الجمهورية الجزائرية الديمقراطية الشعبية

**People's Democratic Republic of Algeria**

وزارة التعليم العالي والبحث العلمي

**Ministry of Higher Education and Scientific Research**

جامعة غرداية

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**University of Ghardaïa**

كلية العلوم والتكنولوجيا

**Faculty of Science and Technology**

قسم الرياضيات والاعلام الي .

Department of Mathematics and informatique

End of study thesis, in order to obtain the diploma

**Master**

**Domain: Science And Technology**

**Branch: Mathematics**

**Specialty: Functional Analysis And Applications**

# Theme

# The (P ,Q) generalized reflexive and anti-reflexive solutions of some matrix equations

**Presented by:**

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**In front of the jury composed of:**

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**Academic year 2021/2022**

# Dedication

*I dedicate this work :*

*To whom I have wounded his hands out of fatigue to lift me up and give me a good life, my dear father, for whom I wish God prolong his life. To whom I prefer it to myself, and why not ; You have sacrificed yourself for me You have always spared no effort to make me happy (My mother is the love of my heart).*

*My only sister, my companion : Aouli.My dear brothers, my big brother Abd-El Malik, my brothers Fateh and Ahmed Al-Amin*

*To my adult parents and all my family*

*To all those who helped me - from near or far - and those who shared with me the emotional moments during the realization of this work and who have warmly supported and encouraged me throughout my journey.*

*To all my friends who have always encouraged me, and to whom I wish them more success.*

M.Chouireb

# THANKS

*In the name of Allah Most Gracious and Most Merciful :*

*First of all, I would like to thank Allah for guiding me along this path and for giving me the strength, courage and will to accomplish this work*

*My thanks also go to all the members of the Jury, for the honor they give me by accepting to be in the Jury of my thesis and to all the teachers of the university.*

*It is particularly pleasant for me to thank my kind promoter Mrs. Guerarra Sihem, for having proposed me this topic, for her orientations which allowed me to carry out my work.*

*My respect and my thanks go then, to the two persons the most dear to me, my father, and my mother for their love, their advice as well as their unconditional support, which allowed me to realize the studies that I wanted to do and consequently this thesis*

*Finally, I would like to express my thanks to all those who have contributed in any way to the realization of this work.*

*Thank you*

M.Chouireb

# ملخص


في هذه المذكرة وضعنا الشروط اللازمة والكافية لوجود (P,Q) الانعكاسي (ضد الانعكاسي )
المعممة  للمعادلة المصفوفاتية AX=B وجملة معادلتين AX=B وXC=D على الترتيب,
المتعلق بالمصفوفة الانعكاس المعممة المزدوجة (P,Q) علاوة على ذلك وضعنا عبارة احسن
حل انعكاسي (ضد انعكاسي) تقريبي



**الكلمات المفتاحية** :الحل الانعكاسي (ضد انعكاسي) المعمم ,معادلة مصفوفاتية ,
معكوس Moor-penrose المعمم , نظيم Frobenius, مصفوفة الانعكاس


## Abstract


In this thesis, we establish necessary and sufficient conditions for the existence of the $(P, Q)$ generalized reflexive (anti-reflexive) solutions and the common $(P, Q)$ generalized reflexive (anti-reflexive) solutions to linear matrix equations $AX = B$ and the system $AX = B$, $XC = D$ respectively, with respect to the generalized reflection matrix dual $(P, Q)$.
Moreover, for all these solutions, we derive the explicit expression of the best solution estimator to a given matrix in the Frobenius norm.



**Keywords:** $(P; Q)$ generalized reflexive solution, $(P; Q)$ generalized anti-reflexive solution, Matrix equation, Moore-penrose generalized inverse, Frobenius norm, generalized reflection matrix.


## Résumé


Dans ce mémoire, nous établissons les conditions nécessaires et suffisantes pour l'existence des solutions réflexives $(P, Q)$ généralisées (anti-réflexives) et des solutions communes réflexives $(P, Q)$ généralisées (anti-réflexives) aux équations matricielles linéaires $AX = B$ et au système $AX = B$, $XC = D$ respectivement, par rapport au couple des matrices de réflexion généralisée $(P, Q)$. De plus, pour toutes ces solutions, on donne l'expression explicite de la meilleure solution estimateure à une matrice donnée avec la norme de Frobenius .



**Mots clés:** $(P; Q)$ generalized réflixive solution, $(P; Q)$ generalized anti-réflixive solution, Matrix equation ,Inverse généralisé de Moore-Penrose, Norme de Frobenius, Matrice de reflection.

# CONTENTS

# INTRODUCTION

In 1920, E. H Moore introduced the notion of the generalized inverse of a matrix. This notion of generalized inverse reformulated by M. Penrose in 1955, where he gave a powerful tool for solving a system of linear equations.

The theory of generalized inverses has its genetic roots mainly in the context of linear problems interpreted by an equation of the type $Ax = b$, Where $A$ is a linear transformation. For the proposed equation to have a solution for $x$, it is necessary that $A$ has an inverse, and as this is not always the case, it may be desirable to looking for a matrix with explicit properties to this inverse, This was stated by. Penrose by intoducing a matrix $A$ verifying the following four equations:

$$AXA = A, \tag{1}$$

$$XAX = X, \tag{2}$$

$$(AX)^* = AX, \tag{3}$$

$$(XA)^* = XA. \tag{4}$$

Thus the systems of linear equations know the appearance of the approached solution as "the solution with least squares", "the solution with minimum norm" , "the solution with least squares and minimum norm", "the solution with least rank". Generalized inverses and Moore-Penrose inverse have been the subject of many researches. See for examples: [1], [21], [15], Thus, a generalized inverse of the matrix $A$ is a matrix having some properties of the inverse matrix of $A$ (when $A$ is invertible). The purpose of the construction of the generalized inverse is to obtain a matrix that can serve as the inverse in some way for a class of matrices wider than those of the invertible matrices. In other words, the generalized inverse exists for any arbitrary matrix, and when a matrix is invertible, then its inverse coincides with its generalized inverse.

The most known matrix equations in matrix theory are the equations :

$$AX = C$$

$$AXB = C$$

$$AXA^* = B$$

$$AX = C,\ XB = D$$

Where $A$, $B$, $C$ and $D$ are known matrices and $X$ is unknown, so that the third equation is a special case of the second equation. It is well known that solving linear matrix equations is a subject of many research in computational mathematic and has been applied in various areas such as control theory, vibration theory, biologie ans so on. In literature, the notion of generalized inverses of matrices was used when Penrose considered general solutions of the matrix equations $AX = B$ and $AXB = C$ see [21], Mitra in [16], [17] gave the conditions to the paire of matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ to have a common solution and a representation of this common solution is given.

Moreover, many problems can be transformed into some linear matrix equations. For example, if the matrix equation is not consistent, researchers often try to find approximate solutions that meet certain optimal criteria. One of the most widely used approach solutions is the least square solution, which is defined as a matrix $X$ which minimises the norm of the difference $AXB - C$ for the first equation $AXB = C$, see [5], [6]. Another kind among the approximated solutions is the generalized reflexive (anti reflexive) solution which is defined as a matrix $X$ satisfies these two equalities : $X = PXQ$ (or $X = -PXQ$). with respect to the matrix dual $(P; Q)$ where $P$ and $Q$ are two generalized reflection matrices, see [4].

The $(P, Q)$ generalized reflexive and anti-reflexive matrices have applications in system and control theory, in engineering, in scientific computations and various other fields. The reflexive and anti-reflexive solutions of a linear matrix equation or system of matrix equations have been studied by many authers. For instance, Liu and Yuan [15] gave some conditions for the existence and the representations for the $(P, Q)$ generalized reflexive and anti-reflexive solutions to matrix equation $AX = B$ with respect to the generalized reflection matrix dual, In [13], Liu establish some conditions for the existence and representations for the common $(P, Q)$ generalized reflexive and anti-reflexive solutions of matrix equations $AX = B$ and $XC = D$. In [7] the author studied the existence of a reflexive solution of the matrix equation $AXB = C$, with respect to the generalized reflection matrix $P$. In [24], the authors derived the extremal ranks of the matrix expression $A - BXC$, also Liu in [14] discussed the extremal ranks of this expression where $X$ is (anti-reflexive) matrix.

The thesis consists of three chapters, in the first chapter we gave some preliminary notions, and since the inverse of Moore-Penrose is the most important tool throughout this work, It was necessary to recall its algebraic properties and its role in the resolution of linear equations.

In the second chapter we give some conditions for the existence and the representations for the common $(P, Q)$ generalized reflexive (anti-reflexive) solutions of matrix equations $AX = B$, where $P$ and $Q$ are two generalized reflection matrices. In addition, among all the solutions of this linear system, we derive the explicit expression of the best estimator solution to a given matrix in the Frobenius norm.

In the third chapter, we are interested by the same purpose of the last chapter but here about the common $(P, Q)$ generalized reflexive (anti-reflexive) solutions of matrix equations $AX = B$ and $XC = D$, where $P$ and $Q$ are two generalized reflection matrices. Also, we establish the explicit formula of the best estimator solution among all solutions of this system in the Frobenius norm.

# RATING

| *Rating* | Definition |
|---|---|
| $\mathbb{K}$ | the field of real or complex numbers. |
| $M_{m\times n}(\mathbb{C})$ | the space of matrices of type $m \times n$ over $\mathbb{C}$. |
| $\mathbb{C}^{m\times n}$ | the space matrices of type $m \times n$ on $\mathbb{C}$. |
| $\mathbb{C}_r^{m\times n}$ | the space matrices of type $m \times n$ on $\mathbb{C}$, of rank r. |
| $A^{-1}$ | the ordinary inverse of $A$. |
| $A^{(1)}$ | the generalized inverse of $A$ . |
| $A^{+}$ | the Moore-Penrose inverse of $A$. |
| $A^{T}$ | the transpose matrix of $A$. |
| $A^{*}$ | the adjoint matrix of $A$. |
| $r(A)$ | rank of matrix $A$. |
| $R(A)$ | the image of the matrix $A$. |
| $N(A)$ | the cor of the matrix $A$. |
| $tr(A)$ | the trace of the matrix $A$. |
| $\|A\|_F$ | the Frobenius norm . |
| $E_A = I - AA^+ \text{ and } F_A = I - A^+A$ | the orthogonal projectors induced by $A$. |

# CHAPTER 1

# PRELIMINARIES

In this chapter, we introduced definitions, theorems and propositions that will be used in the sequal of this thesis.

## 1.1 Basic concepts

### 1.1.1 Linear transformations

In this section, $\mathcal{F}$ denotes the field of real or complex numbers.

**Definition 1.1.1** *Let $V$ and $W$ be two vector spaces over the field $\mathcal{F}$ and let $f : V \longrightarrow W$ an application.*
*Then $\mathcal{F}$ is called lineary if it satisfies the two conditions:*
- *for all vectors $v_1, v_2 \in V; f(v_1 + v_2) = f(v_1) + f(v_2)$.*
- *for any vector $v \in V$ and scalar $t \in \mathcal{F}$ $f(tv) = tf(v)$.*

*We denote $\mathcal{L}(V, W)$, the set of lineary transformations of $V$ in $W$ on $\mathcal{F}$.*

**Definition 1.1.2** *Let $f : V \longrightarrow W$ be a linear transformation*

1. ***The Kernel*** *(or null space) of $f$ is the subset of $V$ noted $N(f) \subset V$*
   $N(f) = \{v \in V, f(v) = 0\}$
2. ***The image*** *of $f$ is the subset of $W$ noted $R(f) \subset W$*
   $R(f) = \{w \in W, \exists v \in V, f(v) = w\}$

### 1.1.2 Matrices associated to linear applications between finite dimensional vector spaces

Let $V$ and $W$ be two finite dimensional vector spaces $n$ and $m$ respectively over the field $\mathcal{F}$
$f : V \longrightarrow W$ a linear application,
and let $\{e_1, e_2, ..., e_n\}$, $\{u_1, u_2, ..., u_m\}$ two bases of $V$ and $W$ respectively.

**Definition 1.1.3** *[10] The matrix of the application $f$
in the bases $\{e_i\}_{i=\overline{1,n}}$ and $\{u_j\}_{j=\overline{1,m}}$ the matrix noted $M(f)_{e_i,u_j}$, belonging to $\mathcal{F}^{m\times n}$, whose columns are the components of the vectors $f(e_1)$, $f(e_2)$ ,..., $f(e_n)$ in the basis $\{u_1, u_2, ..., u_m\}$.
In particular, $\mathbb{C}^{m\times n}$ ($\mathbb{R}^{m\times n}$) denote the set of $m \times n$ complex (real) matrices.
the matrix $A \in \mathcal{F}^{m\times n}$ is square if $m = n$, rectangular otherwise.*

**Proposition 1.1.1** *The application*

$$M : \mathcal{L}(V, W) \longrightarrow \mathcal{F}^{m\times n}$$

$$f \longmapsto M(f)_{e_i,u_j}$$

*is an isomorphism of vector spaces So for all lineaire applications $f$ and $g$ of $V$ in $W$, and for all $\lambda \in \mathcal{F}$ :*

$M(f + g) = M(f) + M(g)$.

$M(\lambda f) = \lambda M(f)$.

*and $M$ is bijective.*

**Definition 1.1.4** *let matrixe $A \in \mathbb{C}^{m\times n}$,*

- *the transpose of $A$ is the matrix $A^t \in \mathbb{C}^{n\times m}$ with $A^t[i, j] = A[j, i]$, for all $i, j$.*
- *the adjoint of $A$ is the matrix $A^* \in \mathbb{C}^{n\times m}$ with $A^*[i, j] = \overline{A[j, i]}$, for all $i$, $j$.*

**Definition 1.1.5** (1) *The square matrix $A \in \mathbb{R}^{n\times n}$ is said to be*

- *symmetric if $A^t = A$.*
- *anti-symmetrical if $A^t = -A$.*
- *orthogonal if $A^t = A^{-1}$.*

*2) A square matrix $A \in \mathbb{C}^{n\times n}$ is said to be*

- *Hermitian (self-adjoint) if $A^* = A$.*
- *anti-hermitian if $A^* = -A$.*
- *unitary if $A^* = A^{-1}$.*

**Proposition 1.1.2** *1. For all matrices $A \in \mathcal{F}^{l\times m}$ and $B \in \mathcal{F}^{m\times n}$, $(AB)^t = B^t A^t$.*

*2. For all matrices* $P \in \mathbb{C}^{l\times m}$ *and* $Q \in \mathbb{C}^{m\times n}$, $(PQ)^* = Q^*P^*$.

### 1.1.3 Partitioned matrices

A notation often used to write a matrix $A(n,p)$ as a jutaposition of sub-matrices or blocks. We then say that A is partitioned. It is of course necessary that the dimensions of the blocks are compatible.

