq \sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi} \mathbb{L}_{\mathbb{F}_k}(\tau)\|z(\tau) - \bar{z}(\tau)\|_{\mathfrak{C}} \\ &\leq \sum_{k=1}^{m} \frac{(\psi(b) - \psi(a))^{\sigma_k}}{\Gamma(\sigma_k + 1)} \|\mathbb{L}_{\mathbb{F}_k}\|\|z(\tau) - \bar{z}(\tau)\|_{\mathfrak{C}}. \end{aligned}$$

Hence, we have

$$\|\mathbb{C}z - \mathbb{C}\bar{z}\|_C \leq \ell_{\psi}^{\sigma_k} \|\mathbb{L}_{\mathbb{F}_k}\|\|z(\tau) - \bar{z}(\tau)\|_{\mathfrak{C}}.$$

Which means that $\mathbb{C}$ is a Lipschitzian on $C(\mathrm{J})$ with Lipschitz constant $\ell_{\psi}^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\|$.

Step 2: We show that $\mathbb{B}$ is completely continuous on $\mathbb{S}_r$. The continuity of $\mathbb{B}$ follows by the continuity of $\mathbb{H}$. Now, it is sufficient to show that $\mathbb{B}$ is uniformly bounded and equicontinuous on $\mathbb{S}_r$. On the other hand, Keeping in mind the definition of the operator $\mathbb{B}$ on $[a, b]$ together with assumption (H3). For any $z \in \mathbb{S}_r$ we can get

$$\begin{aligned}
|\mathbb{B}z(\tau)| \leq & |\mathbb{M}_\psi| + \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} |\mathbb{H}(s, z(s)| \mathrm{ds} \\
\leq & |\mathbb{M}_\psi| \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \Omega(r) p(s) \mathrm{ds} \\
\leq & |\mathbb{M}_\psi| + \Omega(r)\|p\| \int_a^\tau \frac{\psi'(s)(\psi(\tau)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \mathrm{ds} \\
\leq & |\mathbb{M}_\psi| + \frac{(\psi(b)-\psi(a))^\nu}{\Gamma(\nu+1)} \Omega(r)\|p\| \\
= & |\mathbb{M}_\psi| + \Omega(r)\|p\| \ell_\psi^\nu .
\end{aligned}$$

Hence

$$\|\mathbb{B}z\|_C \leq |\mathbb{M}_\psi| + \Omega(r)\|p\| \ell_\psi^\nu .$$

Thus $\|\mathbb{B}z\| \leq \Lambda$ with Λ given in (5.9), for all $z \in \mathbb{S}_r$. This shows that $\mathbb{B}$ is uniformly bounded on $\mathbb{S}_r$.
Now, we will show that $\mathbb{B}(\mathbb{S}_r)$ is an equicontinuous set in $C(\mathrm{J})$.
Let $\tau_1, \tau_2 \in \mathrm{J}$ with $\tau_1 < \tau_2$. Then for any $z \in \mathbb{S}_r$, by (5.6) we get

$$\begin{aligned}
|\mathbb{B}z(\tau_2) - \mathbb{B}z(\tau_1)| \leq & \left| \int_a^{\tau_2} \frac{\psi(s)(\psi(\tau_2)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \mathbb{H}(\tau, z(\tau)) \mathrm{ds} \right. \\
& \left. - \int_a^{\tau_1} \frac{\psi(s)(\psi(\tau_1)-\psi(s))^{\nu-1}}{\Gamma(\nu)} \mathbb{H}(\tau, z(\tau)) \mathrm{ds} \right| \\
\leq & \frac{\Omega(r)\|p\|}{\Gamma(\nu)} \int_a^{\tau_1} \psi'(s) \left[(\psi(\tau_1)-\psi(s))^{\nu-1} - (\psi(\tau_2)-\psi(s))^{\nu-1} \right] \mathrm{ds} \\
& + \frac{\Omega(r)\|p\|}{\Gamma(\nu)} \int_{\tau_1}^{\tau_2} \psi'(s) \left(\psi(\tau_2)-\psi(s) \right)^{\nu-1} \mathrm{ds}. \qquad (3.9)
\end{aligned}$$

It is clear that the right-hand side of (3.9) is independent of z. Therefore, as $\tau_2 \to \tau_1$, inequality (3.9) tends zeros. As consequence of the Arzela-Ascoli theorem, $\mathbb{B}$ is a completely continuous operator on $\mathbb{S}_r$.

Step 3: The hypothesis (c) of Theorem 42 is satisfied.
Let $z \in C(\mathrm{J})$ and $y \in \mathbb{S}_r$ be arbitrary elements such that $z = \mathbb{A}z\mathbb{B}y + \mathbb{C}z$.