**Exemple 1.1.1** *The matrixe* $P = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 \end{bmatrix}$, *can be divided into :*

$$P_{11} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 3 & 3 & 4 \end{bmatrix}, P_{12} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, P_{21} = \begin{bmatrix} 3 & 3 & 4 \end{bmatrix}, P_{22} = \begin{bmatrix} 4 \end{bmatrix}$$

*the partitioned matrix can be written as*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

## 1.2 Complex matrices

### 1.2.1 Conjugate matrix

**Definition 1.2.1** *Let A be a complex matrix, the conjugate matrix of a matrix A with complex coefficients is the matrix* $\bar{A}$ *made up of the conjugate elements of A.*
*More precisely, if we note* $a_{ij}$ *and* $b_{ij}$ *the respective coecients of A and* $\bar{A}$
*then* $b_{ij} = \bar{a_{ij}}$
*for exemple:*

$$\text{if } A = \begin{bmatrix} 2+i & 1-i \\ 4 & 4+3i \end{bmatrix}, \text{ then } \bar{A} = \begin{bmatrix} 2-i & 1+i \\ 4 & 4-3i \end{bmatrix}$$

**Proposition 1.2.1** *Let $A$ and $B$ be any two matrices of $M_{m\times n}(\mathbb{C})$ and $\alpha \in \mathbb{C}$ it is scalar*

1. $\overline{A+B} = \overline{A} + \overline{B}$.
2. $\overline{AB} = \overline{A} \cdot \overline{B}$.
3. $\overline{\alpha A} = \overline{\alpha}\overline{A}$.
4. $\overline{\overline{A}} = A$.
5. *If $A$ is an invertible square matrix* $\overline{(A^{-1})} = (\overline{A})^{-1}$.

### 1.2.2 adjoint matrix

**Definition 1.2.2** *An adjoint matrix (also called a transconjugate matrix) of a matrix $A \in M_{m\times n}(\mathbb{C})$ is the transpose matrix of the conjugate matrix of $A$, and is denoted by $A^*$*
*In the special case where $A \in M_{m\times n}(\mathbb{R})$, its adjoint matrix is therefore simply its transpose matrix. Thus we have :*
$A^* = (\overline{A})^T = \overline{(A)^T}$

**Exemple 1.2.1** $A = \begin{bmatrix} 1-i & 2 \\ 5+i & 4+i \\ i & 3+i \end{bmatrix}$

$$A^* = \begin{bmatrix} 1-i & 2 \\ 5+i & 4+i \\ i & 3+i \end{bmatrix}^* = \begin{bmatrix} 1+i & 5-i & -i \\ 2 & 4-i & 3-i \end{bmatrix}$$

**Proposition 1.2.2** *Let $A, B \in M_{m\times n}(\mathbb{C})$, and $C \in M_{m\times n}(\mathbb{R})$.*
*Then :*

1. $(A^*)^* = A$.
2. $(AB)^* = B^*A^*$.
3. $detA^* = \overline{detA}$.
4. *if $A = A^*$,the matrix $A$ is said to be Hermitian or self-adjoint.*
5. *if $C = C^T$, the matrix $C$ is said to be symmetric .*

6. *if* $A = -A^*$,*the matrix* $A$ *is said to be anti-Hermitian.*

7. *if* $C = -C^T$, *the matrix* $C$ *is said to be anti-symmetric.*

8. *if* $AA^* = A^*A$, *the matrix* $A$ *is said to be normal.*

9. *if* $AA^* = A^*A = I$, *the matrix* $A$ *is said to be unitary.*

10. *if* $CC^T = C^TC = I$, *the matrix* $C$ *is said to be orthogonal.*

### 1.2.3 Frobenius Norm

**Definition 1.2.3** *[1] (Trace of a square matrix) The trace of* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ *is denoted by the sum of the diagonal elements :*

$$tr(A) = \sum_{i=1}^{n} a_{ij}$$

1. $tr(A + B) = tr(A) + tr(B)$.

2. $tr(cA) = c.tr(A)$.

**Definition 1.2.4** *Let* $A \in \mathbb{C}^{n\times m}$, *we define the Frobenius matrix norm denoted by* $\|A\|_F$ *such as :*

$$\|A\|_F = \sqrt{traceA^*A} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$$

### 1.2.4 Rank of a matrix

**Definition 1.2.5** 1. *Let* $\{v_i\}_{i\in I}$ *be a family of vectors, We call the rank of the family* $\{v_i\}$ *the dimension of the space generated by this family.*

2. *Let* $A \in \mathcal{F}^{m\times n}$, *We call the rank of A the rank of the family formed by the column vectors of A, it is also the rank of the family formed by the vector rows.*

**Exemple 1.2.2** *The rank of the matrix* $A = \begin{pmatrix} 1 & 2 & \frac{-1}{2} & 0 \\ 2 & 4 & -1 & 0 \end{pmatrix} \in \mathcal{R}^{2\times 4}(\mathcal{K})$

*is by definition the rank of the family of vectors of*

$$\mathcal{K}^2 : \left\{ v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, v_3 = \begin{pmatrix} \frac{-1}{2} \\ -1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

*All these vectors are collinear to* $v_1$, *so the rank of the family* $\{v_1, v_2, v_3, v_4\}$ *is 1 and so* $r(A) = 1$.

### Full rank factorization

**Proposition 1.2.3** *Let $A \in \mathbb{C}^{m\times n}$ of rank $r$, $r \neq 0$.*
*Then there exist matrices $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$ such that $r(B) = r(C) = r$ and $A = BC$.*
*This decomposition is called a full rank factorization of the matrix $A$.*

**Proof.**
Let $A \in \mathbb{C}^{m\times n}$ of rank $r$, let $\{b_1, b_2, ..., b_r\}$ be a basis of $R(A)$, let $B \in \mathbb{C}^{m\times r}$ whose column vectors are $b_1, b_2, ..., b_r$ so $r(B) = r$.
For a matrix $C \in \mathbb{C}^{r\times n}$, each row vector of $A$ is a linear combination of the row vectors of $C$. of the row vectors of $C$ from which we can write : $A = BC$ for a matrix $C \in \mathbb{C}^{r\times n}$, so $r(C) \leq r$, from the property $r(BC) \leq \min(r(B), r(C))$, we have $r = r(A) \leq r(C)$, consequently $r(C) = r$. ■

**Proposition 1.2.4** *Let $A$, $B \in \mathbb{C}^{m\times n}$ . Then:*
*i)* $r(AB) \leq \min(r(A), r(B))$.
*ii)* $r(A+B) \leq r(A) + r(B)$.

**Proof.**i) the vector of $R(AB)$ is of the form $ABx$ for a certain vector $x$,

and therefore it belongs to $R(A)$, then $R(AB) \subset R(A)$.

accordingly $r(AB) = \dim R(AB) \leq \dim R(A) = r(A)$.

Now, using this fact we have : $r(AB) = r(B^*A^*) \leq r(B^*) = r(B)$.

ii) Let $A = XY$, $B = UV$ full rank factorizations of $A$ and $B$ respectively.

So, $A + B = XY + UV = [X, U]\begin{bmatrix} Y \\ V \end{bmatrix}$ therefore and according to i) $r(A+B) \leq r[X, U]$
Let $\{x_1, ..., x_p\}$ and $\{u_1, ..., u_q\}$ be bases for $R(X)$ and $R(U)$ respectively.
Any vector in the image space of $[X, U]$ can be written as a linear combination of these $p + q$ vectors. so $r[X, U] \leq r(X) + r(U) = r(A) + r(B)$ . ■

### 1.2.5 Idempotent

**Proposition 1.2.5** *1. Any idempotent matrix that different to the identity matrix is non-invertible*

2. *Let $E \in \mathbb{C}^{n\times n}$, the following properties are quivalents :*

   (a) *$E^*$ is an idempotent.*

   (b) *$I - E$ is an idempotent.*

   (c) *$N(E) = R(I - E)$.*

   (d) *if $x \in R(E)$, then $Ex = x$.*

   (e) *$rankE = traceE$.*

   (f) *$E(I - E) = (I - E)E = 0$.*

### 1.2.6 Elementary block matrix operations (EBMO)

In order to conduct explicit formulas for the rank of block matrices, we use the three types of elementary operations on block matrices (abbreviated as EBMO):

1. interchange two row ( column) blocks in a partitioned matrix.
2. multiply a row ( column) block by a non- singular matrix on the left ( on the right) in a partitioned matrix.
3. Add to a row ( column) block multiplied by an appropriate matrix on the left ( right) to another row ( column) block.

## 1.3 The generalized inverse of matrices

**Definition 1.3.1** *If $A$ is a non-singular matrix, then there is a unique inverse noted $A^{-1}$ as*

$$AA^{-1} = A^{-1}A = I$$

*This inverse has the following properties :*

1. *$(A^{-1})^{-1} = A$.*
2. *$(A^T)^{-1} = (A^{-1})^T$.*
3. *$(A^*)^{-1} = (A^{-1})^*$.*
4. *$(AB)^{-1} = B^{-1}A^{-1}$.*

$A^T$ and $A^*$ denote the transpose and conjugate transpose of $A$, respectively. We recall that a real or complex number $\lambda$ is called an eigenvalue of a square matrix $A$, and a non-zero vector $X$ is called an eigenvector of $A$ corresponding to $\lambda$, if $AX = \lambda X$
Consider the linear system

$$Ax = b \tag{1.1}$$

Let $A$ be a square matrix of order $n$, If $A$ is nonsingular, so $ker(A) = 0$, then the solution vector $x$ of the linear equation $Ax = b$ is only determined by $x = A^{-1}b$.
Here, $A^{-1}$ is the inverse matrix of $A$.

**Definition 1.3.2** *Let $A \in \mathbb{C}^{m\times n}$, the matrix $X \in \mathbb{C}^{n\times m}$ if said to be the generalized inverse or(g-inverse) of the matrix $A$ if $AXA = A$.*

If $A$ is square and non singular, then $A^{-1}$ is the unique generalized inverse of $A$, otherwise $A$ has several generalized inverses, we denote by $A^{(1)}$ for a generalized inverse of $A$.

**Theorem 1.3.1** *Let $A \in \mathbb{C}^{m\times n}$ and $X \in \mathbb{C}^{n\times m}$, then the following two conditions are equivalent*
*i) $X$ is a g-inverse of $A$.*
*ii) for any $b \in R(A)$, $x = Xb$ is a solution of $Ax = b$.*

**Proof.**
i)$\Longrightarrow$ii) $\forall b \in R(A)$, $b$ is of the form $b = Ay$, such as $y$, then $A(Xb) = AXAy = Ay = b$.
ii)$\Longrightarrow$i) when $AXb = b$ for all $b \in R(A)$, we have $AXAy = Ay$ for all $y$, than $AXA = A$. ■

**Definition 1.3.3** *Let $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is said to be the reflexive generalized inverse of the matrix $A$, if it satisfies the two conditions:*$\begin{cases} AXA = A \\ XAX = X \end{cases}$

## 1.3.1 Properties of the generalized inverse

Let $A \in \mathbb{C}^{m\times n}$ matrix we give some properties of $A^{(1)}$

**Theorem 1.3.2**
1. *$r(A^{(1)}) \geq r(A) = r(A^{(1)}A) = r(AA^{(1)})$.*
2. *if $A$ is square and nonsingular, then $A^{(1)} = A^{-1}$ is unique.*
3. *$AA^{(1)}$ and $A^{(1)}A$ are idempotent.*

**Proof.**

1. For two matrices $B$ and $C$, we have $r(BC) \leq min r(A), r(B)$, then

$$r(A) \geq r(AA^{(1)}) \geq r(AA^{(1)}A) = r(A) \Longrightarrow r(A) = r(AA^{(1)})$$

$$r(A) \geq r(A^{(1)}A) \geq r(AA^{(1)}A) = r(A) \Longrightarrow r(A) = r(A^{(1)}A)$$

   Hence

$$r(A) = r(A^{(1)}A) = r(AA^{(1)})$$

   On the other hand

$$r(A^{(1)}) \geq r(AA^{(1)}) \geq r(AA^{(1)}A) = r(A)$$

   .

2. We have $AA^{(1)}A = A$
   If $A$ is non-singular, then multiplying by $A^{-1}$ on both the left and the right would give

$$A^1 = A^{-1}$$

3.

$$(AA^{(1)})^2 = (AA^{(1)}A)A^{(1)} = AA^{(1)}$$

$$(A^{(1)}A)^2 = (A^{(1)}AA^{(1)})A = A^{(1)}A$$

■

**Lemma 1.3.1** *Let $A \in \mathbb{C}_r^{m \times n}$ So we at :*

1. $A^{(1)}A = I_n$ *if and only if* $r = n$.
2. $AA^{(1)} = I_m$ *if and only if* $r = m$.

**Exemple 1.3.1** *Determine a generalized inverse of* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

*Let* $A^{(1)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

*for* $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

*you must have* $a + 2b + 2c + 4d = 1$ *then*

$A = \begin{bmatrix} 1 - 2b - 2c - 4d & b \\ c & d \end{bmatrix}$ *where* $a, b$ *and* $c$ *are arbitrary*

*for example if we choose* $d = -2,\ b = 2,\ c = 0,$ *we will find*

$A^{(1)} = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix}$

**Theorem 1.3.3** *Let* $H = AA^-$, $F = A^-A$, *then :*

*1)* $H^2 = H$ *and* $F^2 = F$.

*2)* $r(H) = r(F) = r(A)$.

*3)* $r(A^-) \geq r(A)$.

*4)* $r(A^-AA^-) = r(A)$.

**Proof.**

It is clear from the definition of the generalized inverse

2) $r(A) \geq r(AA^-) = r(H)$, et $r(A) = r(AA^-A) = r(HA) \leq r(H)$,

from which we conclude $r(A) = r(H)$

$r(F) = r(A)$ proven in a similar way.

3) $r(A) = r(AA^-A) \leq r(AA^-) \leq r(A^-)$.

4) $r(A^-AA) = r(A^-A)$, then $r(A^-AA^-) = r(A^-A) = r(A)$. ■

**Exemple 1.3.2** *Determine a generalized inverse of* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

*Let* $A^- = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *from* $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

*you must have* $a + b + c + d = 1,$ *then* $A^- = \begin{bmatrix} a & b \\ c & 1 - a - b - c \end{bmatrix}$,

*or* $a$, $b$ *and* $c$ *are arbitrary.*

**Exemple 1.3.3** *In particular the second Method for computing the generalized inverse, construction of 1-inverse for any matrix $A \in \mathbb{C}^{m\times n}$ is simplified by transforming $A$ to the Hermitian normal form as shown in the following theorem :*

**Theorem 1.3.4** *Let $A \in \mathbb{C}_r^{m\times n}$ and let $E \in \mathbb{C}_m^{m\times m}$ and $P \in \mathbb{C}_n^{n\times n}$ as*

$$EAP = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix}, \tag{1.1}$$

*for all $L \in \mathbb{C}^{(n-r)\times(m-r)}$ matrices $n \times m$*

$$X = P \begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix} E \tag{1.2}$$

*is a 1-inverse of $A$. Let $A \in \mathbb{C}^{m\times n}$, and let $T_0 = [A \quad I_m]$. $E$ tronsform $A$ to the Hermitian normal form note $EA$, we use Elimination of Gausses, which $ET_0 = [EA \quad A]$ let*

$$A = \begin{bmatrix} 0 & 2i & i & 0 & 4+2i & 1 \\ 0 & 0 & 0 & -3 & -6 & -3-3i \\ 0 & 2 & 1 & 1 & 4-4i & 1 \end{bmatrix} \textit{with} \quad r(A) = 2 \tag{1.3}$$

$$[EA \quad E] = \left[\begin{array}{cccccc|ccc} 0 & 2i & i & 0 & 4+2i & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -3-3i & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 4-4i & 1 & 0 & 0 & 1 \end{array}\right] : \begin{cases} L_1 \mapsto (\frac{1}{2i})L_1 \\ L_3 \mapsto 2L_1 - L_3 \end{cases}$$

*then*

$$= \left[\begin{array}{cccccc|ccc} 0 & 1 & \frac{1}{2} & 0 & 1-2i & -\frac{1}{2}i & -\frac{1}{2}i & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -3-3i & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1+i & i & 0 & 1 \end{array}\right] : L_2 \mapsto (-\tfrac{1}{3})L_2$$

$$= \left[\begin{array}{cccccc|ccc} 0 & 1 & \frac{1}{2} & 0 & 1-2i & -\frac{1}{2}i & -\frac{1}{2}i & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1+i & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & \frac{1}{3} & 1 \end{array}\right] : L_3 \mapsto L_3 - L_2$$

*then*

$$EA = \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 & 1-2i & -\frac{1}{2}i \\ 0 & 0 & 0 & 1 & 2 & 1+i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.4}$$

*and*

$$E = \begin{bmatrix} -\frac{1}{2}i & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ i & \frac{1}{3} & 1 \end{bmatrix}$$

*for all* $L \in \mathbb{C}^{(n-r)\times(m-r)}$ *we choose the permutation matrix* $P$ *as*

$$P = \begin{bmatrix} 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 1 & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix}$$

*Then*

$$EAP = \begin{bmatrix} 1 & 0 & \vdots & 0 & \frac{1}{2} & 1-2i & -\frac{1}{2}i \\ 0 & 1 & \vdots & 0 & 0 & 2 & 1+i \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 0 & 0 & 0 \end{bmatrix} \textit{ We choose } L = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \in \mathbb{C}^{4\times 1}$$

$$X = P \begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix} E \;=\; \begin{bmatrix} 0 & 0 & \vdots & 1 & 0 & 0 & 0 \\ 1 & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \vdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 1 & 0 \\ 0 & 0 & \vdots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ \cdots & \cdots & \vdots & \cdots \\ 0 & 0 & \vdots & \alpha \\ 0 & 0 & \vdots & \beta \\ 0 & 0 & \vdots & \gamma \\ 0 & 0 & \vdots & \delta \end{bmatrix} \begin{bmatrix} -\frac{1}{2}i & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ i & \frac{1}{3} & 1 \end{bmatrix}$$

*Then*

$$= \begin{bmatrix} i\alpha & \frac{1}{3}\alpha & \alpha \\ -\frac{1}{2} & 0 & 0 \\ i\beta & \frac{1}{3}\beta & \beta \\ 0 & -\frac{1}{3} & 0 \\ i\gamma & \frac{1}{3}\gamma & \gamma \\ i\delta & \frac{1}{3}\delta & \delta \end{bmatrix}$$

*For example if we choose :* $\alpha = -1,\ \beta = i,\ \gamma = 3i,\ \delta = -6.$ *we will find a generalized invese verifying*

$$X = \begin{bmatrix} -i & -\frac{1}{3} & -1 \\ -\frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{3}i & i \\ 0 & -\frac{1}{3} & 0 \\ -3 & 3 & 3i \\ -6i & -2 & -6 \end{bmatrix}$$

### 1.3.2 The generalized inverse and linear equations:

In this section we represent solutions to linear equations involving the generalized inverses of matrices.

**Theorem 1.3.5** *[1] Let* $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}, D \in \mathbb{C}^{m\times q}$.
*Then the matrix equation*

$$AXB = D \tag{1.5}$$

*is consistent if and only if, for some* $A^{(1)}, B^{(1)}$,

$$AA^{(1)}DB^{(1)}B = D, \tag{1.6}$$

*in which case the general solution is*

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)} \tag{1.7}$$

*for arbitrary* $Y \in \mathbb{C}^{n\times p}$.

**Proof.**If (1.6) holds, then $X = A^{(1)}DB^{(1)}$ is a solution of (1.5).

Conversely, if $X$ is any solution of (1.5), then

$$D = AXB = AA^{(1)}AXBB^{(1)}B = AA^{(1)}DB^{(1)}B.$$

Moreover, it follows from (1.6) and the definition of $A^{(1)}$ and $B^{(1)}$ that every matrix $X$ of the form (1.7) satisfies (1.5). On the other hand, let $X$ be any solution of (1.5). Then, clearly

$$X = A^{(1)}DB^{(1)} + X - A^{(1)}AXBB^{(1)},$$

which is of the form (1.7). ■

The following characterization of the set $A\{1\}$, in terms of an arbitrary element element $A^{(1)}$ of this set.[6]

**Corollary 1.3.1** *[1] Let $A \in \mathbb{C}^{m\times n}$, $A^{(1)} \in A\{1\}$. then*

$$A\{1\} = \left\{A^{(1)} + Z - A^{(1)}AZAA^{(1)},\, Z \in \mathbb{C}^{n\times m}\right\} \tag{1.4}$$

**Proof.**The set $A\{1\}$ is obtained by writing $Y = A(1) + Z$ in the set of solutions of $AXA = A$ as given by Theorem 1.3.5. ■

Specializing Theorem 1.3.5 to ordinary systems of linear equations gives:

**Corollary 1.3.2** *let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m}$ Then the equation $Ax = b$ is consistent if and only if $AA^{(1)}b = b$ in this case the general solution is given by*

$$x = A^{(1)}b + (I - A^{(1)}A)y \tag{1.8}$$

*or $y \in \mathbb{C}^{n}$ is arbitrary*

**Proof.**The sufficient condition, it is obvious

The necessary condition can be demonstrated by the substitution of $Ax = b$ in $AA^{(1)}Ax = Ax$

■

## 1.4 The generalized Moore-Penrose inverse:

In this section, we will present the well known generelezd inverse of matrices which is the Moore-Penrose inverse and its properties and applications.

**Definition 1.4.1** *[1] E. H Moore introduced the notion of a generalized inverse of a matrix in 1920, the application of this notion to the solution of systems of linear equations led to a great interest in this subject. In 1955 Penrose demonstrated that, for any matrix A (square or rectangular) with real or complex elements, there exists a unique matrix X satisfying the four equations :*

$$AXA = A. \tag{1.9}$$

$$XAX = X. \tag{1.10}$$

$$(AX)^* = AX. \tag{1.11}$$

$$(XA)^* = XA. \tag{1.12}$$

*This unique generalized inverse is commonly known as the Moore-Penrose generalized inverse and is often referred to as* $A^+$.

*If* $A$ *is nonsingular, the matrix* $X = A^{-1}$ *satisfied the four equations trivially, then the Moore-Penrose inverse of a nonsingular matrix is the same as the ordinary inverse.*

*From equations* (1.11) *and* (1.12) *above, we have* $(AA^+)^* (I_n - AA^+) = 0$

*and* $(A^+A)^* (I_m - A^+A) = 0$ *so* $P_A = AA^+$ *and* $P_{A^+} = A^+A$ *are orthogonal projectors on* $R(A)$ *and* $R(A^*)$ *respectively.*

*These are the orthogonal projectors on* $R(A)$ *and* $R(A^*)$ *respectively.*

### 1.4.1 Uniqueness and existence

**Existence:**

**Theorem 1.4.1** *if* $A = BC$ *or* $A \in \mathbb{C}^{n\times m}$, $B \in \mathbb{C}^{m\times r}$, $C \in \mathbb{C}^{r\times n}$, *and* $r = r(A) = r(B) = r(C)$, *then :*

$$A^+ = C^*(CC^*)^{-1}(B^*B)^{-1}B^*$$

.

**Proof.** We conclude that $B^*B$ and $CC^*$ are matrices of rank r, because according to the properties of rank we have

$$r(B) = r(B^*B) = r(CC^*) = r(C) = r$$

We take

$$X = C^*(CC^*)^{-1}(B^*B)B^*$$

Then we have :

$AX = BCC^*(CC^*)^{-1}(B^*B)B^* = B(B^*B)B^*$ then $(AX)^* = AX$

also $XA = C^*(CC^*)^{-1}(B^*B)B^*BC = C^*(CC^*)^{-1}$ then

$(XA)^* = XA$ To verify (1.9) an (1.10) We use $XA = C^*(CC^*)^{-1}$ we obtain

$A(AX) = BC(C^*(CC^*)^{-1}) = BC = A$ and $(XA)X = C^*(CC^*)^{-1}C^*(CC^*)^{-1}(B^*B)^{-1}B^* = (CC^*)^{-1}(B^*B)^{-1}B^* = X$

We notice that the matrix X satisfies the four Moore-Penrose equations . Thus $X = A^+$ by denition. ■

**Uniqueness :**

We assume that $X_1$ and $X_2$ are two Moore-Penrose inverses of $A$, then according to the definition of $A^+$ we have

$$X_1 = X_1 (AX_1) = X_1 X_1^* (A^*) = X_1 X_1^* A^* (X_2^* A^*)$$

$$= X_1 (X_1^* A^*) AX_2 = X_1 (AX_1A) X_2 = X_1 (A) X_2$$

$$= X_1 (AX_2A) X_2 = (X_1A) A^* X_2^* X_2 = (A^* X_1^* A^*) X_2^* X_2$$

$$= (A^* X_2^*) X_2 = X_2AX_2$$

$$= X_2$$

**Exemple 1.4.1** *Let* $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, $r(A) = 1$ *et* $A = BC$ *or* $B \in \mathbb{C}^{2\times 1}$ *and* $C \in \mathbb{C}^{1\times 3}$.

*We can take :* $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

*then,*

$B^*B = [5], C^*C = [6]$ *Thus,* $A^+ = \frac{1}{30} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{30} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$ *Special case,*

*if* $A \in \mathbb{C}^{m\times n}$ *et* $r(A) = 1$ .

*then* $A^+ = (\frac{1}{\alpha})A^*$ *or* $\alpha = trace(A^*A) = \sum_{i=1}^{m} \sum_{=1}^{n} |a_{ij}|^2$

### 1.4.2 Main properties of the generalized Moore-Penrose inverse

**Theorem 1.4.2** *Let* $A \in \mathbb{C}^{m\times n}$. *then,*

*1)* $(A^+)^+ = A$.

*2)* $(A^+)^* = (A^*)^+$.

*3) if* $\lambda \in \mathbb{C}$, $(\lambda A)^+ = \lambda^+ A^+$ *or* $\lambda^+ = \dfrac{1}{\lambda}$ *if* $\lambda \neq 0$, *and* $\lambda^+ = 0$ *if* $\lambda = 0$.

*4)* $A^* = A^*AA^+ = A^+AA^*$.

*5)* $(A^*A)^+ = A^+(A^*)^+$.

*6)* $A^+ = (A^*A)^+ A^* = A^*(AA^*)^+$.

*7)* $(UAV)^+ = V^*A^+U^*$, *où* $U$, $V$ *are arbitrary matrices.*

**Theorem 1.4.3** *If* $A \in \mathbb{C}^{m\times n}$, *then*

*1)* $R(A) = R(AA^+) = R(AA^*)$.

*2)* $R(A^+) = R(A^*) = R(A^+A) = R(A^*A)$.

*3)* $R(I - AA^+) = N(AA^+) = N(A^*) = N(A^+) = R(A)^\perp$.

*4)* $R(I - A^+A) = N(A^+A) = N(A) = R(A^*)^\perp$.

We need a convenient notation for a generalized inverse satisfying certain specified equations

.

**Definition 1.4.2** *For any* $A \in \mathbb{C}^{m\times n}$, $A\{i,j,...,k\}$ *denotes the set of matrices* $X \in \mathbb{C}^{n\times m}$, *which satisfy equations* $(i)$, $(j)$,..., $(k)$ *among equations (1.9)-(1.12).*
*the matrix* $X \in A\{i, j,...,k\}$ *is called* $\{i, j,...,k\}$ *-inverse of* $A$ *and also noted by* $A^{(i,\, j,...,k)}$.
*In particular, a matrix* $X \in \mathbb{C}^{n\times m}$ *of the set* $A\{1\}$ *is called a g-inverse of* $A$ *and denoted by* $A^{(1)}$. *Thus, in the following we will note* $A^{(1)}$ *for a generalized inverse of* $A$ *instead of* $A^-$

The following properties of the conjugate transpose will be used :

$$
\begin{gathered}
A^{**} = A, \\
(A+B)^* = A^* + B^*, \\
(\lambda A)^* = \lambda A^*, \\
(BA)^* = A^*B^*, \\
AA^* = 0 \quad \textit{implies} \quad A = 0,
\end{gathered}
$$

The last of these follows from the fact that the trace of $AA^*$ is the sum of the squares of the moduli of the elements of $A$. From the last two we obtain the rule

$$BAA^* = CAA^* \quad \textit{implies} \quad BA = GA. \tag{1.13}$$

since

$$(BAA^* - CAA^*)(B - C)^* = (BA - CA)(BA - GA)^*.$$

Similarly

$$BA^*A = CA^*A \quad \textit{implies} \quad BA^* = GA^*. \tag{1.14}$$

**Theorem 1.4.4** *[21] The four equations*

$$AXA = A.$$

$$XAX = X.$$

$$(AX)^* = AX.$$

$$(XA)^* = XA.$$

*have a unique solution for any A*

**Proof.**I first show that equations (1.10) and (1.11) are equivalent to the single equation

$$XX^*A^* = X. \tag{1.15}$$

Equation (1.15) follows from (1.10) and (1.11), since it is merely (1.11) substituted in (1.10). Conversely, (1.15) implies $AXX^*A^* = AX$, the left-hand side of which is hermitian. Thus (1.11) follows, and substituting (1.11) in (1.15) we get (1.10).
Similarly, (1.9) and (1.12) can be replaced by the equation

$$XAA^* = A^*. \tag{1.16}$$

Thus it is sufficient to find an $X$ satisfying (1.15) and (1.16). Such an $X$ will exist if a $B$ can be found satisfying

$$BA^*AA^* = A^*$$

For then $X = BA^*$ satisfies (1.16). Also, we have seen that (1.14) implies $A^*X^*A^* = A^*$ and therefore $BA^*X^*A^* = BA^*$. Thus $X$ also satisfies (1.15).
Now the expressions $A^*A, (A^*A)^2, (A^*A)^3, ...$ cannot all be linearly independent, i.e. there exists a relation

$$\lambda_1 A^*A + \lambda_2(A^*A)^2 + ... + \lambda_k(A^*A)^k = 0. \tag{1.17}$$

where $\lambda_1, ..., \lambda_k$ are not all zero.
Let $\lambda_r$, be the first non-zero $\lambda$ and put

$$B = -\lambda_r^{-1}\left\{\lambda_{r+1}I + \lambda_{r+2}A^*A + ... + \lambda_k(A^*A)^{k-r-1}\right\}$$

Thus (1.17) gives $B(A^*A)^{r+1} = (A^*A)^r$, and applying (1.13) and (1.14) repeatedly we obtain $BA^*AA^* = A^*$, as required