Then we have

$$\begin{aligned}
|z(\tau)| &\leq |\mathbb{A}z(\tau)|\, |\mathbb{B}y(\tau)| + |\mathbb{C}z(\tau)| \\
&\leq |\mathbb{G}(\tau, z(\tau))| \left\{ \mathbb{M}_\psi + I_{a^+}^{\nu;\psi} |\mathbb{H}(\tau, y(\tau))| \right\} + \sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi} |\mathbb{F}_k(\tau, z(\tau))| \\
&\leq \big(|\mathbb{G}(\tau, z(\tau)) - \mathbb{G}(\tau, 0)| + |\mathbb{G}(\tau, 0)| \big) \left\{ \mathbb{M}_\psi + I_{a^+}^{\nu;\psi} |\mathbb{H}(\tau, y(\tau))| \right\} \\
&+ \sum_{k=1}^{m} I_{a^+}^{\sigma_k;\psi} \big| \big(|\mathbb{F}_k(\tau, z(\tau)) - \mathbb{F}_k(\tau, 0)| + |\mathbb{F}_k(\tau, 0)| \big) \\
&\leq \big(\|\mathbb{L}_{\mathbb{G}}\| \|z\|_C + \mathbb{G}^* \big) \big[|\mathbb{M}_\psi| + \Omega(r) \|p\| \ell_\psi^\nu \big] + \ell_\psi^{\sigma_k} \big(\|\mathbb{L}_{\mathbb{F}_k}\| \|z\|_C + \mathbb{F}_k^* \big).
\end{aligned}$$

Thus,

$$|z(\tau)| \leq \big(\|\mathbb{L}_{\mathbb{G}}\| \|z\|_C + \mathbb{G}^* \big) \Lambda + \ell_\psi^{\sigma_k} \big(\|\mathbb{L}_{\mathbb{F}_k}\| \|z\|_C + \mathbb{F}_k^* \big).$$

Taking the supremum over τ,

$$\|z\| \leq \frac{\mathbb{G}^* \Lambda + \ell_\psi^{\sigma_k} \mathbb{F}_k^*}{1 - \|\mathbb{L}_{\mathbb{G}}\| \Lambda - \ell_\psi^{\sigma_k} \|\mathbb{L}_{\mathbb{F}_k}\|} \leq r.$$

Step 4: Finally we show that $\delta M + \xi < 1$, that is, (d) of Theorem 42 holds. Since

$$M = \|B(S)\| = \sup_{z \in S} \left\{ \sup_{\tau \in J} |Bz(t)| \right\} \leq \Lambda,$$

and so

$$\|\mathbb{L}_{\mathbb{G}}\| M + \ell_\psi^{\sigma_k} \|\mathbb{L}_{\mathbb{F}_k}\| \leq \|\mathbb{L}_{\mathbb{G}}\| \Lambda + \ell_\psi^{\sigma_k} \|\mathbb{L}_{\mathbb{F}_k}\| < 1,$$

with $\delta = \|\mathbb{L}_{\mathbb{G}}\|, \xi = \ell_\psi^{\sigma_k} \|\mathbb{L}_{\mathbb{F}_k}\|$. Thus all the conditions of Theorem 42 are satisfied and hence the operator equation $z = \mathcal{A}z\mathcal{B}z + \mathcal{C}z$ has a solution in $\mathbb{S}_r$. As a result, problem (5.1) has a solution on J. $\square$

5.3 Application

In this section, we present an example to show the applicability of the main result.

Example 71. *Consider the following hybrid fractional integro-differential equation:*

$$\begin{cases} {}^c\mathbb{D}_{a^+}^{\frac{1}{2};\psi} \left[\frac{z(\tau) - \sum_{k=1}^{m} \mathbb{I}_{a^+}^{\sigma_k;\psi} \mathbb{F}_k(\tau, z(\tau))}{\mathbb{G}(\tau, z(\tau))} \right] = \frac{1}{\sqrt{25+t^2}} \left(\frac{|z|}{(4|z|+1)} + \frac{z^2}{|z|+1} + \frac{1}{4} \right), \tau \in \mathrm{J} := [0,1], \\ z(a) = 0, \end{cases} \tag{5.10}$$

We take

$$\nu = \frac{1}{2}, \quad m = 3, \quad \sigma_1 = \frac{1}{2}, \quad \sigma_2 = \frac{3}{2}, \quad \sigma_3 = \frac{5}{2},$$

$$\begin{aligned}\sum_{k=1}^{3} \mathbb{I}_{a^+}^{\sigma_k;\psi} \mathbb{F}_k(\tau, z(\tau)) &= \mathbb{I}_{a^+}^{\frac{1}{2};\psi} \frac{\tau}{10} \left(z(\tau) + e^{-\tau}\right) \\ &+ \mathbb{I}_{a^+}^{\frac{3}{2};\psi} \frac{\tau \cos \tau}{12(1+e^{\tau})} \left(\frac{|z(\tau)|}{1+|z(\tau)|} + \frac{\tau}{\tau+1}\right) \\ &+ \mathbb{I}_{a^+}^{\frac{5}{2};\psi} \frac{3 \sin \pi\tau}{4+\tau} \left(\frac{|z(\tau)|}{5+|z(\tau)|} + \cos \tau\right),\end{aligned}$$