To show that $X$ is unique, we suppose that $X$ satisfies (1.15) and (1.16) and that $Y$ satisfies $Y = A^*Y^*Y$ and $A^* = A^*AY$. These last relations are obtained by respectively substituting (1.14) in (1.12) and (1.13) in (1.11). (They are (1.14) and (1.15) with $Y$ in place of $X$ and the reverse order of multiplication and must, by symmetry, also be equivalent to (1.11), (1.12), (1.13) and (1.14).) Now

$$X = XX^*A^* = XX^*A^*AY = XAY = XAA^*Y^*Y = A^*Y^*Y = Y$$

The unique solution of (1.9), (1.10), (1.11) and (1.12) will be called the generalized inverse of $A$ (abbreviated g.i.) and written $X = A^+$. (Note that $A$ need not be a square matrix and may even be zero. ) I shall also use the notation $\lambda^+$ for scalars, where $\lambda^+$ means $\lambda^{-1}$ if $\lambda \neq 0$ and if $A = 0$

In the calculation of $A^+$ it is only necessary to solve the two unilateral linear equations $XAA^* = A^*$ and $A^*AY = A^*$. By putting $A^* = XAY$ and using the fact that $XA$ and $AY$ are hermitian and satisfy $AXA = A = AYA$ we observe that the four relations $AA^+A = A, A^+AA^+ = A^+, (AA^+)^* = AA^*$ and $(A^+A)^* = A^+A$ are satisfied. Relations satisfied by $A^+$ include

$$\text{and} \quad \begin{cases} A^+A^{+*}A^* = A^+ = A^*A^{+*}A^+ \\ A^+AA^* = A^* = A^*AA^+ \end{cases} \tag{1.18}$$

these being (1.15), (1.16) and their reverses. ■

### 1.4.3 Characterization of $A\{1,3\}$ and $A\{1,4\}$

Recall that $X$ is an element of the set $A\{1,3\}$ if $X$ verifies the equations (1.9) and (1.11) among the four Penrose equations so, $\begin{cases} AXA = A \\ (AX)^* = AX \end{cases}$

Also, $X$ is an element of the set $A\{1,4\}$ if $X$ verifies equations (1.9) and (1.12) among the four Penrose equations, i.e. $\begin{cases} AXA = A \\ (XA)^* = XA \end{cases}$

### 1.4.4 The least squares solutions of a linear system not consistent

For the matrix $A \in \mathbb{C}^{m\times n}$, and the vector $b \in \mathbb{C}^m$, the linear system

$$Ax = b \tag{1.19}$$

is consistent, i.e. has a solution for $x$, if and only if $b \in R(A)$. In other words, the residual vector,

$$r = b - Ax$$

is nonzero for all $x \in \mathbb{C}^n$, and it may be desirable to find an approximate solution of $Ax = b$, i.e., find a vector $x$ such that such that the residual vector $r$ is minimized. Often used in particular in statistical applications is the least square solution least squares solution of $Ax = b$, which is defined by:

**Definition 1.4.3** *Suppose that $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$ then the vector $u \in \mathbb{C}^n$ is called the least squares solution of $Ax = b$, if and only if $\|Au - b\| \leq \|Av - b\|$ for all $v \in \mathbb{C}^n$.*

**Definition 1.4.4** *A vector $u$ is called a least square and minimum norm solution of $Ax = b$ if $u$ is a least square solution of $Ax = b$ and $\|u\| \leq \|w\|$, for all other least-squares solutions $w$.*

if $b \in R(A)$, then the notions of "solution" and "least square solution" obviously coincide. The following theorem shows that $\|Ax - b\|$ is minimized by choosing $x = Xb$, where $X \in A\{1,3\}$. Thus we establish a relation between the $\{1,3\}$ inverses and the least squares solutions of $Ax = b$.

**Theorem 1.4.5** *[1] Let $A \in \mathbb{C}^{m\times n}, b \in \mathbb{C}^m$.*
*Then $\|Ax - b\|$ is smallest when $x = A^{(1,3)}b$, where $A^{(1,3)} \in A\{1,3\}$.*
*Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all $b$, $\|Ax - b\|$ is smallest when $x = Xb$, then $X \in A\{1,3\}$.*

**Proof.**From

$$b = (P_{R(A)} + P_{R(A)^{\perp}})b. \tag{1.20}$$

$$b - Ax = (P_{R(A)}b - Ax) + P_{N(A^*)}b.$$

$$\|Ax - b\|^2 = \|Ax - P_{R(A)}b\|^2 + \|P_{N(A^*)}b\|^2, \tag{1.21}$$

Evidently, (1.19) assumes its minimum value if and only if

$$Ax = P_{R(A)}b, \tag{1.22}$$

which holds if $x = A^{(1,3)}b$ for any $A^{(1,3)} \in A\{1,3\}$, since

$$AA^{(1,3)} = P_{R(A)}. \tag{1.23}$$

Conversely, if $X$ is such that for all $b$, $\|Ax - b\|$ is smallest when $x = Xb$, (1.22) gives $AXb = P_{R(A)}b$ for all $b$, and therefore

$$AX = P_{R(A)}$$

■

When the system $Ax = b$ has a multiplicity of solutions for $x$, there is a unique solution of minimum-norm.

The following theorem relates minimum-norm solutions of $Ax = b$ and $\{1,4\}$-inverses of $A$, characterizing each of these two concepts in terms of the other.

**Theorem 1.4.6** *[1] Let $A \in \mathbb{C}^{m\times n}, b \in \mathbb{C}^{m}$. If $Ax = b$ has a solution for $x$, the unique solution for which $\|x\|$ is smallest is given by*

$$x = A^{(1,4)}b,$$

*where $A^{(1,4)} \in A\{1,4\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ is such that, whenever $Ax = b$ has a solution, $x = Xb$ is the solution of minimum-norm, then $X \in A\{1,4\}$.*

**Proof.**If $Ax = b$ is consistent, then for any $A^{(1,4)} \in A\{1,4\}$, $x = A^{(1,4)}b$ is a solution, lies in $R(A^*)$ and thus, is the unique solution in $R(A^*)$, and thus the unique minimum-norm solution.

Conversely, let $X$ be such that, for all $b \in R(A)$, $x = Xb$ is the solution of $Ax = b$ of minimum-norm.

Setting b equal to each column of $A$, in turn, we conclude that

$$XA = A^{(1,4)}A$$

and $X \in A\{1,4\}$ ■

## 1.5 Research and results on some matrix equation involving generalized inverses

In this section we will derive some research and results obtained recently about some well known matrix equations.

**Theorem 1.5.1** *[24] Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$ be given. Then,* *(a) There exists an* $X \in \mathbb{C}^{n\times q}$ *such that*

$$AX = C \tag{1.24}$$

*if and only if* $R(C) \subseteq R(A)$*, or equivalently,* $AA^{+}C = C$ *.*

*In this case, the general solution of (1.10) can be written in the following parametric form*

$$X = A^{+}C + F_A V \tag{1.25}$$

*where* $V \in \mathbb{C}^{n\times q}$ *is arbitrary*

*(a) There exists an* $X \in \mathbb{C}^{n\times p}$ *such that*

$$AXB = C \tag{1.26}$$

*if and only if* $R(C) \subseteq R(A)$ *and* $R(C^*) \subseteq R(B^*)$*, or equivalently,* $AA^{+}CB^{+}B = C$*.*

*In this case, the general solution of (1.12) can be written as*

$$X = A^{+}CB^{+} + F_A V_1 + V_2 E_B, \tag{1.27}$$

*where* $V_1, V_2 \in \mathbb{C}^{n\times p}$ *are arbitrary.*

**Theorem 1.5.2** *[24] Let* $A_j \in \mathbb{C}^{m_j\times n}, B_j \in \mathbb{C}^{p\times q_j}$ *and* $C_j \in \mathbb{C}^{m_j\times q_j}$ *be given,* $j = 1, 2$*. Then*

*(a)[18] There exists an* $X \in \mathbb{C}^{n\times p}$ *such that*

$$A_1XB_1 = C_1 \quad and \quad A_2XB_2 = C_2 \tag{1.28}$$

*if and only if*

$$R(C_j) \subseteq R(A_j), R(C_j^*) \subseteq R(B_j^*), r = \begin{bmatrix} C_1 & 0 & A_1 \\ 0 & -C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r[B_1, B_2], j = 1, 2. \tag{1.29}$$

*(b) [22]Under (1.29), the general common solution of (1.28) can be written as*

$$X = X_0 + F_A V_1 + V_2 E_B + F_{A_1} V_3 E_{B_2} + F_{A2} V_4 E_{B_1}, \tag{1.30}$$

*where $X_0$ is a special solution of (1.28) ,*

$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, B = [B_1 \quad B_2]$ *and the four matrices $V_1, ..., V_4 \in \mathbb{C}^{n\times p}$ are arbitrary.*

**Theorem 1.5.3** *[24] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_H^m$ be given. Then*

*(a)[9] There exists an $X \in \mathbb{C}_H^n$ such that*

$$AXA^* = B \tag{1.31}$$

*if and only if $R(B) \subseteq R(A)$, or equivalently, $AA^+B = B$*

*(b) [23] Under $R(B) \subseteq R(A)$, the general Hermitian solution of (1.31) can be written as*

$$X = A^+B(A^+)^* + F_A V + V^* F_A \tag{1.32}$$

*where $V \in \mathbb{C}^{n\times n}$ is arbitrary.*

*(c) [1] There exists an $X \in \mathbb{C}^{n\times n}$ such that*

$$AXX^*A^* = B \geq 0 \tag{1.33}$$

*if and only if $R(B) \subseteq R(A)$.*

*In this case, the general solution of(1.33) can be written in the following parametric form*

$$XX^* = \left(A^+B^{\frac{1}{2}} + F_A W\right)\left(A^+B^{\frac{1}{2}} + F_A W\right)^* \tag{1.34}$$

*where $W \in \mathbb{C}^{n\times m}$ is arbitrary*

***Theorem 1.5.4*** *[9] Let $A$ and $B$ be given matrices in $\mathbb{C}^{m\times n}$ such that the equation*

$$AX = B \tag{1.35}$$

*is consistent. The equation (1.35) has a Hermitian solution if and only if*

$$BA^* \text{ is Hermitian} \tag{1.36}$$

*in which case a general Hermitian solution is*

$$X = A^-B + B^*(A^-)^* - AAB^*(A)^* + (I - A^-A)U(I - A^-A)^*, \tag{1.37}$$

*where $A^-$ is an arbitrary g-inverse of $A$ and $U$ an arbitrary Hermitian matrix in $\mathbb{C}^{n\times n}$*

***Proof.*** *Let $X$ be an Hermitian solution of (1.35). Then $BA^* = AXA^*$ is clearly Hermitian. This shows the necessity of (1.36).*
*For sufficiency, check that when $BA^*$ is Hermitian so are $AB^*$ and $X_0 = AB + B^*(A)^* - A^-AB^*(A)^*$ and that $X_0$ satisfies (1.35).*
*A general Hermitian solution is obtained by adding to $X_0$ a general Hermitian solution of the homogeneou equation $AX = 0$.* ■

***Theorem 1.5.5*** *[9] The equation (1.35) has a nonnegative definite solution if and only if*

$$BA^* \text{ is nonnegative } definite \text{ }, rankBA^* = rankB, \tag{1.38}$$

*in which case the nonnegative definite solution is given by :*

$$X = B^*(BA^*)^-B + (I - A^-A)U(I - A^-A)^* \tag{1.39}$$

*where $(BA^*)^{-1}$ and $A^-$ are arbitrary g-inverses of $BA^*$ and $A$ respectively and $U$ is an arbitrary nonnegative definite matrix in $\mathbb{C}^{n\times n}$.*

***Proof.*** *Let $X$ be a nonnegative definite solution of (1.35) . Then $BA^* = AXA^*$ is clearly nonnegative definite and rank $BA^* = rankAXA^* = rankAX = rankB$ This shows the necessity of (1.36).*
*For sufficiency, check that when (1.36) is true, $B^*(BA^*)^-B$ is invariant under the choice of a g-inverse of $BA^*$.*
*Since a nonnegative definite matrix such as $BA^*$ has a nonnegative definite g-inverse, it is seen that $X_0 = B^*(BA^*)B$ is nonnegative definite. Also, $AX_0 = AB^*(BA^*)B = BA^*(BA^*)B = B$. Sufficiency of (1.36) is thus established.* ■

***Theorem 1.5.6*** *[1] If $A, B, C,$ and $D$ are given, then $AX = B$, $XC = D$ has a solution in*

$\mathbb{C}^{n\times m}$ *if and only if*

$$BC = AD, B = AA^{+}B, \quad \text{and} \quad D = DC^{+}C \tag{1.40}$$

*Moreover, its general solution can be expressed as*

$$X = A^{+}B + (I_n - A^{+}A)DC^{+} + (I_n - A^{+}A)F(I_m - CC^{+}), \forall F \in \mathbb{C}^{n\times m}. \tag{1.41}$$

***Proof.*** *The condition is obviously necessary. To show that it is sufficient, put*

$$X = A^{+}C + DB^{+} - A^{+}ADB^{+}$$

*which is a solution if the required conditions $AA^{+}C = C, DB^{+}B = D, AD = CB$ are satisfied.*

■

***Theorem 1.5.7*** *[11]Let $A$ and $C$ be given matrices in $\in \mathbb{C}^{m\times n}$ and $B$, $D$ be given matrices in $\in \mathbb{C}^{n\times p}$ such that the equation*

$$AX = B, XC = D \tag{1.42}$$

*are consistent:*

*a) These equations have a common Hermitian solution if and only if*

$$M = \begin{pmatrix} BA^{*} & BC \\ D^{*}A^{*} & D^{*}C \end{pmatrix} \tag{1.43}$$

*is Hermitian, in which case a general Hermitian solution is*

$\binom{A}{C^{*}}^{-}\binom{B}{D^{*}} + \left(B^{*}\vdots D\right)\left[\binom{A}{C^{*}}^{-}\right]^{*} - \binom{A}{C^{*}}^{-}M\left[\binom{A}{C^{*}}^{-}\right]^{*} + \left[I - \binom{A}{C^{*}}^{-}\binom{A}{C^{*}}\right]U\left[I - \binom{A}{C^{*}}^{-}\binom{A}{C^{*}}\right]^{*}$

*where $U$ is an arbitrary Hermitian matrix in $\mathbb{C}^{n\times n}$.*

*b) These equations have a common nonnegative definite solution if and only if*

$$M \text{ is nonnegative definite and } rankM = rank(B^{*}\vdots D), \tag{1.44}$$

*in which case a general nonnegative definite solution is*

$$(B^{*}\vdots D)M\begin{pmatrix} B \\ D^{*} \end{pmatrix}\left[I - \begin{pmatrix} A \\ C^{*} \end{pmatrix}^{-}\begin{pmatrix} A \\ C^{*} \end{pmatrix}\right]U\left[I - \begin{pmatrix} A \\ C^{*} \end{pmatrix}^{-}\begin{pmatrix} A \\ C^{*} \end{pmatrix}\right]^{*} \tag{1.45}$$

*where $U$ is an arbitrary nonnegative definite matrix in $\mathbb{C}^{n\times n}$.*

***Proof.*** *Observe that for $AX = B$, $XC = D$ to have a common Hermitian (nonnegative definite) solution it is necessary and sufficient that the equation*

$$\begin{pmatrix} A \\ C^* \end{pmatrix} X \begin{pmatrix} B \\ D^* \end{pmatrix}$$

*has a Hermitian ( nonnegative definite) solution* ■

## 1.6 Generalized reflexive and anti-reflexive matrices:

**Definition 1.6.1** *Reflection matrix : is a matrix that is used to reflect an object over a line or plane.*

**Properties:**

A reflection across an axis followed by a reflection in a second axis not parallel to the first one results in a total motion that is a rotation around the point of intersection of the axes, by an angle twice the angle between the axes.

The matrix for a reflection is orthogonal with determinant -1 and eigenvalues $-1, 1, 1, ..., 1$. The product of two such matrices is a special orthogonal matrix that represents a rotation. Every rotation is the result of reflecting in an even number of reflections in hyperplanes through the origin, and every improper rotation is the result of reflecting in an odd number. Thus reflections generate the orthogonal group, and this result is known as the Cartan–Dieudonne theorem. Similarly the Euclidean group, which consists of all isometries of Euclidean space, is generated by reflections in affine hyperplanes.

**Definition 1.6.2** *[2] Let $P$ be some generalized reflection matrix of dimension $n$.*
- *Reflexive matrices : A matrix $A \in \mathbb{C}^{n\times n}$ is said to be reflexive with respect to $P$ if $A = PAP$.*
- *anti-reflexive matrices : A matrix $A \in \mathbb{C}^{n\times n}$ is said to be anti-reflexive with respect to $P$ if $A = -APA$.*

**Definition 1.6.3** *[2] Let $P$ and $Q$ be two generalized reflection matrices of dimension $n$ and $m$, respectively.*
- *Generalized reflexive matrices : A matrix $A \in \mathbb{C}^{n\times m}$ is said to be generalized reflexive with respect to the matrix pair $(P, Q)$ if $A = PAQ$.*
- *Generalized anti-reflexive matrices : A matrix $A \in \mathbb{C}^{n\times m}$ is said to be generalized anti-reflexive with respect to the matrix pair $(P, Q)$ if $A = -PAQ$.*

the matrix $A$, real or complex, is a generalized reflexive matrix with respect to $(P, Q)$ since $A = PAQ$

**Definition 1.6.4** *[2] Let $\mathbb{C}_r^{n\times m}(P,Q)$ and $\mathbb{C}_a^{n\times m}(P,Q)$, where the order of $P$ and $Q$ in the matrix pair is important, be two subsets of the space $\mathbb{C}^{n\times m}$ defined by*

$$\mathbb{C}_r^{n\times m}(P,Q) = \left\{A \mid A \in \mathbb{C}^{n\times m} \quad and \quad A = PAQ\right\} \tag{1.46}$$

$$\mathbb{C}_a^{n\times m}(P,Q) = \left\{A \mid A \in \mathbb{C}^{n\times m} \quad and \quad A = -PAQ\right\} \tag{1.47}$$

*where $P$ and $Q$ are two generalized reflection matrices of dimension $n$ and $m$, respectively.*

Here we use the subscript $r(a)$ to reflect the generalized reflexive (antireflexive) nature of the subsets . Note that if $m = n$ and $Q = P$, then $\mathbb{C}_r^{n\times m}(P,Q)$ and $\mathbb{C}_a^{n\times m}(P,Q)$ reduce to $\mathbb{C}_r^{n\times m}(P)$ and $\mathbb{C}_a^{n\times m}(P)$, respectively.
In the case where $m = 1$ and $Q = 1$, $\mathbb{C}_r^{n\times m}(P,Q)$ and $\mathbb{C}_a^{n\times m}(P,Q)$ become $\mathbb{C}_r^{n}(P)$ and $\mathbb{C}_a^{n}(p)$, respectively.

### 1.6.1 Some research on reflexive and anti reflexive matrices

**Theorem 1.6.1** *[4] Let $P$ and $Q$ be two generalized reflection matrices of dimensions $n$ and $m$, respectively, and $\alpha$, $\beta \in \mathbb{C}$ .*

1. *If $A$ and $B$ are both in $\mathbb{C}_r^{n\times m}(P,Q)$, then*

$$(\alpha A^+ + \beta B^+) \in \mathbb{C}_r^{m\times n}(Q,P).$$
$$(\alpha A^* + \beta B^*) \in \mathbb{C}_r^{m\times n}(Q,P).$$
$$A^*B \in \mathbb{C}_r^{m\times m}(Q), \quad and \quad AB^* \in \mathbb{C}_r^{n\times n}(P).$$

2. *If $A$ and $B$ are both in $\mathbb{C}_a^{n\times m}(P,Q)$, then*

$$(\alpha A^+ + \beta B^+) \in \mathbb{C}_a^{m\times n}(Q,P).$$
$$(\alpha A^* + \beta B^*) \in \mathbb{C}_a^{m\times n}(Q,P).$$
$$A^*B \in \mathbb{C}_r^{m\times m}(Q), \quad and \quad AB^* \in \mathbb{C}_r^{n\times n}(P).$$

3. *If $A$ is in $\mathbb{C}_r^{n\times m}(P,Q)$ and $B$ is in $\mathbb{C}_a^{n\times m}(P,Q)$, or vice versa, then*

$$(\alpha A^*A + \beta B^*B) \in \mathbb{C}_r^{m\times m}(Q,P).$$
$$(\alpha AA^* + \beta BB^*) \in \mathbb{C}_a^{n\times n}(Q,P).$$
$$A^*B \in \mathbb{C}_r^{m\times m}(Q), \quad and \quad AB^* \in \mathbb{C}_a^{n\times n}(P).$$

**Proof.**The generalized inverse of a matrix $A \in \mathbb{C}^{n\times m}$ is typically defined to be the unique matrix $X$ that satisfies the following four Moore-Penrose conditions [21]

$(a)AXA = A$, $(b)XAX = X$, $(c)(AX)^* = AX$, and $(d)(XA)^* = XA$.

To prove part 1, we shall first prove that both $A^+$ and $B^+$ are in $\mathbb{C}_r^{m\times n}(Q,P)$. Sub-stitution of $A = PAQ$ into the condition $(a)$ with $A^+$ replacing $X$ yields

$$PAQA^+PAQ = PAQ \tag{1.48}$$

since $A^+$ is the generalized inverse of $A$ .

Premultiplying and postmultiplying both sides of (3) by $P^{-1}$ and $Q^{-1}$, respectively, we have

$$AYA = A,$$

where $Y = QA^+P$. Observe that $Y$ satisfies the first Moore-Penrose condition. By using the fact that both $P$ and $Q$ are unitary Hermitian matrices, it can easily be shown that $Y$ also satisfies the other three Moore-Penrose conditions. Therefore, $Y$ is a generalized inverse of $A$. Since the Moore-Penrose inverse is known to be unique, we conclude that $A^+ = Y$ and, therefore,

$$A^+ = QA^+P \in \mathbb{C}_r^{m\times n}(Q,P).$$

Likewise,

$$B^+ = QB^+P \in \mathbb{C}_r^{m\times n}(Q,P).$$

Accordingly, $(\alpha A^+ + \beta B^+) \in \mathbb{C}_a^{m\times n}(Q,P)$, The proof for the rest requires no further knowledge and is, therefore, omitted. Analogous proof can also be obtained for parts 2 and 3. ■

**Theorem 1.6.2** *[4] Given two generalized reflection matrices $P$ of dimension $n$ and $Q$ of dimension $m$, any matrix $A \in \mathbb{C}^{n\times m}$ can be decomposed into two parts $U$ and $V$, $U + V = A$, such that $U \in \mathbb{C}_r^{n\times m}(P,Q)$ and $V \in \mathbb{C}_a^{n\times m}(P,Q)$.*

**Proof.**Take

$$U = \frac{1}{2}(A + PAQ) \quad \textit{and} \quad V = \frac{1}{2}(A - PAQ) \tag{1.49}$$

and employ the involutory property $P^2 = I$ *and* $Q^2 = I$. The proof is trivial and, thus, omitted.

Two special instances of this theorem can be found in [5] ,[3], where one is obtained by setting $m = 1$ and $Q = 1$ for vectors and the other is the special case when $m = n$ and $Q = P$ for square matrices. ■

**Theorem 1.6.3** *[4] Given a linear least-squares problem*

$\min_x \|Ax - b\|$, $A \in \mathbb{C}^{n\times m}, x \in \mathbb{C}^m$ , $b \in \mathbb{C}^n$, $m \leq n$,.

*where $A$ is assumed to have full column rank , i.e., $rank(A) = m$ , let $\tilde{x}$ be the unique solution to the problem and $\tilde{r} = b - A\tilde{x}$ , the residual .*

1. *If $A \in \mathbb{C}_r^{n\times m}(P,Q)$, then*

$$\tilde{x} \in \mathbb{C}_r^m(Q) \quad \textit{and} \quad \tilde{r} \in \mathbb{C}_r^n(P) if b \in \mathbb{C}_r^n(P), \tag{1.50}$$

$$\tilde{x} \in \mathbb{C}_a^m(Q) \quad \textit{and} \quad \tilde{r} \in \mathbb{C}_a^n(P) if b \in \mathbb{C}_a^n(P), \tag{1.51}$$

2. *If $A \in \mathbb{C}_a^{n\times m}(P,Q)$, then*

$$\tilde{x} \in \mathbb{C}_r^m(Q) \quad \textit{and} \quad \tilde{r} \in \mathbb{C}_a^n(P) if b \in \mathbb{C}_a^n(P), \tag{1.52}$$

$$\tilde{x} \in \mathbb{C}_a^m(Q) \quad \textit{and} \quad \tilde{r} \in \mathbb{C}_r^n(P) if b \in \mathbb{C}_r^n(P), \tag{1.53}$$

**Proof.**The proof for part 2 is analogous to that for part 1. Therefore, we need only prove part 1. From the assumption that $A \in \mathbb{C}_r^{n\times m}(P,Q)$, we have $A = PAQ$, where $P$ and $Q$ are, by definition, generalized reflection matrices and thus

$$P = P^* = P^{-1} \quad \textit{and} \quad Q = Q^* = Q^{-1}.$$

Since $rank(A) = m$, $A^+$ can be expressed as

$$\begin{aligned} A^+ &= (A^*A)^{-1} A^* = (QA^*PPAQ)^{-1} QA^*P \\ &= (QA^*AQ)^{-1} QA^*P = Q(A^*A)^{-1} QQA^*P \\ &= QA^+P. \end{aligned}$$

It follows that if $b = Pb$, we have

$$\tilde{x} = A^+b = QA^+Pb = QA^+b = Q\tilde{x}$$

and

$$\tilde{r} = b - A\tilde{x} = Pb - PAQQ\tilde{x} = P(b - A\tilde{x}) = P\tilde{r}.$$

Analogously, if $b = -Pb$, then

$$\tilde{x} = A^{+}b = QA^{+}Pb = -QA^{+}b = -Q\tilde{x}$$

and

$$\tilde{r} = b - A\tilde{x} = -Pb - PAQ(-Q\tilde{x}) = -P\left(b - A\tilde{x}\right) = -P\tilde{r}$$

. This completes our proof. ■

**Exemple 1.6.1** *Consider the linear least-squares solution to the following over determined linear system:*

$$Ax = b, \quad where \quad A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \\ 2 & 4 \\ 3 & 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad and \quad b = \begin{bmatrix} 16 \\ 15 \\ 16 \\ 15 \end{bmatrix} \tag{1.54}$$

*Let* $P = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}$ *and* $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

*where $I_2$ is the identity matrix of dimension 2. It is easy to see that $A = PAQ$ and $b = Pb$. we know that $x = Qx$, i.e., $x_1 = x_2$ Solving (1.54) is therefore, equivalent to solving*

$$\begin{bmatrix} 6 \\ 4 \end{bmatrix} [x_1] = \begin{bmatrix} 16 \\ 15 \end{bmatrix} \tag{1.55}$$

*The least-squares solution to (1.55) is $x_1 = 3$.*
*Accordingly, the solution to the original problem is $x_1 = x_2 = 3$ , which can be verified by solving the normal equation $A^T Ax = A^T b$ .*
*The residual $r$,$r = b - Ax = [-2, 3, -2, 3]^T$, is obviously reflexive with respect to $P$, as expected, since $r = Pr$ It deserves mentioning that the generalized reflexivity property of the matrix $A$ usually comes from physical models with some sort of reflexive symmetry.*
*The vector $b$, nevertheless, could be arbitrary and will not have any special form in general . This, however, should not impose any difficulty since given $P$, any vector can be decomposed*

*into a reflexive and an antireflexive part.*

*Once the decomposition is performed. Theorem* 1.8.1 *can be employed to take advantage of the reflexivity and antireflexivity present in the problem, as shown in the next example where we choose $b$ to be neither reflexive nor antireflexive.*

**Remark 1.6.1** *Note that the converse of* $(1.50), (1.51), (1.52),$ *and* $(1.53)$ *does not hold in general.*

*For example, let $b$ be some vector that is neither reflexive nor antireflexive with respect to $P$, i.e., $b \notin \mathbb{C}_r^n(P)$ and $b \notin \mathbb{C}_a^n(P)$.*

# CHAPTER 2

# THE $(P, Q)$ GENERALIZED REFLEXIVE AND ANTI-REFLEXIVE SOLUTION OF AX=B

In this chapter, we establish some new conditions for the existence and the representations for the $(P, Q)$ generalized reflexive and anti-reflexive solutions to matrix equation $AX = B$ with respect to the generalized reflection matrix dual $(P, Q)$. Moreover, in corresponding solution sets of the equation, the explicit expressions of the nearest matrix to a given matrix in the Frobenius norm have been provided.

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices. For $A \in \mathbb{C}^{n\times n}$, its trace will be denoted by $tr(A)$. For $A \in \mathbb{C}^{m\times n}$, its conjugate transpose, Frobenius norm and Moore-Penrose inverse will be denoted by $A^*$, $\|A\|_F = \sqrt{tr(AA^*)} = \sqrt{tr(A^*A)}$ and $A^+$ respectively.

In represents the identity matrix of size $n$. For convenience, we denote $E_A = I - AA^+$ and $F_A = I - A^+A$.

A matrix $P \in \mathbb{C}^{n\times n}$ is called a generalized reflection matrix if $P^* = P$ and $P^2 = I$. Chen [4] and Chen and Sameh [2] defined two subspaces of matrix:

$$\begin{aligned}
\mathbb{C}_r^{n\times m}(P, Q) &= \{A \mid A \in \mathbb{C}^{n\times m} \quad and \quad A = PAQ\} \\
\mathbb{C}_a^{n\times m}(P, Q) &= \{A \mid A \in \mathbb{C}^{n\times m} \quad and \quad A = -PAQ\}
\end{aligned}$$

where $P, Q$ are generalize reflection matrices. The matrices $A \in \mathbb{C}_r^{m\times n}(P, Q), B \in \mathbb{C}_a^{m\times n}(P, Q)$ are said to be a $(P, Q)$ generalized reflexive and $(P, Q)$ generalized antireflexive matrices respectively with respect to the generalized reflection matrix dual $(P, Q)$.