$$\begin{aligned}\psi(\tau, z(\tau)) &= \frac{\tau}{2}(\tau+1), \tau \in [0,1], \\ \mathbb{G}(\tau, z(\tau)) &= \frac{6\sqrt{\pi}\sin^2(\pi\tau)}{(\tau+5)} \frac{z(\tau)}{1+z(\tau)} + \frac{1}{2}, \\ \mathbb{H}(\tau, z(\tau))) &= \frac{1}{\sqrt{36+t^2}} \left(\frac{|z|}{(4|z|+1)} + \frac{z^2}{|z|+1} + \frac{1}{4}\right).\end{aligned}$$

We can show that

$$\begin{aligned}|\mathbb{F}_1(\tau, z(\tau)) - \mathbb{F}_1(\tau, \overline{z}(\tau))| &\leq \frac{\tau}{10}|z - \overline{z}|, \\ |\mathbb{F}_2(\tau, z(\tau)) - \mathbb{F}_2(\tau, \overline{z}(\tau))| &\leq \frac{\tau}{12(1+e^{\tau})}|z - \overline{z}|, \\ |\mathbb{F}_3(\tau, z(\tau)) - \mathbb{F}_3(\tau, \overline{z}(\tau))| &\leq \frac{3}{20+5\tau}|z - \overline{z}|, \\ |\mathbb{G}(\tau, z(\tau))\mathbb{G}(\tau, \overline{z}(\tau)) - | &\leq \frac{6\sqrt{\pi}}{(\tau+5)}|z - \overline{z}|, \\ |\mathbb{H}(\tau, z(\tau))) - \mathbb{H}(\tau, \overline{z}(\tau)))| &= \frac{1}{\sqrt{36+t^2}}(|z| + \frac{1}{2}),\end{aligned}$$

where

$$\Omega(|z|) = |z| + 1, \quad p(\tau) = \frac{1}{\sqrt{36+t^2}}.$$

Hence we have

$$\mathbb{L}_{\mathbb{G}}(\tau) = \frac{6\sqrt{\pi}}{(\tau+5)}, \quad \mathbb{F}_1 = \frac{\tau}{10}, \quad \mathbb{F}_2 = \frac{\tau}{12(1+e^{\tau})}, \quad \mathbb{F}_3 = \frac{3}{20+\tau}.$$

Then

$$\|\mathbb{L}_{\mathbb{G}}\| = \frac{6\sqrt{\pi}}{5}, \quad \|\mathbb{L}_{\mathbb{F}_1}\| = \frac{1}{10}, \quad \|\mathbb{L}_{\mathbb{F}_2}\| = \frac{1}{12(1+e)}, \quad \|\mathbb{L}_{\mathbb{F}_3}\| = \frac{3}{20}, \quad \|p\| = \frac{1}{6}, \quad l_{\psi}^{\nu} = \frac{2}{\sqrt{\pi}},$$

$$l_{\psi}^{\sigma_k}\|\mathbb{L}_{\mathbb{F}_k}\| = \frac{81(1+e)+25}{450\sqrt{\pi}(1+e)}, \quad l_{\psi}^{\sigma_k}\mathbb{F}_k^* = \frac{58}{75\sqrt{\pi}e^2}, \quad \mathbb{M}_{\psi} = \frac{234(1+e)+100}{225\sqrt{\pi}(1+e)},$$

and

$$\mathbb{F}_k^* = \sup_{z\in J}|\mathbb{F}_k(\tau,0)| = \frac{1}{5e^2}, \quad \mathbb{G}^* = \sup_{z\in J}|\mathbb{G}(\tau,0)| = \frac{1}{2}, \quad k = 1,2,3.$$

By using Matlab program, it follows by (5.7) *and* (5.8) *that the constant* r *satisfies the inequality* $0.7411 < r < 0.9970$. *As all the assumptions of Theorem* (70) *are satisfied then the problem* (5.10) *has at least one solution on* J.

Conclusion and Perspective

The main goals of this thesis is to investigate the existence and uniqueness of solutions to certain boundary value problems for of a class of fractional differential equations is established by using some fixed point theorems, Banach contraction principal theorem, Krasnoselskii's, Scheafer, D'hage, Leray-Shouder fixed point theorem, stability Ulam-Hayers

In the future researches, a first we intend to study some boundary problems, the application of certain other methods by combining the technique of measure of noncompactness, topological degree, monotone iterative technique.

Concerning the second research is concerned dynamics fractional systems, we propose the models mathematical biology such as: Endemic model (covid-19), and models in mathematical ecology, by combin numerical methods of resolution, Homotopy perturbation methods, ADM method.

Concerning the third research, is concerned of application the fractional calculus in machine learning, such as the dynamics of Hopfield-type natural networks.

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