The $(P, Q)$ generalized reflexive and anti-reflexive matrices have applications in system and control theory, in engineering, in scientific computations and various other fields (see for example [4],[2]) .

Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{m\times k}$ be known, and $X \in \mathbb{C}^{n\times k}$ be variable, consider the following equation

$$AX = B \tag{2.1}$$

It is well known that (2.1) is consistent if and only if $AA^{+}B = B$, in this case, a general solution can be given by $X = A^{\dagger}B + F_A V$ where $V \in \mathbb{C}^{n\times n}$ is arbitrary. Khatri and Mitra [11] investigated the Hermitian solution and nonnegative definite solution to (2.1). The Re-nonnegative definite (Re-nnd) and Re-positive definite (Re-pd) solutions were also investigated in [12] and [25]. For the $(P,Q)$ generalized reflexive and anti-reflexive solutions to (2.1) when $n = k$, Zhang et al. [26] considered the following two problems by using the structure properties of matrices : the first problem is to consider the $(P,Q)$ generalized reflexive and anti-reflexive solutions with respect to the generalized reflection matrix dual $(P,Q)$, the other is to consider the matrix nearness problem

$$\min_{X\in S_X} \|X - C\|_F \tag{2.2}$$

where $C \in \mathbb{C}^{n\times k}$ is a given matrix, and $S_X$ is the solution set of Eq. (2.1). Specially, when $P = Q$, Peng and Hu [20] studied these two problems. Using the same method, Cvetkovic-Ilic [7] and Peng et al. [19] considered the reflexive and anti-reflexive solutions to $AXB = C$, Zhou and Yang [27] studied the Hermitian reflexive solutions and the anti-Hermitian reflexive solutions to matrix equations $(AX = B, XC = D)$ .

Motivated by the above work, in this paper, we restudy these two problem investigated by Zhang et al.[19]. By using the Moore-Penrose inverse, we present sufficient and necessary conditions for the existence and the expressions of the $(P,Q)$ generalized reflexive and anti-reflexive solutions to $AX = B$. The matrix nearness problem (2.2) is also considered.

Before giving the main results, we first introduce several lemmas as follows. The following two results can be easy to verify by the definitions.

**Lemma 2.0.1** *[15]*

*Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{k\times n}$. If $BA^* = 0$, then*

$$\begin{pmatrix} A \\ B \end{pmatrix}^{+} = \left(A^{+}B^{+}\right).$$

**Lemma 2.0.2** *[15] Let $A, B \in \mathbb{C}^{m\times n}$. If $BA^* = 0 (or B^*A = 0)$, then*

$$\| A + B \|_F^2 = \| A \|_F^2 + \| B \|_F^2$$

**Lemma 2.0.3** *[15] Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be two generalized reflection matrices, and $A \in \mathbb{C}^{m\times n}$ be variable. Then every $m \times n(P,Q)$ generalized reflexive matrix can be written as $A+PAQ$, and every $m\times n(P,Q)$ generalized anti-reflexive matrix can be written as $A = -PAQ$.*

## 2.1 The solution of the matrix Equation $AX = B$

In this section, our purpose is to establish some new conditions for the existence and representations for the $(P,Q)$ generalized reflexive and anti-reflexive solutions to $AX = B$.
For convenience, the following notations will be used in this paper. For $A \in \mathbb{C}^{m\times n}, P \in \mathbb{C}^{n\times n}$ is a generalized reflection matrix, we set

$$A_1(P) = \begin{pmatrix} A(I+P) \\ A(I-P) \end{pmatrix}, A_2(P) = \begin{pmatrix} A(I-P) \\ A(I+P) \end{pmatrix} \tag{2.3}$$

and denote $[A_i(P)]^*$ and $[A_i(P)]^+$ by $A_i^*(P)$ and $A_i^+(P)(i = 1,2)$ for short respectively.
Next, we will give some properties on $A_1(P)$ and $A_2(P)$ defined by (2.3).

**Lemma 2.1.1** *[15] Let $P \in \mathbb{C}^{n\times n}$ be a generalized reflection matrix, and $A \in \mathbb{C}^{m\times n}$. Then*

1. $F_{A_1(P)} \in \mathbb{C}_r^{n\times n}(P,P)$, *and* $AF_{A_1(P)} = 0$ .
2. $F_{A_2(P)} \in \mathbb{C}_r^{n\times n}(P,P)$, *and* $AF_{A_2(P)} = 0$ .
3. $F_{A_1(P)} = F_{A_2(P)}$.

**Proof.**In view of Lemma 2.0.1, we have
$A_1^+(P) = ([A(I+P)]^+[A(I-P)]^+)$
$F_{A_1(P)} = I_n - [A(I+P)]^+A(I+P) - [A(I-P)]^+A(I-P)$
Hence,

$$\begin{aligned} PF_{A_1(P)}P &= I_n - P[A(I+P)]^+A(I+P)P - P[A(I-P)]^+A(I-P)P \\ &= I_n - [A(I+P)]^+A(I+P) - [A(P-I)]^+A(P-I) \\ &= I_n - [A(I+P)]^+A(I+P) - [A(I-P)]^+A(I-P) \\ &= F_{A_1(P)} \end{aligned}$$

means that $F_{A_1(P)}$ is a $(P,P)$ generalized reflexive matrix with respect to the generalized reflection matrix P.
Moreover

$$\begin{aligned} AF_{A_1(P)} &= A - A[A(I+P)]^+A(I+P) - A[A(I-P)]^+A(I-P) \\ &= A - \tfrac{1}{2}[A(I+P)+A(I-P)][A(I+P)]^+A(I+P) \\ &- \tfrac{1}{2}[A(I+P)+A(I-P)][A(I-P)]^+A(I-P) \\ &= 0 \end{aligned}$$

Similarly, the statements (2) and (3) can be deduced. ■

In order to discuss the $(P,Q)$ generalized reflexive and anti-reflexive solutions to $AX = B$, we first consider a special case $B = 0$.

**Theorem 2.1.1** *[15] Let $A \in \mathbb{C}^{m\times n}$ be given. Then*

1. *The general solution $X \in \mathbb{C}_r^{n\times k}(P,Q)$ to matrix equation $AX = 0$ can be expressed as*

$$X = F_{A_1(P)}V \tag{2.4}$$

*where $V \in \mathbb{C}_r^{n\times k}(P,Q)$ is arbitrary.*

2. *The general solution $X \in \mathbb{C}_a^{n\times k}(P,Q)$ to matrix equation $AX = 0$ can be expressed as*

$$X = F_{A_1(P)}W$$

*where $W \in \mathbb{C}_a^{n\times k}(P,Q)$ is arbitrary.*

**Proof.**It follows from Lemma 2.1.1 that $X = F_{A_1(P)}V$ is a $(P,Q)$ generalized reflexive solution to $AX = 0$ with $V \in \mathbb{C}_r^{n\times k}(P,Q)$.

On the other hand, suppose $Y$ is an arbitrary $(P,Q)$ generalized reflexive solution to $AX = 0$, i.e., $AY = 0$ and $PYQ = Y$, which implies $A_{1(P)}Y = 0$. Therefore, $Y = F_{A_1(P)}Y$, which is of the form (2.4) . That is to say, each $(P,Q)$ generalized reflexive solution to $AX = 0$ can be formed by (2.4).

Similarly, statement (2) can be verified. ■

**Theorem 2.1.2** *[15] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$ be given. Then*

1. *$AX = B$ has a solution $X \in \mathbb{C}_r^{n\times k}(P,Q)$ if and only if $A_1(P)A_1^+(P)B_1(Q) = B_1(Q)$. In this case, a general solution $X$ can be written as*

$$X = A_1^+(P)B_1(Q) + F_{A_1(P)}V, \tag{2.5}$$

*where* $V \in \mathbb{C}_r^{n\times k}(P,Q)$ *is arbitrary.*

2. $AX = B$ *has a solution* $X \in \mathbb{C}_a^{n\times k}(P,Q)$ *if and only if* $A_1(P)A_1^{+}(P)B_2(Q) = B_2(Q)$. *In this case, a general solution* $X$ *can be written as*

$$X = A_1^{+}(P)B_2(Q) + F_{A_1(P)}W,$$

*where* $W \in \mathbb{C}_a^{n\times k}(P,Q)$ *is arbitrary.*

**Proof.**(1) IF: If $A_1(P)A_1^{+}(P)B_1(Q) = B_1(Q)$, then

$$\begin{aligned}\begin{pmatrix} A(I+P)A_1(P)A_1^{+}(P)B_1(Q) \\ A(I-p)A_1(P)A_1^{+}(P)B_1(Q)\end{pmatrix} &= \begin{pmatrix} AA_1^{+}(P)B_1(Q) + APA_1^{+}(P)B_1(Q) \\ AA_1^{+}(P)B_1(Q) - APA_1^{+}(P)B_1(Q)\end{pmatrix} \\ &= \begin{pmatrix} B(I+Q) \\ B(I-Q)\end{pmatrix} = \begin{pmatrix} B+BQ \\ B-BQ\end{pmatrix}\end{aligned} \tag{2.6}$$

gives that $AA_1^{+}(P)B_1(Q) = B, i.e., A_1^{+}(P)B_1(Q)$ is a solution to $AX = B$ . By computation, we have

$$\begin{aligned} PA_1^{+}(P)B_1(Q)Q &= P\left([A(I+P)]^{+}[A(I-P)]^{+}\right)\begin{pmatrix} B(I+Q) \\ B(I-Q)\end{pmatrix} Q \\ &= P[A(I+P)]^{+}B(I+Q)Q + P[A(I-P)]^{+}B(I-Q)Q \\ &= [A(I+P)]^{+}B(I+Q) + [A(I-P)]^{+}B(I-Q) \\ &= A_1^{+}(P)B_1(Q).\end{aligned}$$

So, $A_1^{+}(P)B_1(Q)$ is a $(P,Q)$ generalized reflexive solution to $AX = B$.
For any $(P,Q)$ generalized reflexive solution $X$ to $AX = B$,
we have $A\left[X - A_1^{+}(P)B_1(Q)\right] = 0$. Then, (2.5) is followed by Theorem 2.1.1.

ONLY IF: (2.5) is a $(P,Q)$ generalized reflexive solution to $AX = B$, then $AA_1^{+}(P)B_1(Q) = B$, and $APA_1^{+}(P)B_1(Q) = BQ$ .
According to (2.6), $A_1(P)A_1^{+}(P)B_1(Q) = B_1(Q)$ is evident.
The proof of statement (2) is similar to (1), hence, the details are omitted. ■

## 2.2 The best solution estimator to a given matrix

In this section, we deduce the explicit expressions of the nearness solution to a given matrix, where we will find the solution $X$ of the problem

$$\min_{X\in S_X} \|X-C\|_F \tag{2.7}$$

where $C$ is given.

First, we verify the following results.

**Lemma 2.2.1** *[15] Let $C\in\mathbb{C}^{n\times k}, F_{A_1(P)}\in\mathbb{C}_r^{n\times n}(P,P), V\in\mathbb{C}_r^{n\times n}(P,Q)$ and $W\in\mathbb{C}_a^{n\times k}(P,Q)$. Then*

$$\|F_{A_1(P)}V-F_{A_1(P)}C\|_F^2=\|F_{A_1(P)}V-F_{A_1(P)}PCQ\|_F^2 \tag{2.8}$$

$$=\|F_{A_1(P)}V-\frac{1}{2}F_{A_1(P)}(C+PCQ)\|_F^2+\frac{1}{2}tr\left[F_{A_1(P)}(CC^*-CQC^*P)F_{A_1(P)}\right] \tag{2.9}$$

*Similarly ,*

$$\|F_{A_1(P)}W-F_{A_1(P)}C\|_F^2=\|F_{A_1(P)}W-F_{A_1(P)}PCQ\|_F^2 \tag{2.10}$$

$$=\|F_{A_1(P)}W-\frac{1}{2}F_{A_1(P)}(C+PCQ)\|_F^2+\frac{1}{2}tr\left[F_{A_1(P)}(CC^*-CQC^*P)F_{A_1(P)}\right] \tag{2.11}$$

**Proof.**Since $\|.\|_F$ is an unitarily invariant norm, according to the assumptions, (2.8) is obvious. Moreover, we have

$$\begin{aligned}
tr\left(F_{A_1}(P)VQC^*PF_{A_1(P)}\right) &= tr\left(PF_{A_1(P)}VQC^*PF_{A_1(P)}P\right)=tr\left(F_{A_1(P)}VC^*F_{A_1(P)}\right)\\
tr\left(F_{A_1(P)}PCQV^*F_{A_1(P)}\right) &= tr\left(PF_{A_1(P)}PCQV^*F_{A_1(P)}P\right)\\
&= tr\left(F_{A_1(P)}CV^*F_{A_1(P)}\right)\\
tr\left(F_{A_1(P)}CQC^*PF_{A_1(P)}\right) &= tr\left(PF_{A_1(P)}CQC^*PF_{A_1(P)}P\right)\\
&= tr\left(F_{A_1(P)}PCQC^*F_{A_1(P)}\right)\\
tr\left(F_{A_1(P)}PCC^*PF_{A_1(P)}\right) &= tr\left(F_{A_1(P)}CC^*F_{A_1(P)}\right)
\end{aligned}$$

Therefore

$\|F_{A_1(P)}V-\frac{1}{2}F_{A_1(P)}(C+PCQ)\|_F^2$

$=tr\left\{\left[F_{A_1(P)}V-\frac{1}{2}F_{A_1(P)}(C+PCQ)\right]\left[F_{A_1(P)}V-\frac{1}{2}F_{A_1(P)}(C+PCQ)\right]^*\right\}$

$$\begin{aligned}
&= tr\{F_{A_1(P)}VV^*F_{A_1(P)} - \tfrac{1}{2}F_{A_1(P)}VC^*F_{A_1(P)} - \tfrac{1}{2}F_{A_1(P)}VQC^*PF_{A_1(P)} - \tfrac{1}{2}F_{A_1(P)}CV^*F_{A_1(P)} \\
&- \tfrac{1}{2}F_{A_1(P)}PCQV^*F_{A_1(P)} + \tfrac{1}{4}F_{A_1(P)}CC^*F_{A_1(P)} + \tfrac{1}{4}F_{A_1(P)}CQC^*PF_{A_1(P)} \\
&+ \tfrac{1}{4}F_{A_1(P)}PCQC^*F_{A_1(P)} + \tfrac{1}{4}F_{A_1(P)}PCC^*F_{A_1(P)}\} \\
&= tr\{F_{A_1(P)}VV^*F_{A_1(P)} - F_{A_1(P)}VC^*F_{A_1(P)} - F_{A_1(P)}CV^*F_{A_1(P)} \\
&+ \tfrac{1}{2}F_{A_1(P)}CC^*F_{A_1(P)} + \tfrac{1}{2}F_{A_1(P)}CPC^*PF_{A_1(P)}\} \\
&= \|F_{A_1(P)}V - F_{A_1(P)}C\|_F^2 - \tfrac{1}{2}tr[F_{A_1(P)}(CC^* - CQC^*P)F_{A_1(P)}]
\end{aligned}$$

Hence, (2.9) is evident. Equation (2.10) and (2.11) can be verified similarly ■

The following theorem give the explicit expressions of the solutions of the matrix nearness problem (2.2)

**Theorem 2.2.1** *[15] Given a matrix $C \in \mathbb{C}^{n\times k}$.*

1. *Assume the solution set $S_X \subseteq \mathbb{C}_r^{n\times k}(P,Q)$ of Equation. $AX = B$ is nonempty, then the matrix nearness problem* (2.4) *has an unique solution $\hat{X}$ in $S_X$, which can be written as*

$$\hat{X} = A_1^+(P)B_1(Q) + \frac{1}{2}F_{A_1(P)}(C + PCQ) \tag{2.12}$$

2. *Assume the solution set $S_X \subseteq \mathbb{C}_a^{n\times k}(P,Q)$ of Eq .* (1.19) *is nonempty , then the matrix nearness problem* (2.4) *has an unique solution $\hat{X}$ in $S_X$ , which can be written as*

$$\hat{X} = A_1^+(P)B_2(Q) + \frac{1}{2}F_{A_1(P)}(C + PCQ)$$

**Proof.**(1) Let $X \in S_X$ it follows from Lemma 2.0.2 and (2.5) that

$$\begin{aligned}
\|X - C\|_F^2 &= \|A_1^+(P)B_1(Q) + F_{A_1}(P)V - C\|_F^2 \\
&= \|A_1^+(P)B_1(Q) - A_1^+(P)A_1(P)C\|_F^2 + \|F_{A_1(P)}V - F_{A_1(P)}C\|_F^2 \\
&= \|A_1^+(P)B_1(Q) - A_1^+(P)A_1(P)\|_F^2 \\
&\quad + \tfrac{1}{2}tr\left[F_{A_1(P)}(CC^* - CQC^*P)F_{A_1(P)}\right] \\
&\quad + \|F_{A_1(P)}V - \tfrac{1}{2}F_{A_1(P)}(C + PCQ)\|_F^2
\end{aligned}$$

Therefore, there exists $\hat{X} \in S_X$ such that the matrix nearness problem (2.2) holds if and only if there exists $V \in \mathbb{C}_r^{n\times k}(P,Q)$ such that

$$\min_{V\in\mathbb{C}_r^{n\times k}(P,Q)} \|F_{A_1(P)}V - \frac{1}{2}F_{A_1(P)}(C + PCQ)\|_F$$

Obviously , matrix equation $F_{A_1(P)}V = \frac{1}{2}F_{A_1(P)}(C + PCQ)$ is consistent for $V \in \mathbb{C}_r^{n\times k}(P,Q)$, for example, take $V = \frac{1}{2}(C + PCQ)$. Hence,

$$\min_{V\in\mathbb{C}_r^{n\times k}(P,Q)} \|F_{A_1(P)}V - \frac{1}{2}F_{A_1(P)}(C + PCQ)\|_F = 0$$

■

Therefore, (2.12) is evident.

Similarly, statement (2) can be obtained.

**Exemple 2.2.1** *Consider the linear matrix equation* $AX = 0$

*Where* $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \in \mathbb{C}^{4\times 3}$, *and Let* $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{3\times 3}, Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$ *be two generalized reflection matrices.*

*In this example we want to give the reflexive solution of the matrix equation* $AX = 0$, *by theorem 2.1.1, the general reflexive solution of* $AX = 0$ *is given by* $X = F_{A_1(P)}V$ *where* $V$ *is arbitrary reflexive matrix with appropriate size.*

*So firstly we compute* $A_1^+(P)$, *we have*

$$A_1(P) = \begin{bmatrix} A(I+P) \\ A(I-P) \end{bmatrix}.$$

$$A_1(P)=\begin{bmatrix} A(I+P) \\ A(I-P) \end{bmatrix}=\begin{bmatrix} A+AP \\ A-AP \end{bmatrix}=\begin{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$A_1(P)=\begin{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \end{bmatrix}$$

$$A_1^+(P)=\begin{bmatrix} (A(I+P))^+ & (A(I-P))^+ \end{bmatrix}=\begin{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \end{bmatrix}^+ \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}^+ \end{bmatrix}=$$

$$(A(I+P))^+=\begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \end{bmatrix}^+=?$$

$$(A(I+P)) = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = DC \textit{ is the full rank factorization}$$

*(see proposition 1.2.3) such that* $D = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix}$ *and*

$C = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

*So*

$$(A(I+P))^{+} = C^{*}(CC^{*})^{-1}(D^{*}D)^{-1}D^{*}.$$

$$(CC^{*})^{-1} = \left( \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)^{-1} = \frac{1}{2}$$

$$(D^{*}D)^{-1} = \left( \begin{bmatrix} 2 & 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right)^{-1} = \frac{1}{16}$$

$$\begin{aligned} (A(I+P))^{+} &= C^{*}(CC^{*})^{-1}(D^{*}D)^{-1}D^{*}. \\ &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} \frac{1}{16} \end{bmatrix} \times \begin{bmatrix} 2 & 2 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} \end{bmatrix} \end{aligned}$$

$$Now\ (A\,(I-P))^{+} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}^{+} =?$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = DC$$

*is the full rank factorization of* $(A\,(I-P)) = C^{*}(CC^{*})^{-1}(D^{*}D)^{-1}D^{*}$.

*So*

$$\begin{aligned}(A\,(I-P))^{+} &= C^{*}(CC^{*})^{-1}(D^{*}D)^{-1}D^{*}. \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \frac{1}{16} \times \begin{bmatrix} 2 & 2 & 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

*So,*

$$A_1^+(P) = \begin{bmatrix} (A(I+P))^+ & (A(I-P))^+ \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \end{bmatrix}^+ & \begin{bmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}^+ \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{A_1(P)} = I_3 - A_1^+(P)\,A_1(P) = I_3 - \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

*So the reflexive solution of the matrix equation $AX = 0$ is $X = F_{A_1(P)}V$*

*Where* $V = \begin{bmatrix} -1 & -3 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix} \in \mathbb{C}_r^{3\times3}(P,Q)$ *is arbitrary.*

$$\begin{aligned} X &= F_{A_1(P)}V \\ &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \\ 1 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & -2 & \frac{1}{2} \end{bmatrix} \end{aligned}$$

**Exemple 2.2.2** *Consider the linear matrix equation* $AX = B$

*Where* $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \in \mathbb{C}^{4\times3}$, $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} \in \mathbb{C}^{4\times2}$ *and Let* $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{C}^{3\times3}$ , $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{C}^{2\times2}$ *two generalized reflection matrices*

*In this example we want to compute a* ***reflexive solution*** *of the matrix equation* $AX = B$. *By theorem* 2.1.2, $AX = B$ *has a solution* $X \in \mathbb{C}_r^{n\times k}(P,Q)$ *if and only if*

$$A_1(P)A_1^{+}(P)B_1(Q) = B_1(Q)$$

. *In this case, a general solution* $X$ *can be written as*

$$X = A_1^{+}(P)B_1(Q) + F_{A_1(P)}V$$

*where* $V$ *is arbitrary reflexive matrix with appropriate size.*

*So from the previous example we have*

$$A_1^+(P) = \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_1(P) = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

*and*

$$F_{A_1(P)} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

*Now we calculate* $B_1(Q)$

$$B_1(Q) = \begin{bmatrix} B(I+Q) \\ B(I-Q) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \\ 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \end{bmatrix}$$

*we have*

$$A_1(P)A_1^{+}(P)B_1(Q) = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \\ 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \\ 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \end{bmatrix} = B_1(Q) \tag{2.13}$$

*So from the condition (2.13) the equation $AX = B$ have a reflexive solution $X \in \mathbb{C}_r^{3\times 2}(P,Q)$*

*defined by*

$$X = A_1^{+}(P)B_1(Q) + F_{A_1(P)}V$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \\ 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix}$$

*Where* $V = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix} \in \mathbb{C}_r^{3\times 2}(P,Q)$ *is chosen.*

*Here we are looking that $X$ is a reflexive solution for $AX = B$ because*

$$PXQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix} = X$$

*2. Now we will compute the* ***anti-reflexive solution*** *of $AX = B$ .*

*By theorem 2.1.2, $AX = B$ has a solution $X \in \mathbb{C}_a^{n\times k}(P,Q)$ if and only if*

$$A_1(P)A_1^{+}(P)B_2(Q) = B_2(Q)$$

*. In this case, a general solution $X$ can be written as*

$$X = A_1^{+}(P)B_2(Q) + F_{A_1(P)}W$$

*where $W$ is arbitrary anti-reflexive matrix with appropriate size.*

$$B_2\left(Q\right)=\begin{bmatrix} B\left(I-Q\right) \\ B\left(I+Q\right) \end{bmatrix}=\begin{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}=\begin{bmatrix} 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \\ 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \end{bmatrix}$$

*We have*

$$A_1(P)A_1^{+}(P)B_2(Q) = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 0 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \\ 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \\ 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \end{bmatrix} = B_2\left(Q\right) \tag{2.14}$$

*So from the condition (2.14) $AX = B$ have an anti-reflexive soltion $X \in \mathbb{C}_a^{3\times 2}(P,Q)$ defined by*

$$X = A_1^{+}(P)B_2(Q) + F_{A_1(P)}W$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \\ 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -4 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{3}{2} \\ 2 & 0 \\ 0 & -\frac{7}{2} \end{bmatrix}$$

*Where* $W = \begin{bmatrix} 0 & -4 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{C}_a^{3\times 2}(P,Q)$ *is chosen*

*Here we are looking that $X$ is an anti reflexif solution because*

$$PXQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{3}{2} \\ 2 & 0 \\ 0 & -\frac{7}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{3}{2} \\ -2 & 0 \\ 0 & \frac{7}{2} \end{bmatrix} = -X$$

*Assume that* $C = \begin{bmatrix} 1 & -2 \\ 1 & 3 \\ 5 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$ *is a given matrix*

*1. Now, we want to give the best matrix estimator to the matrix $C$, by theorem 2.2.1, in the set* *of the reflexive solutions,* ***the best matrix estimator*** *to the matix $C$ is given by*

$$\hat{X} = A_1^{+}(P)B_1(Q) + \frac{1}{2}F_{A_1(P)}(C + PCQ)$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \\ 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \left( \begin{bmatrix} 1 & -2 \\ 1 & 3 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \\ 10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$$

*2. Also in the set of the anti-reflexive solutions,* ***the best matrix estimator*** *to the matix $C$ is given by*

$$\hat{X} = A_1^{+}(P)B_2(Q) + \frac{1}{2}F_{A_1(P)}(C + PCQ)$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{16} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 4 \\ 0 & 4 \\ 0 & -4 \\ 4 & 0 \\ 4 & 0 \\ 4 & 0 \\ -4 & 0 \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \left( \begin{bmatrix} 1 & -2 \\ 1 & 3 \\ 5 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 5 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$$

CHAPTER 3

# THE COMMON $(P, Q)$ GENERALIZED REFLEXIVE AND ANTI-REFLEXIVE SOLUTIONS TO $AX = B$ AND $XC = D$

In this chapter we establish some conditions for the existence and the representations for the common $(P, Q)$ generalized reflexive and anti-reflexive solutions of matrix equations $AX = B$ and $XC = D$, where P and Q are two generalized reflection matrices.
Moreover, in the set of solutions of the equations, the explicit expression of the best matrix estimator to a given matrix in the Frobenius norm has been presented

Let $A \in \mathbb{C}^{m\times n}, B \in \mathbb{C}^{m\times k}, C \in \mathbb{C}^{k\times l}$ and $D \in \mathbb{C}^{n\times l}$ be known, and $X \in \mathbb{C}^{n\times k}$ be variable matrix, consider the following equations

$$AX = B, XC = D \tag{3.1}$$

It is well known that (1.1) has a common solution if and only if

$$AA^{+}B = B, \quad DC^{+}C = D, \quad BC = AD, \quad and \quad X = A^{+}B + F_{A}DC^{+} + F_{A}VE_{C}, \tag{3.2}$$

where $V$ is variable. For the constrained common solutions of (1.1), such as common Hermitian solution, common Re-nnd solution, common Hermitian reflexive solution and anti-Hermitian reflexive solution, were studied by Khatri and Mitra [11], Liu [12], Zhou and Yang [27] respectively. In[8], the authors investigated the common $(P, Q)$ generalized reflexive solution to (1.1). For the generalized reflexive and anti-reflexive solutions of $AX = B$, they were discussed in [15, 20, 26]. And for the generalized reflexive and anti-reflexive solutions of $AXB = C$, they were considered in [7, 19].
Another problem is the so called matrix nearness problem. Suppose the solution set composed

by the common $(P,Q)$ generalized reflexive or anti-reflexive solution of (3.1) is nonempty, and denoted by $S_E$.

Consider the matrix nearness problem

$$\min_{X \in S_E} \|X - X_0\|_F \tag{3.3}$$

where $X_0 \in \mathbb{C}^{n\times k}$ is a given matrix, this problem was also studied in [8]. Motivated by the above work, in this paper, we restudy the common $(P,Q)$ generalized reflexive and anti-reflexive solutions of matrix equations $AX = B$ and $XC = D$, present some new conditions for the existence and the representations for the common solutions. Then discuss the matrix nearness problem (3.3) .

Before giving the main results, we first introduce several lemmas as follows. The following two results can be easy to verify by the definitions.

## 3.1 The solution of the matrix equations $AX = B, XC = D$

In this section, our purpose is to establish some new conditions for the existence and representations for the common $(P,Q)$ generalized reflexive and anti-reflexive solutions of matrix equations $AX = B$ and $XC = D$. For convenience, the following notations will be used in this paper. For $A \in \mathbb{C}^{m\times n}$, generalized reflection matrices $P \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{m\times m}$, we set

$$A_1(P) = \begin{pmatrix} A(I+P) \\ A(I-P) \end{pmatrix}, A_2(P) = \begin{pmatrix} A(I+P) \\ A(I-P) \end{pmatrix}$$

$$A_{3Q} = ((I+Q)A(I-Q)A)\,, A_{4Q} = ((I-Q)A(I+Q)A)$$

And denote $(A_{iP})^+(or(A_{iQ})^+)$ by $A^+_{iP}(orA^+_{iQ})(i = 1,2,3,4)$ for short respectively. Next, we will give some properties on $A_{1P}, A_{2P}, A_{3Q}$ and $A_{4Q}$ defined by above.

**Lemma 3.1.1** *[13] Let $P \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{m\times m}$ e two generalized reflection matrices, $A \in \mathbb{C}^{m\times n}$. Then*

*(1)$F_{A_1P}, F_{A_2P} \in \mathbb{C}^{n\times n}_r(P,P)$, and $AF_{A_1P} = AF_{A_2P} = 0$.*

*(2)$E_{A_3Q}, E_{A_4Q} \in \mathbb{C}^{m\times m}_r(Q,Q)$, and $E_{A_3Q}A = E_{A_4Q}A = 0$.*

*(3)$F_{A_1P} = F_{A_2P}, E_{A_3Q} = E_{A_4Q}$.*

**Proof.**The results on$A_{1P}$ and $A_{2P}$ are provided by Lemma 2.1.1, and the results on $A_{3Q}$ and $A_{4Q}$ can be proved similarly ■

Recall that the next lemma was given in the second chapter, and we give it also here in order to reduce the reflexive (anti-reflexive) solution of matrix equation $XC = D$

**Lemma 3.1.2** *[13] Let $A \in \mathbb{C}_a^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$ be given. Then*

1. *$AX = B$ has a solution $X \in \mathbb{C}_r^{n\times k}(P, Q)$ if and only if $A_{1P}A_{1P}^{+}B_{1Q} = B_{1Q}$. In this case, a general solution $X$ can be written as*

$$X = A_{1P}^{+}B_{1Q} + F_{A_{1P}}V, \tag{3.4}$$

*where $V \in \mathbb{C}_r^{n\times k}(P, Q)$ is arbitrary.*

2. *$AX = B$ has a solution $X \in \mathbb{C}_a^{n\times k}(P, Q)$ if and only if $A_{1P}A_{1P}^{+}B_{2Q} = B_{2Q}$. In this case, a general solution $X$ can be written as*

$$X = A_{1P}^{+}B_{2Q} + F_{A_{1P}}W,$$

*where $W \in \mathbb{C}_a^{n\times k}(P, Q)$ is arbitrary.*

**Remark 3.1.1** *[13] According to the statement (1) in Lemma 3.1.2, a equivalent condition for the existence for the $(P, Q)$ generalized reflexive (anti-reflexive) solution to $AX = B$ is that $A_{1P}Y = B_{1Q}(A_{1P}Z = B_{2Q})$ is consistent, and the reflexive solution is given by $\frac{1}{2}(Y + PYQ)$, the anti-reflexive solution is $\frac{1}{2}(Z - PZQ)$*

An alternative result can be obtained by a similar approach.

**Corollary 3.1.1** *[13] Let $C \in \mathbb{C}^{k\times l}$ and $D \in \mathbb{C}^{n\times l}$ be given. Then*

1. *$XC = D$ has a solution $X \in \mathbb{C}_r^{n\times k}(P, Q)$ if and only if $D_{3P}C_{3Q}^{+}C_{3Q} = D_{3P}$. In this case, a general solution $X$ can be written as*

$$X = D_{3P}C_{3Q}^{+} + VE_{C_{3Q}}$$

.

*where $V \in \mathbb{C}_r^{n\times k}(P, Q)$ is arbitrary.*

2. *$XC=D$ has a solution $X\in\mathbb{C}_a^{n\times k}(P,Q)$ if and only if $D_{4P}C_{3Q}^{+}C_{3Q}=D_{4P}$. In this case, a general solution $X$ can be written as*

$$X=D_{4P}C_{3Q}^{+}+VE_{C_{3Q}},$$

*where $W\in\mathbb{C}_a^{n\times k}(P,Q)$ is arbitrary.*

Next, we give the main results of this paper on the common $(P,Q)$ generalized reflexive and anti-reflexive solutions of matrix equations $AX=B$ and $XC=D$.

**Theorem 3.1.1** *[13] Let $P\in\mathbb{C}^{n\times n}$ and $Q\in\mathbb{C}^{k\times k}$ be two generalized reflection matrices, $A\in\mathbb{C}^{m\times n}, B\in\mathbb{C}^{m\times k}, C\in\mathbb{C}^{k\times l}$ and $D\in\mathbb{C}^{n\times l}$ be given . Then*
*(1)$AX=B, XC=D$ has a common $(P,Q)$ generalized reflexive solution if and only if*

$$A_{1P}A_{1P}^{+}B_{1Q}=B_{1Q}, D_{3P}C_{3Q}^{+}C_{3Q}=D_{3P}, BC=AD \quad and \quad BQC=APD \tag{3.5}$$

*In which case, a general common $(P,Q)$ generalized reflexive solution is given by*

$$X=A_{1P}^{+}B_{1Q}+F_{A_{1P}}D_{3P}C_{3Q}^{+}+FA_{1P}WE_{C_{3Q}}, \tag{3.6}$$

*where $W\in\mathbb{C}_r^{n\times k}(P,Q)$ is arbitrary.*
*(2)$AX=B, XC=D$ has a common $(P,Q)$ generalized anti-reflexive solution if and only if $A_{1P}A_{1P}^{+}B_{2Q}=B_{2Q}, D_{4P}C_{3Q}^{+}C_{3Q}=D_{4P}, BC=AD$ and $BQC=-APD$*
*In which case, a general common $(P,Q)$ generalized reflexive solution is given by*

$$X=A_{1P}^{+}B_{2Q}+FA_{1P}D_{4P}C_{3Q}^{+}+F_{A_{1P}}ZE_{C_{3Q}},$$

*where $Z\in\mathbb{C}_a^{n\times k}(P,Q)$ is arbitrary.*

**Proof.**In the above, we show that $AX=B$ has a solution $X\in\mathbb{C}_r^{n\times k}(P,Q)$ if and only if $A_{1P}Y=B_{1Q}$ is solvable. In the same way, $XC=D$ has a solution $X\in\mathbb{C}_r^{n\times k}(P,Q)$ if and only if $YC_{3Q}=D_{3P}$ is solvable.
Hence, it is obvious that $AX=B, XC=D$ has a common $(P,Q)$ generalized reflexive solution if and only if $A_{1P}Y=B_{1Q}$ and $YC_{3Q}=D_{3P}$ has a common solution, and $X=\frac{1}{2}(Y+PYQ)$. Applying (3.2), (3.5) is evident.
Together with Lemma 3.1.1 and the fact that $FA_{1P}D_{3P}C_{3Q}^{+}\in\mathbb{C}_r^{n\times k}(P,Q)$, (3.6) is achieved.
Similarly methods show the statement (2). The proof is complete ■

## 3.2 The best solution estimator to a given matrix

In this section, we deduce the explicit expressions of the solution of the matrix nearness problem (3.3).

First, we verify the following results.

**Lemma 3.2.1** *[13] Let* $X_0 \in \mathbb{C}^{n\times k}, F_{A_{1P}} \in \mathbb{C}_r^{n\times n}(P,P), E_{C_{3Q}} \in \mathbb{C}_r^{k\times k}(Q,Q), W \in \mathbb{C}_r^{n\times k}(P,Q)$ *and* $Z \in \mathbb{C}_a^{n\times k}(P,Q)$. *Then*

$$\|F_{A_{1P}}WE_{C_{3Q}} - F_{A_{1P}}X_0E_{C_{3Q}}\|_F^2 = \|F_{A_{1P}}WE_{C_{3Q}} - F_{A_{1P}}PX_0QE_{C_{3Q}}\|_F^2 \tag{3.7}$$

$$= \|F_{A_{1P}}WE_{C_{3Q}} - \frac{1}{2}F_{A_{1P}}(X_0 + PX_0Q)E_{C_{3Q}}\|_F^2 + \frac{1}{2}tr\left[F_{A_{1P}}X_0E_{C_{3Q}}(X_0 - PX_0Q)^*F_{A_{1P}}\right] \tag{3.8}$$

*Similarly,*

$$\|F_{A_{1P}}ZE_{C_{3Q}} - F_{A_{1P}}X_0E_{C_{3Q}}\|_F^2 = \|F_{A_{1P}}ZE_{C_{3Q}} - F_{A_{1P}}PX_0QE_{C_{3Q}}\|_F^2 \tag{3.9}$$

$$= \|F_{A_{1P}}ZE_{C_{3Q}} - \frac{1}{2}F_{A_{1P}}(X_0 + PX_0Q)E_{C_{3Q}}\|_F^2 + \frac{1}{2}tr\left[F_{A_{1P}}X_0E_{C_{3Q}}(X_0 - PX_0Q)^*F_{A_{1P}}\right] \tag{3.10}$$

**Proof.**Since $\|.\|_F$ is an unitarily invariant norm, according to the assumptions, (3.7) is obvious. Moreover, we have

$$\begin{aligned}
tr\left(F_{A_{1P}}WE_{C_{3Q}}QX_0^*PF_{A_{1P}}\right) &= tr\left(PF_{A_{1P}}PPWQQE_{C_{3Q}}QX_0^*PF_{A_{1P}}P\right), \\
&= tr\left(PF_{A_1P}WE_{C_{3Q}}X_0^*F_{A_1P}\right), \\
tr\left(F_{A_{1P}}PX_0QE_{C_{3Q}}W^*F_{A_{1P}}\right) &= tr\left(F_{A_1P}X_0^*E_{C_{3Q}}W^*F_{A_1(P)}\right), \\
tr\left(F_{A_{1P}}X_0E_{C_{3Q}}QX_0^*PF_{A_1P}\right) &= tr\left(F_{A_1P}PX_0QE_{C_{3Q}}X_0^*F_{A_1P},\right). \\
tr\left(F_{A_1P}PX_0E_{C_{3Q}}X_0^*PF_{A_1P}\right) &= tr\left(F_{A_1P}X_0E_{C_{3Q}}X_0^*F_{A_1P}\right),
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|F_{A_1P}WE_{C_{3Q}} - \tfrac{1}{2}F_{A_1P}(X_0 + PX_0Q)E_{C_{3Q}}\|_F^2 \\
&= tr\left\{\left[F_{A_1P}WE_{C_{3Q}} - \tfrac{1}{2}F_{A_1P}(X_0 + PX_0Q)E_{C_{3Q}}F_{A_1P}WE_{C_{3Q}} - \tfrac{1}{2}(X_0 + PX_0Q)E_{C_{3Q}}\right]^*\right\} \\
&= tr\{F_{A_1P}WE_{C_{3Q}}W^*F_{A_1P} - \tfrac{1}{2}F_{A_1P}WE_{C_{3Q}}(X_0 + PX_0Q)^*F_{A_1P} - \tfrac{1}{2}F_{A_1P}(X_0 + PX_0Q)E_{C_{3Q}}W^*F_{A_1P} \\
&+ \tfrac{1}{4}F_{A_1P}X_0E_{C_{3Q}}X_0^*F_{A_1P} + \tfrac{1}{4}F_{A_1P}X_0E_{C_{3Q}}QX_0^*PF_{A_1P} \\
&+ \tfrac{1}{4}F_{A_1P}X_0QE_{C_{3Q}}X_0^*F_{A_1P} + \tfrac{1}{4}F_{A_1P}PX_0E_{C_{3Q}}X_0^*PF_{A_1P}\}
\end{aligned}$$

Hence, (3.8) is evident. (3.9) and (3.10) can be verified similarly ■

The following theorem gives the explicit expressions of the solutions of the matrix nearness problem (3.3).

**Theorem 3.2.1** *[13] Give a matrix $X_0 \in \mathbb{C}^{n\times k}$*
(1)*Assume the solution set $S_E \subseteq \mathbb{C}_r^{n\times k}(P,Q)$ of Eq. (3.1) is nonempty, then the matrix nearness problem (3.3) has an unique solution $\hat{X}$ in $S_E$, which can be written as*

$$\hat{X} = A_{1P}^{+}B_{1Q} + F_{A_{1P}}D_{3P}C_{3Q}^{+} + \frac{1}{2}F_{A_{1P}}(X_0 + PX_0Q)E_{C_{3Q}}. \tag{3.11}$$

(2)*Assume the solution set $S_E \subseteq \mathbb{C}_a^{n\times k}(P,Q)$ of Eq. (3.1) is nonempty, then the matrix nearness problem (3.3) has an unique solution $\hat{X}$ in $S_E$, which can be written as*

$$\hat{X} = A_{1P}^{+}B_{2Q} + F_{A_{1P}}D_{4P}C_{3Q}^{+} + \frac{1}{2}F_{A_{1P}}(X_0 + PX_0Q)E_{C_{3Q}}$$

.

**Proof.**(1) Let $X \in S_E$, it follows from Lemma 2.0.2, Lemma 3.2.1 and (3.5) that

$$\begin{aligned}
\|X - X_0\|_F^2 &= \|A_{1P}^{+}B_{1Q} + F_{A_1P}W - D_{3P}C_{3Q}^{+} + F_{A_1P}WE_{C_{3Q}} - X_0\|_F^2 \\
&= \|A_{1P}^{+}B_{1Q} - A_{1P}^{+}A_{1P}X_0\|_F^2 + \|F_{A_1P}D_{3P}C_{3Q}^{+} + F_{A_1P}WE_{C_{3Q}}X_0 - F_{A_1P}X_0\|_F^2 \\
&= \|A_{1P}^{+}B_{1Q} - A_{1P}^{+}A_{1P}X_0\|_F^2 + \|F_{A_1P}D_{3P}C_{3Q}^{+} - F_{A_1P}X_0C_{3Q}C_{3Q}^{+}\|_F^2 \\
&\quad + \|F_{A_1P}WE_{C_{3Q}} - F_{A_1P}X_0EC_{3Q}\|_F^2 \\
&\quad + \|A_{1P}^{+}B_{1Q} - A_{1P}^{+}A_{1P}X_0\|_F^2 + \|F_{A_1P}D_{3P}C_{3Q}^{+} - F_{A_1P}X_0C_{3Q}C_{3Q}^{+}\|_F^2 \\
&\quad + \tfrac{1}{2}tr\left[F_{A_1P}X_0E_{C_{3Q}}(X_0 - PX_0Q)^{*}F_{A_1P}\right] \\
&\quad + \|F_{A_1P}WE_{C_{3Q}} - \tfrac{1}{2}F_{A_1P}(X_0 - PX_0Q)E_{C_{3Q}}Q\|_F^2
\end{aligned}$$

Therefore, there exists $\hat{X} \in S_E$ such that the matrix nearness problem (3.3) holds ifand only if there exists $W \in \mathbb{C}_r^{n\times k}(P,Q)$ such that

$$\min_{V\in\mathbb{C}_r^{n\times k}(P,Q)} \|F_{A_1P}WE_{C_{3Q}} - \frac{1}{2}F_{A_1P}(X_0 + PX_0Q)\|_F$$

Obviously, we can take $W = \frac{1}{2}(X_0 + PX_0Q)$ .

Hence,

$$\min_{V\in\mathbb{C}_r^{n\times k}(P,Q)} \|F_{A_1P}WE_{C_{3Q}} - \frac{1}{2}F_{A_1P}(X_0 + PX_0Q)\|_F = 0$$

Therefore, (3.11) is evident .

Similarly, statement (2) can be obtained ■

# CONCLUSION

The equations $AX = B$, $XC = D$ are of the most well-known matrix equations in matrix theory and its applications. For this importance, In this work we considered the $(P, Q)$ generalized reflexive (anti-reflexive) solutions for the matrix equation $AX = B$, and the system $AX = B$, $XC = D$ where we gave necessary and sufficient conditions for the existence of the $(P, Q)$ generalized reflexive (anti-reflexive) solutions and the common $(P, Q)$ generalized reflexive (anti-reflexive) solutions to these matrix equations respectively, with respect to the generalized reflection matrix dual $(P, Q)$. Also for all solutions exist, we derive the explicit expression of the best solution estimator to a given matrix in the Frobenius norm.

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